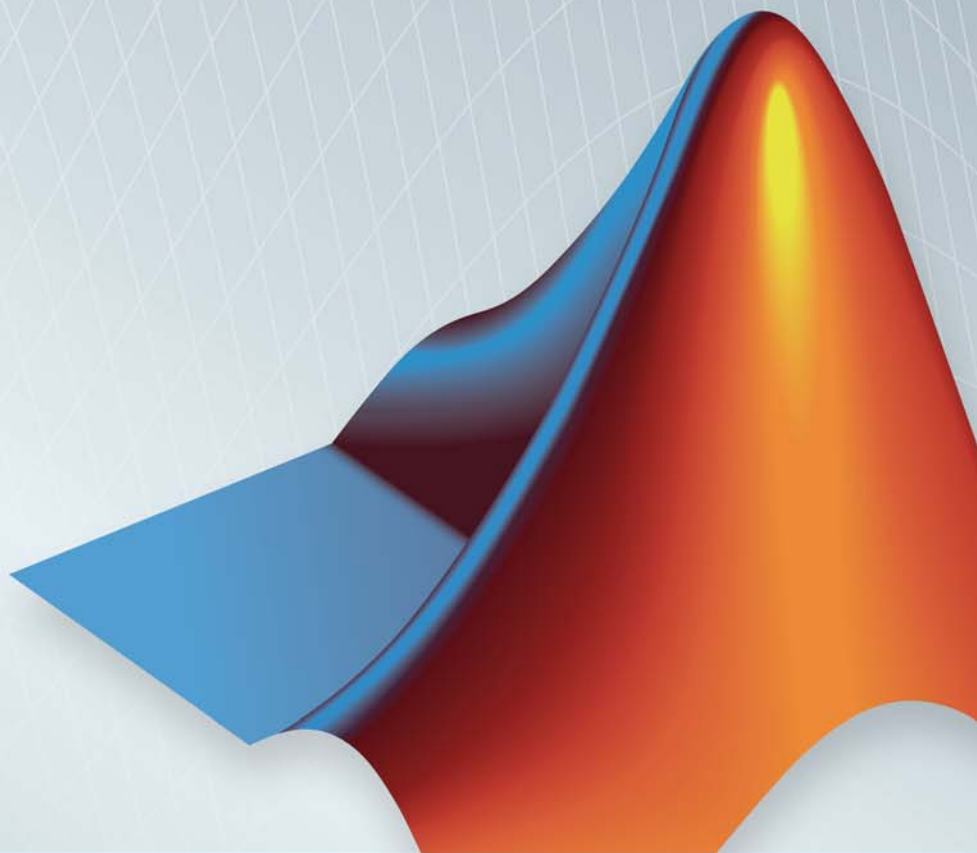


**MuPAD®**

User's Guide

**R2014a**



**MATLAB®**



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*MuPAD® User's Guide*

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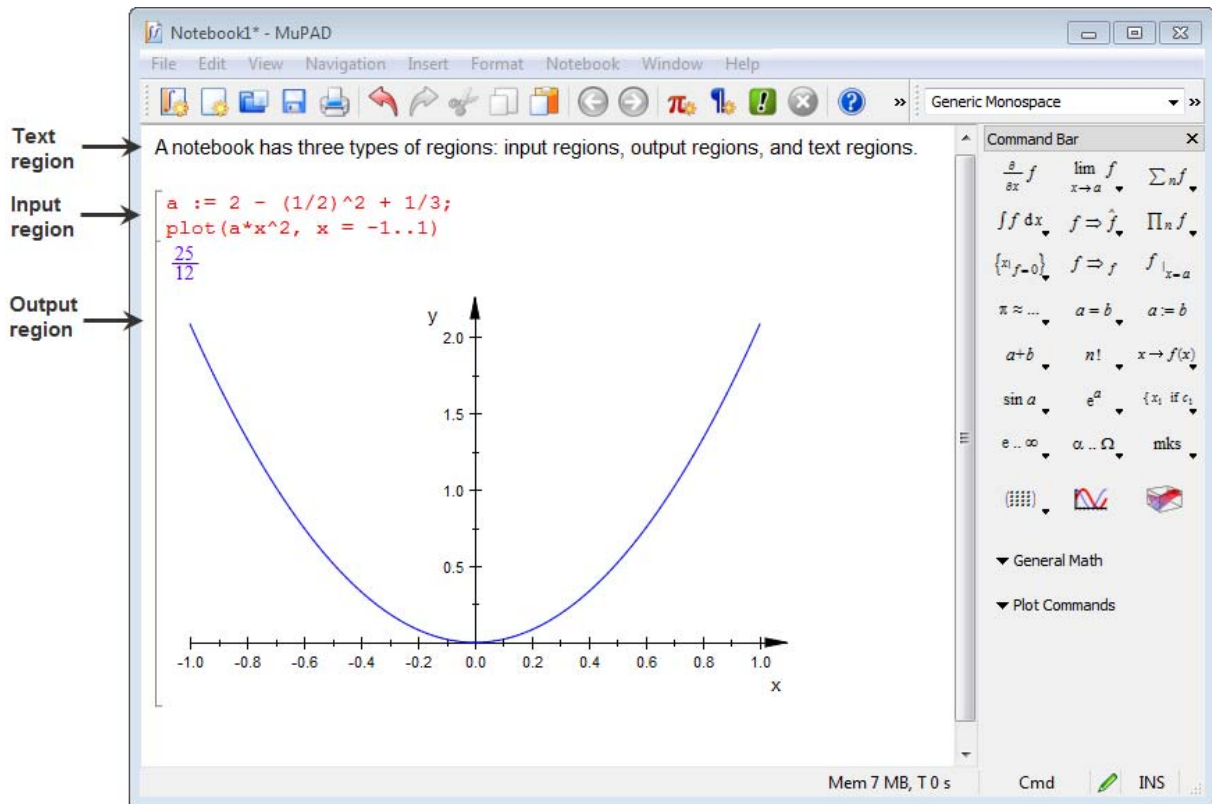
# Getting Started

---

- “Desktop Overview” on page 1-2
- “Evaluate Mathematical Expressions and Commands” on page 1-4
- “Quickly Access Standard MuPAD Functions” on page 1-7
- “Access Help for Particular Command” on page 1-15
- “Perform Computations” on page 1-20
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- “Debug MuPAD Code Using the Debugger” on page 1-184

## Desktop Overview

A MuPAD® notebook has three types of regions: input regions, output regions, and text regions.



In the input regions, marked by grey brackets, you can type mathematical expressions and commands in the MuPAD language. For example, type the following expression and press **Enter** to evaluate the result:

$$3 \cdot 2^{10} + \frac{1}{3} - 3$$

$$\frac{9208}{3}$$

The results (including graphics) appear in a new output region. The default font color for input regions is red, and the default font color for output regions is blue. To customize default settings, see [Changing Default Format Settings](#).

When you evaluate an expression in the bottom input region, MuPAD inserts a new input region below. To insert new input regions in other parts of a notebook:

- 1 Select the place in a notebook where you want to insert a new input region
- 2 Insert a new input region:
  - To insert an input region below the cursor position, select **Insert>Calculation** from the main menu.
  - To insert an input region above the cursor position, select **Insert>Calculation Above** from the main menu.

You can type and format text in a notebook similar to working in any word processing application. To start a new text region, click outside the gray brackets and start typing.

Also, to insert a new text region, you can select **Insert>Text Paragraph** or **Insert>Text Paragraph Above**. You cannot insert a text region between adjacent input and output regions.

You can exchange data between different regions in a notebook. For example, you can:

- Copy expressions and commands from the text regions to the input regions and evaluate them.
- Copy expressions and commands from the input regions to the text regions.
- Copy results including mathematical expressions and graphics from the output regions to the text regions.
- Copy results from the output regions to the input regions. Mathematical expressions copied from the output regions appear as valid MuPAD input commands.

You cannot paste data into the output regions. To change the results, edit the associated input region and evaluate it by pressing **Enter**.

## Evaluate Mathematical Expressions and Commands

### In this section...

“Working in a Single Input Region” on page 1-4

“Working with Multiple Input Regions” on page 1-5

### Working in a Single Input Region

To evaluate an expression or execute a command in a notebook, press **Enter**:

$3 \cdot 2^{10} + 1/3 - 3$

$\frac{9208}{3}$

The results appear in the same grey bracket below the input data. By default, the commands and calculations you type appear in red color, the results appear in blue.

To suppress the output of a command, terminate a command with a colon. This allows you to hide irrelevant intermediate results. For example, assign the factorial of 123 to the variable *a*, and the factorial of 132 to the variable *b*. In MuPAD, the assignment operator is `:=` (the equivalent function is `_assign`). The factorial operator is `!` (the equivalent function is `fact`). Terminate these assignments with colons to suppress the outputs. Here MuPAD displays only the result of the division *a/b*:

`a := 123!: b := 132!: a/b`

$\frac{1}{9206492916741120000}$

`delete a, b:`

You can enter several commands in an input region separating them by semicolons or colons:

`a+b; a*b; a^b`

$$a+b$$

$$a b$$

$$a^b$$

To start a new line in an input region, press **Ctrl+Enter** or **Shift+Enter**.

## Working with Multiple Input Regions

If you have several input regions, you can go back to previous calculations and edit and reevaluate them. If you have a sequence of calculations in several input regions, the changes in one region do not automatically propagate throughout other regions. For example, suppose you have the following calculation sequence:

$$y := \exp(2*x)$$

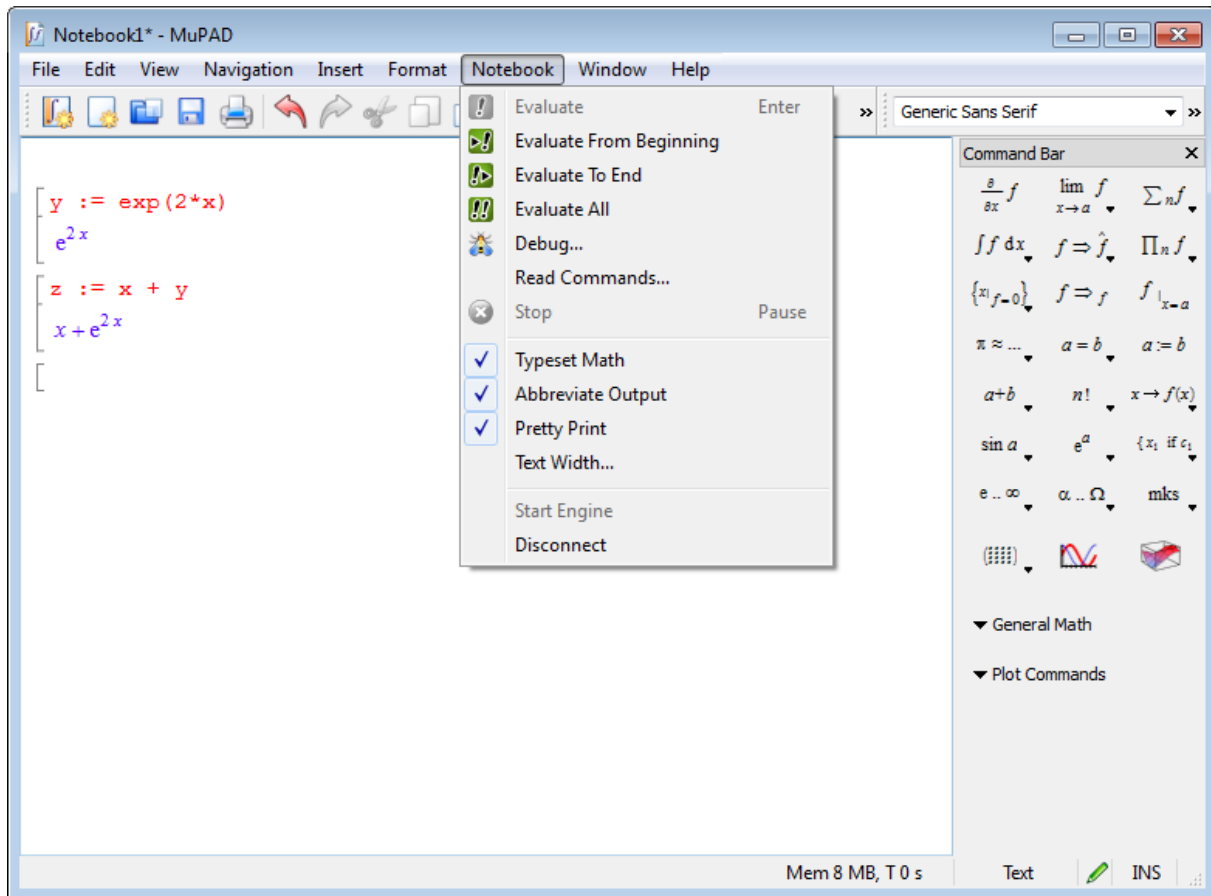
$$e^{2x}$$

$$z := x + y$$

$$x + e^{2x}$$

If you change the value of the variable  $y$ , the change does not automatically apply to the variable  $z$ . To propagate the change throughout different input regions, select **Notebook** from the main menu. From here you can:

- Select **Evaluate** to evaluate calculations in one input region.
- Select **Evaluate From Beginning** to evaluate calculations in the input regions from the beginning of a notebook to the cursor position.
- Select **Evaluate To End** to evaluate calculations in the input regions from the cursor position to the end of a notebook.
- Select **Evaluate All** to evaluate calculations in all input regions in a notebook.



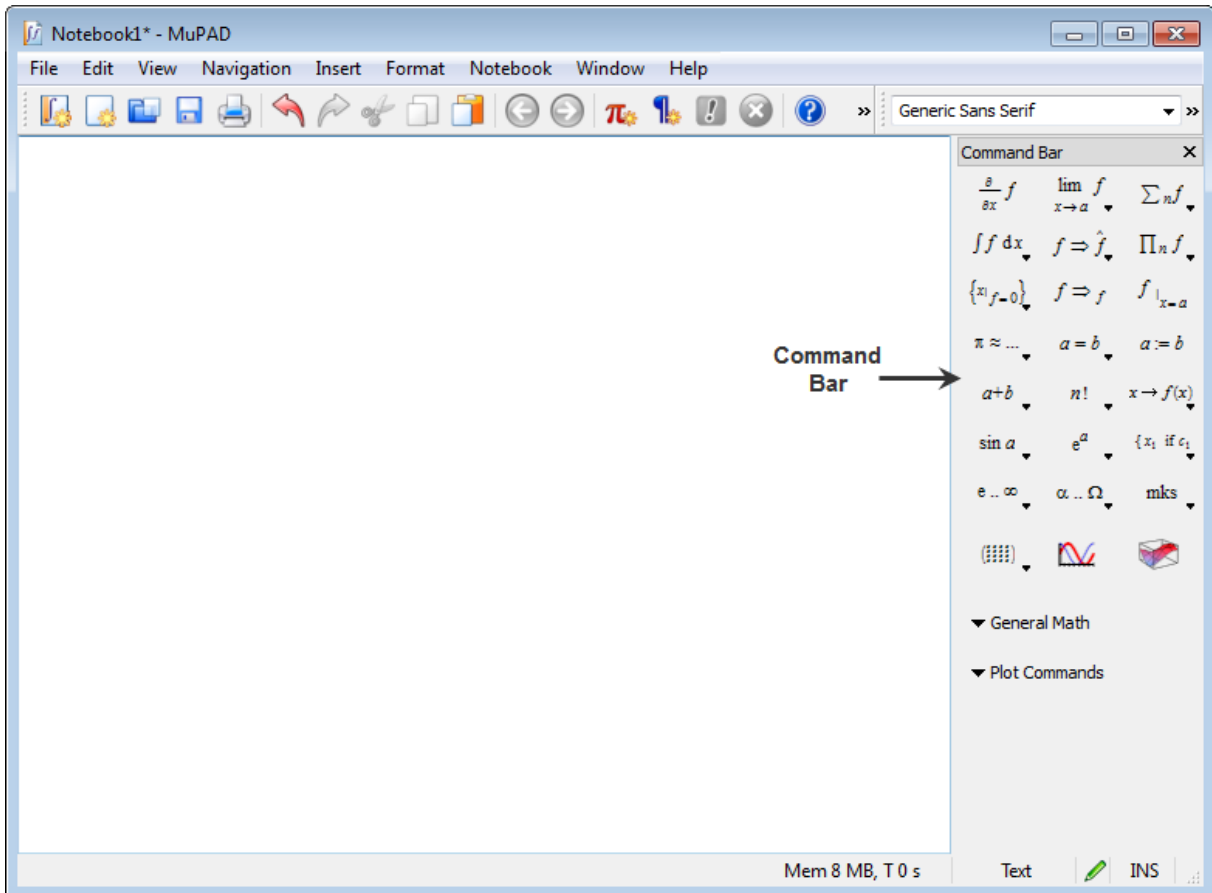
Also, you can propagate the change throughout multiple input regions by pressing **Enter** in each input region.



## Quickly Access Standard MuPAD Functions

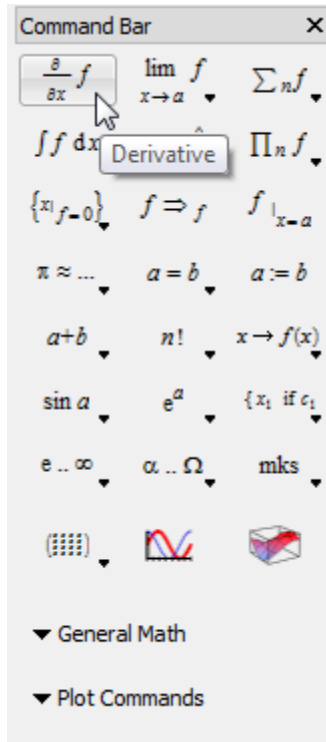
To eliminate syntax errors and to make it easy to remember the commands and functions, MuPAD can automatically complete the command you start typing. To automatically complete the command, press **Ctrl+space**.

You also can access common functions through the Command Bar.



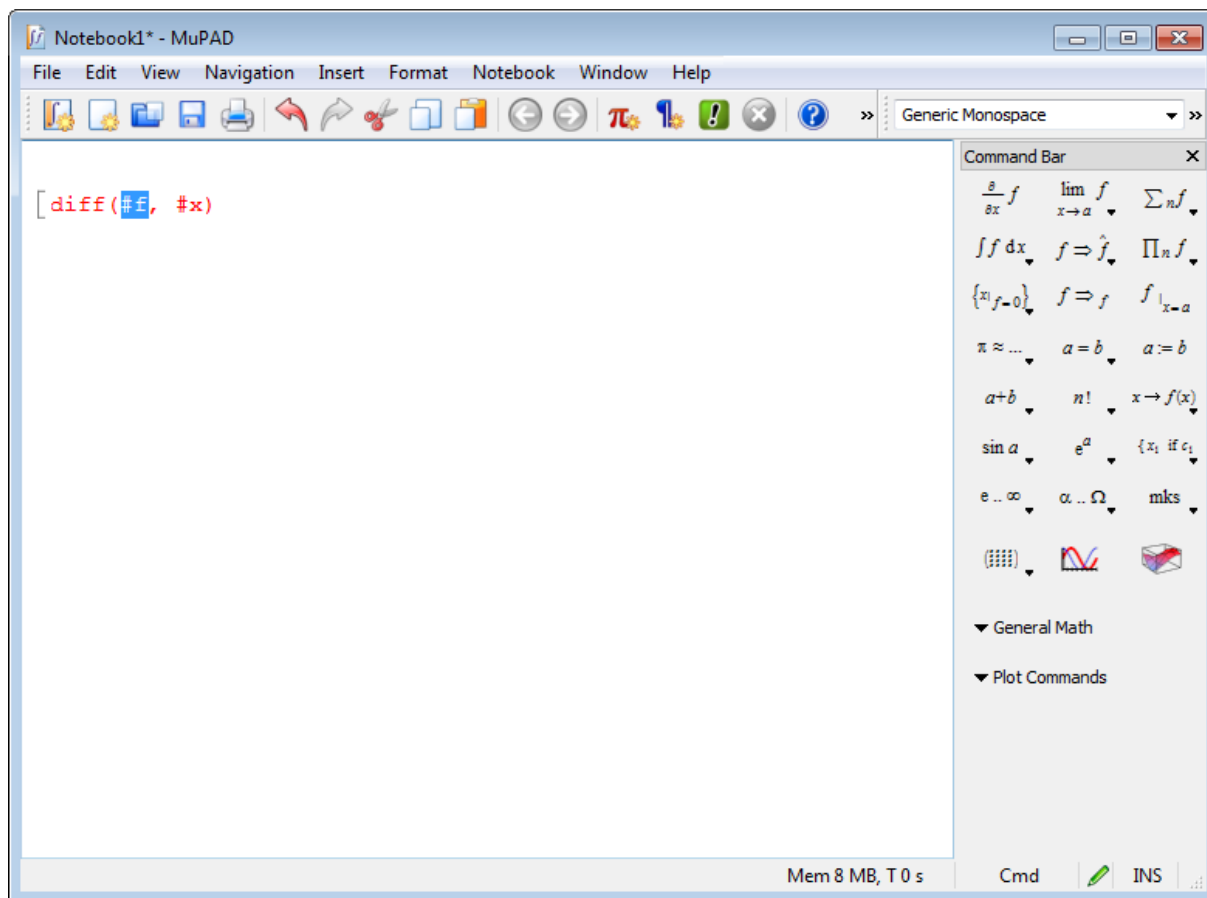
If you do not see the Command Bar, select **View>Command Bar**.

The buttons on the Command Bar display the function labels. To see the name of the function that the button presents, hover your cursor over the button.

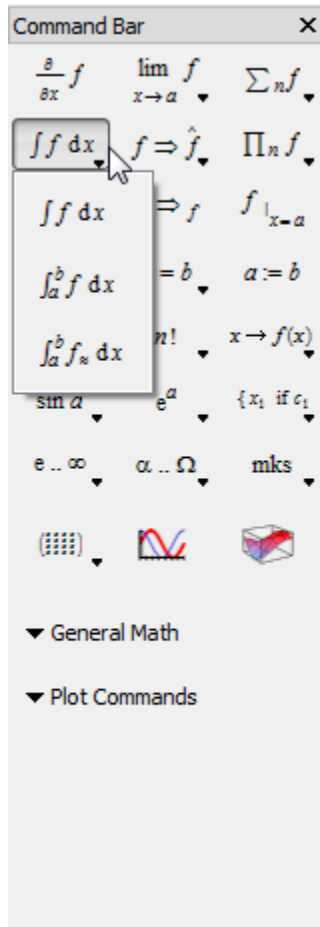


To insert a function:

- 1 Point the cursor at the place in an input region where you want to insert a function.
- 2 Click the button corresponding to the function.
- 3 Insert the parameters instead of the # symbols. You can switch between the parameters by pressing the **Tab** key.

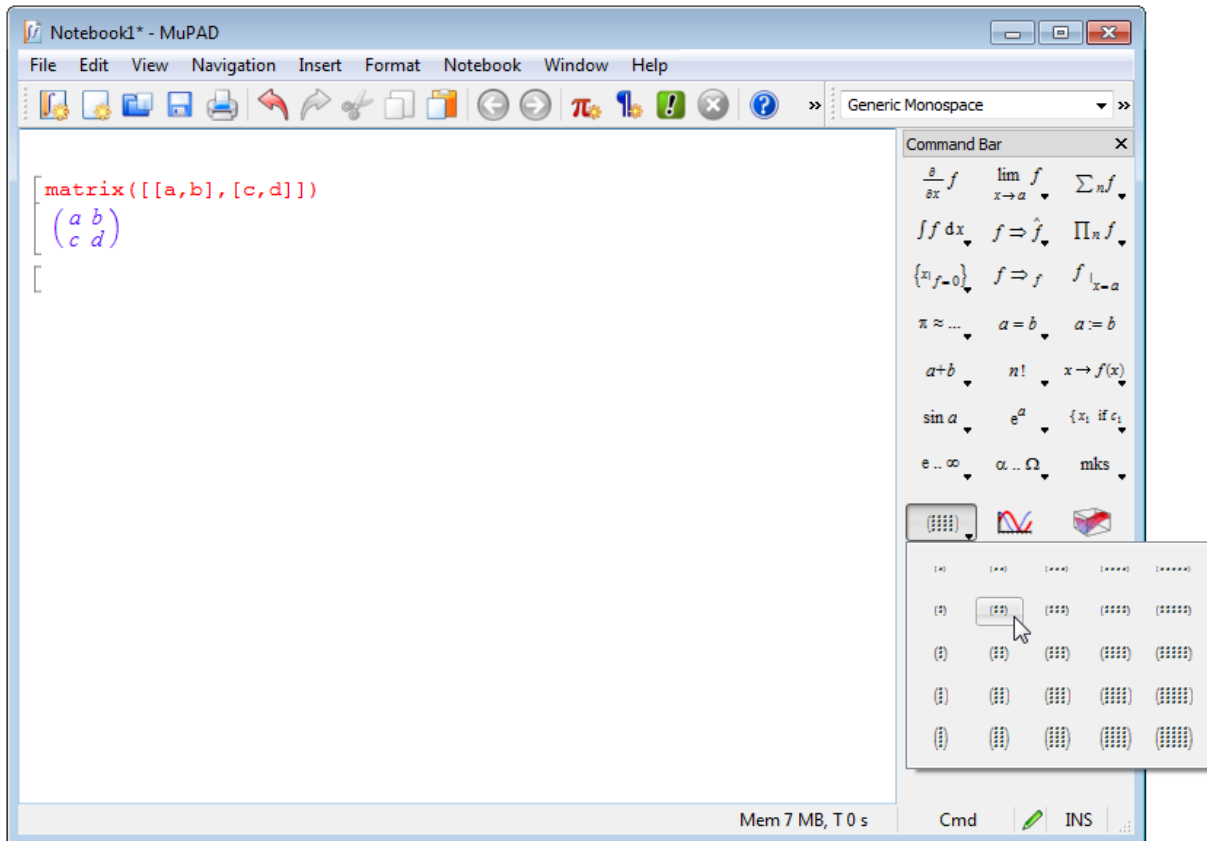


Most of the buttons on the Command Bar include a drop-down menu with a list of similar functions. The buttons display a small triangle in the bottom right corner. Click the button to open the list of functions.

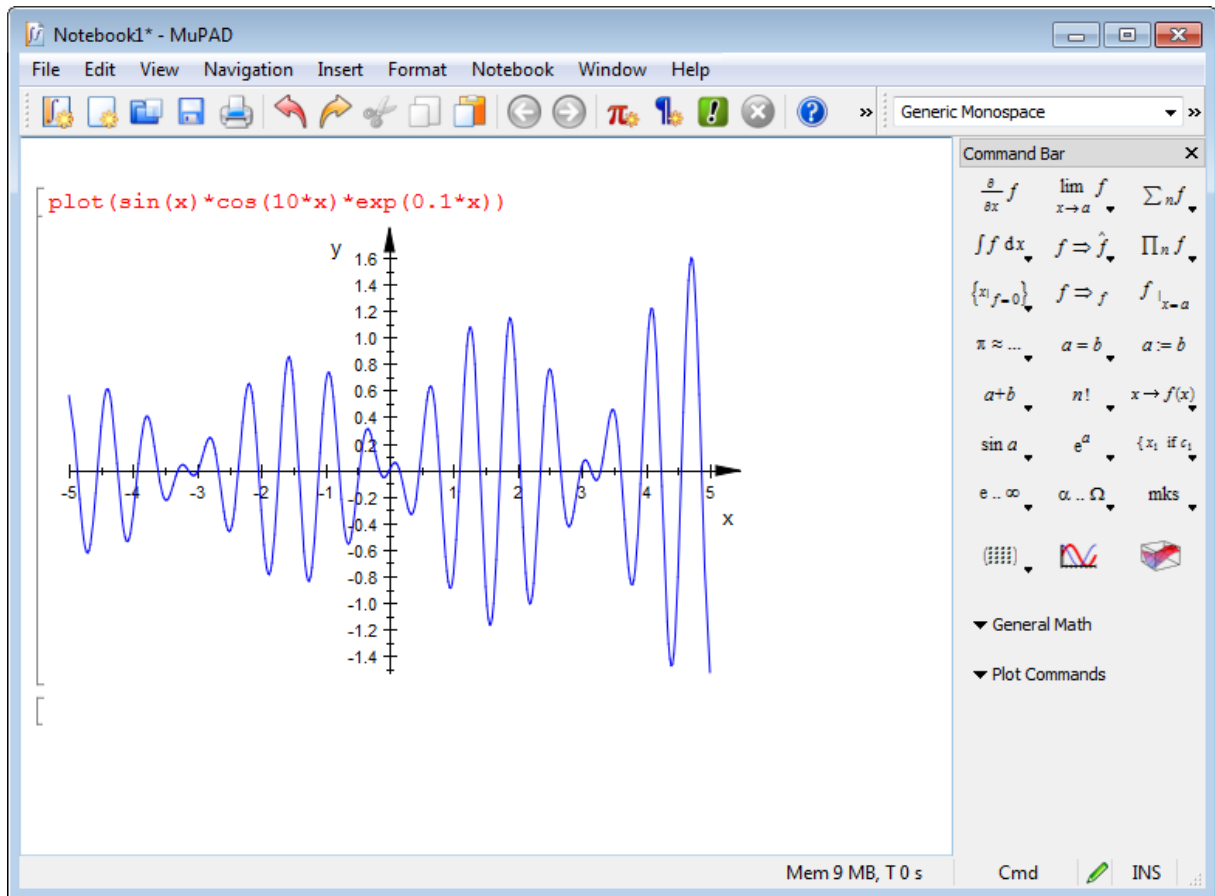


Using the Command Bar, you also can create the following:

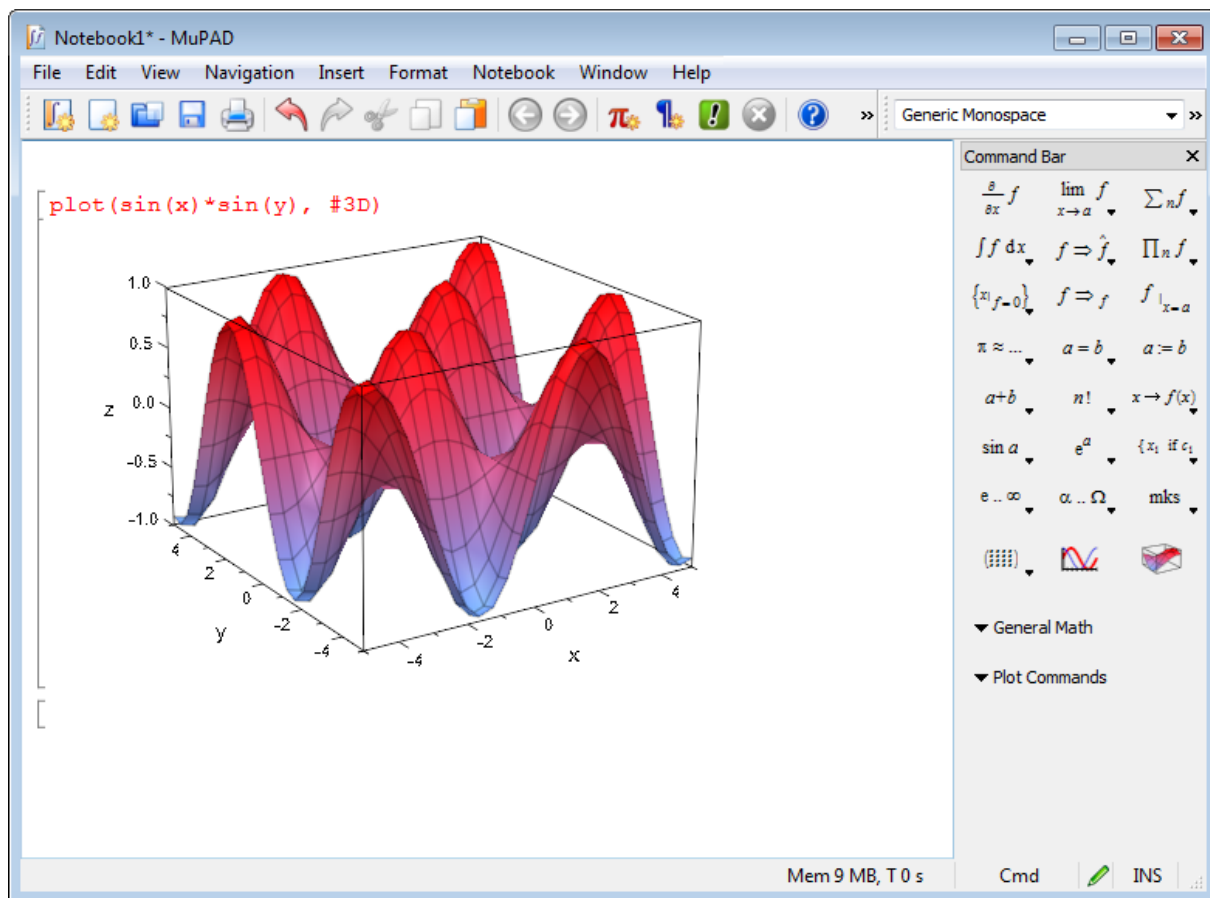
- Vectors and matrices



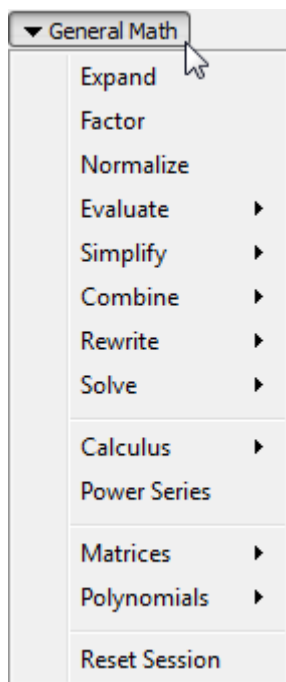
- 2-D plots and animations



- 3-D plots



General Math and Plot Commands menus at the bottom of the Command Bar display the categorized lists of functions.



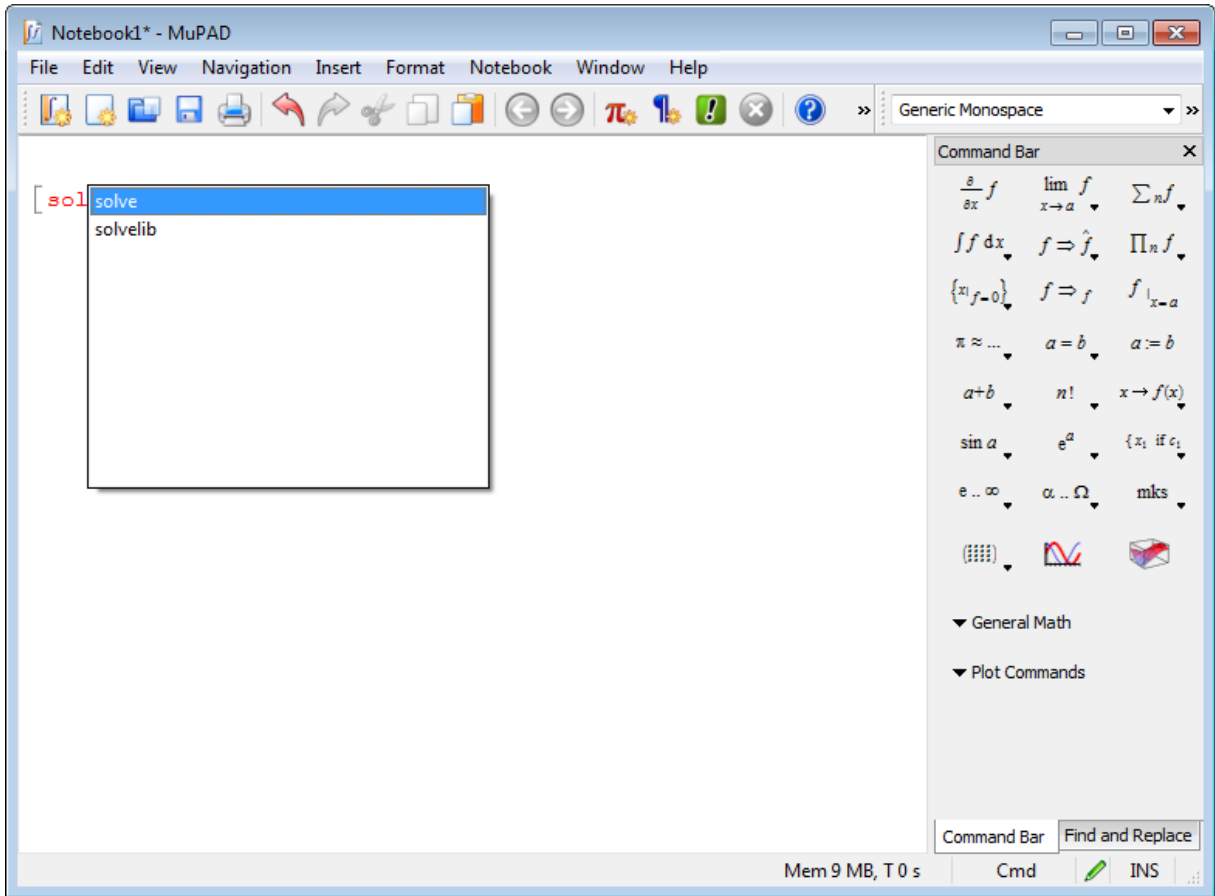


## Access Help for Particular Command

In this section...
“Autocomplete Commands” on page 1-15
“Use Tooltips and the Context Menu” on page 1-16
“Use Help Commands” on page 1-18

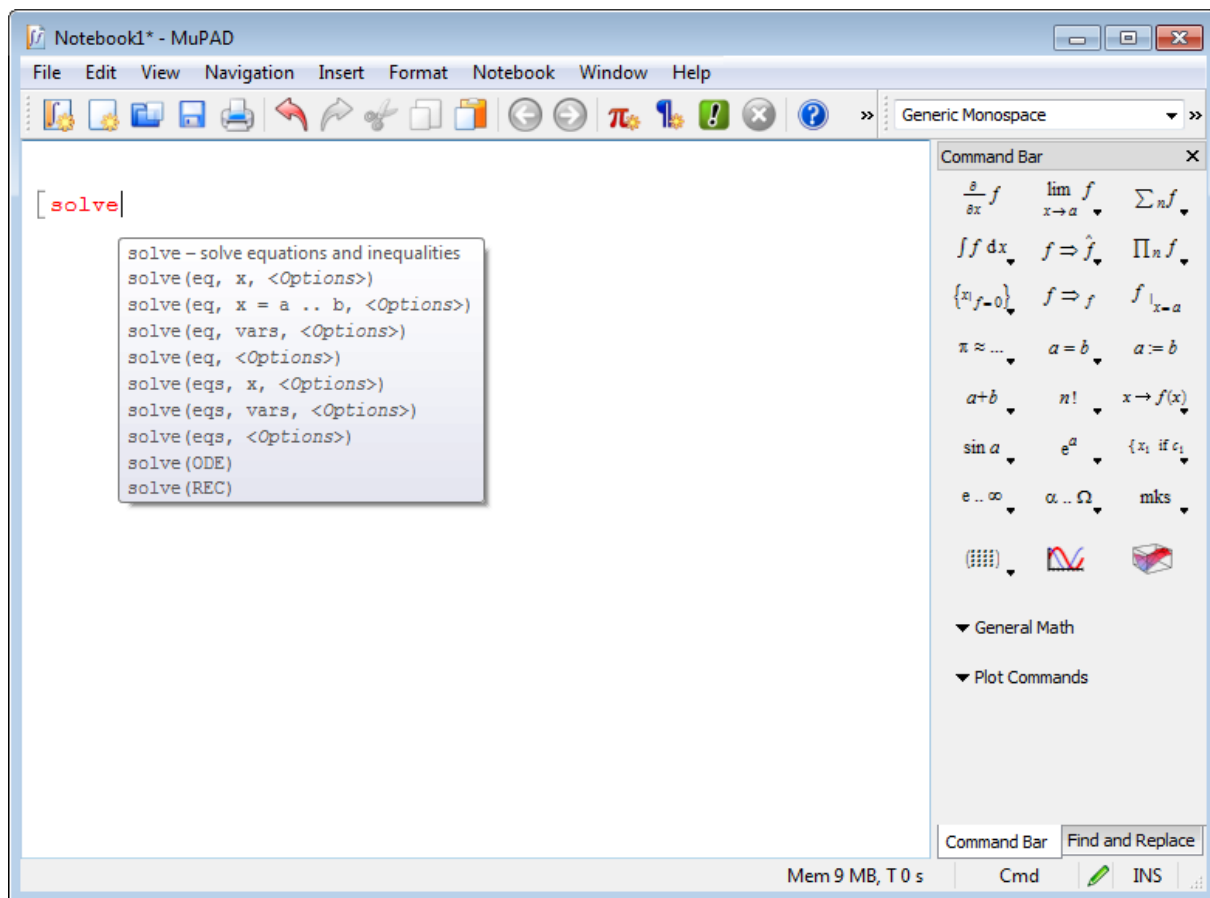
### Autocomplete Commands

MuPAD helps you complete the names of known commands as you type them so that you can avoid spelling mistakes. Type the first few characters of the command name, and then press **Ctrl+space**. If there is exactly one name of a command that starts with these letters, MuPAD completes the command. If more than one name starts with the characters you typed, MuPAD displays a list of all names starting with those characters.

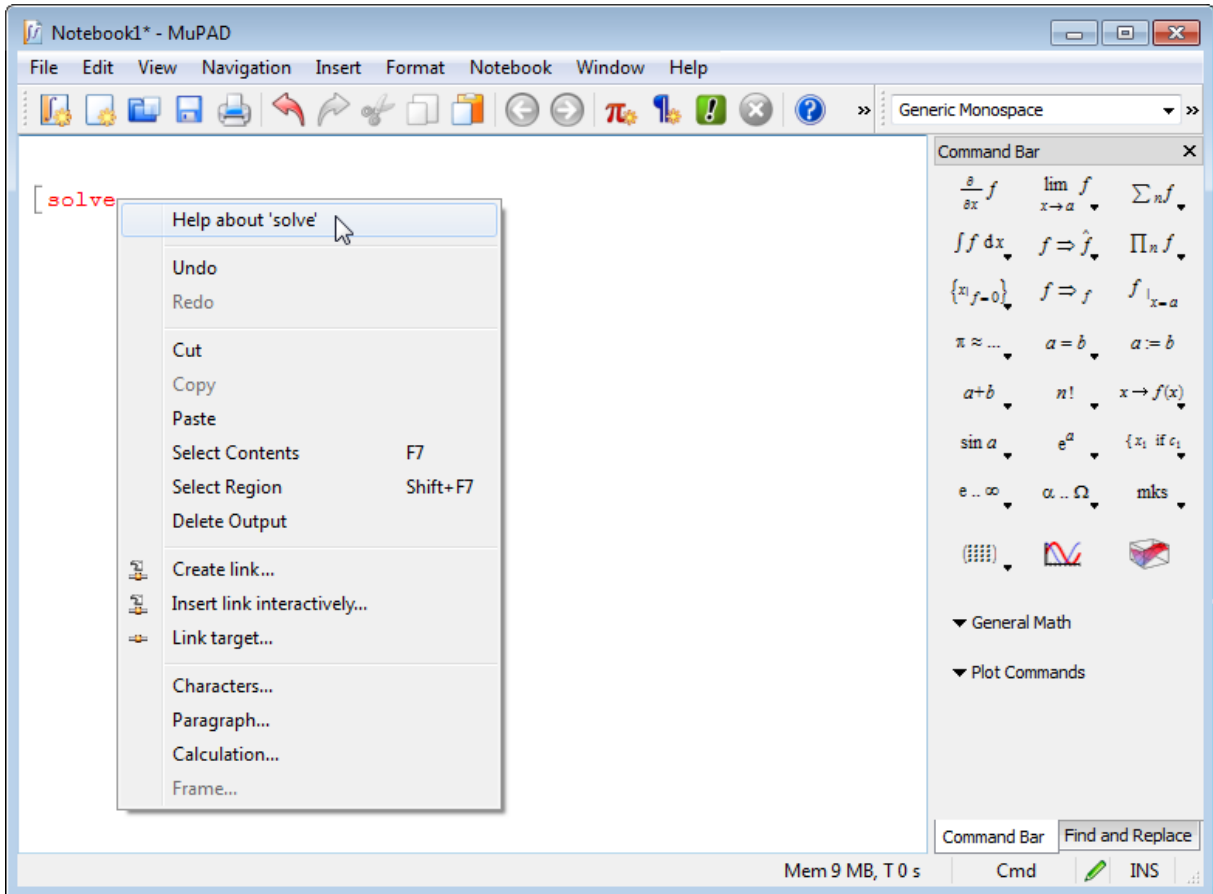


## Use Tooltips and the Context Menu

To get a brief description and acceptable syntax for a function, type the function name in a notebook and hover your cursor over the command.



For more detailed information, right-click the name of a command and select **Help about** from the context menu.



## Use Help Commands

You can get a brief description of a command and a list of acceptable input parameters using `info`:

```
info(solve)
```

```
solve -- solve equations and inequalities [try ?solve for options]
```

For more detailed information about the command and its input parameters, use the ? command:

`?solve`

## Perform Computations

In this section...
“Compute with Numbers” on page 1-20
“Differentiation” on page 1-25
“Integration” on page 1-28
“Linear Algebra” on page 1-29
“Solve Equations” on page 1-33
“Manipulate Expressions” on page 1-36
“Use Assumptions in Your Computations” on page 1-39

### Compute with Numbers

#### Types of Numbers

Using MuPAD, you can operate on the following types of numbers:

- Integer numbers
- Rational numbers
- Floating-point numbers
- Complex numbers

By default, MuPAD assumes that all variables are complex numbers.

#### Compute with Integers and Rationals

When computing with integers and rational numbers, MuPAD returns integer results

$$2 + 2$$

4

or rational results:

$$(1 + (5/2*3)) / (1/7 + 7/9)^2$$

$$\frac{67473}{6728}$$

If MuPAD cannot find a representation of an expression in an integer or rational form, it returns a symbolic expression:

$$56^{(1/2)}$$

$$\sqrt{56}$$

### Compute with Special Mathematical Constants

You can perform exact computations that include the constants

$e = \exp(1) = 2.718\dots$  and  $\pi = 3.1415\dots$ :

$$2 * (\exp(2) / \text{PI})$$

$$\frac{2 e^2}{\pi}$$

For more information on the mathematical constants implemented in MuPAD, see “Constants”.

### Approximate Numerically

By default, MuPAD performs all computations in an exact form. To obtain a floating-point approximation to an expression, use the `float` command.

For example:

$$\text{float}(\text{sqrt}(56))$$

$$7.483314774$$

The accuracy of the approximation depends on the value of the global variable `DIGITS`. The variable `DIGITS` can assume any integer value between 1 and  $2^{29} + 1$ . For example:

```
DIGITS:=20: float(sqrt(56))
```

```
7.4833147735478827712
```

The default value of the variable DIGITS is 10. To restore the default value, enter:

```
delete DIGITS
```

When MuPAD performs arithmetic operations on numbers involving at least one floating-point number, it automatically switches to approximate numeric computations:

```
(1.0 + (5/2*3))/(1/7 + 7/9)^2
```

```
10.02868609
```

If an expression includes exact values such as  $e$  or  $\sin(2)$  and floating-point numbers, MuPAD approximates only numbers:

```
1.0/3*exp(1)*sin(2)
```

```
0.3333333333 e sin(2)
```

To approximate an expression with exact values, use the `float` command:

```
float(1.0/3*exp(1)*sin(2))
```

```
0.8239088907
```

or use floating-point numbers as arguments:

```
1.0/3*exp(1.0)*sin(2.0)
```

```
0.8239088907
```



You also can approximate the constants  $\pi$  and  $e$ :

```
DIGITS:=30: float(PI); float(E); delete DIGITS
```

```
3.14159265358979323846264338328
```

```
2.71828182845904523536028747135
```

### Work with Complex Numbers

In the input regions MuPAD recognizes an uppercase I as the imaginary

unit  $\sqrt{-1}$ . In the output regions, MuPAD uses a lowercase i to display the imaginary unit:

```
sqrt(-1), I^2
```

```
i -1
```

Both real and imaginary parts of a complex number can contain integers, rationals, and floating-point numbers:

```
(1 + 0.2*I)*(1/2 + I)*(0.1 + I/2)^3
```

```
0.0988 - 0.1144 i
```

If you use exact expressions, for example,  $\sqrt{2}$ , MuPAD does not always return the result in Cartesian coordinates:

```
1/(sqrt(2) + I)
```

$$\frac{1}{\sqrt{2} + i}$$

To split the result into its real and imaginary parts, use the `rectform` command:

```
rectform(1/(sqrt(2) + I))
```

$$\frac{\sqrt{2}}{3} - \frac{i}{3}$$

The functions `Re` and `Im` return real and imaginary parts of a complex number:

```
Re(1/(2^(1/2) + I))
```

$$\frac{\sqrt{2}}{3}$$

```
Im(1/(2^(1/2) + I))
```

$$-\frac{1}{3}$$

The function `conjugate` returns the complex conjugate:

```
conjugate(1/(2^(1/2) + I))
```

$$\frac{1}{\sqrt{2} - i}$$

The function `abs` and `arg` return an absolute value and a polar angle of a complex number:

```
abs(1/(2^(1/2) + I));
```

```
arg(1/(2^(1/2) + I))
```

$$\frac{\sqrt{3}}{3}$$

$$-\arctan\left(\frac{\sqrt{2}}{2}\right)$$

## Differentiation

### Derivatives of Single-Variable Expressions

To compute the derivative of a mathematical expression, use the `diff` command. For example:

```
f := 4*x + 6*x^2 + 4*x^3 + x^4: diff(f, x)
```

$$4x^3 + 12x^2 + 12x + 4$$

### Partial Derivatives

You also can compute a partial derivative of a multivariable expression:

```
f := y^2 + 4*x + 6*x^2 + 4*x^3 + x^4: diff(f, y)
```

$$2y$$

### Second- and Higher-Order Derivatives

To find higher order derivatives, use a nested call of the `diff` command

```
diff(diff(diff(sin(x), x), x), x)
```

$$-\cos(x)$$

or, more efficiently:

```
diff(sin(x), x, x, x)
```

$$-\cos(x)$$

You can use the sequence operator `$` to compute second or higher order derivatives:

```
diff(sin(x), x $ 3)
```

$-\cos(x)$

### Mixed Derivatives

`diff(f, x1, x2, ...)` is equivalent to `diff(...diff(diff(f, x1), x2)...)` . The system first differentiates `f` with respect to `x1`, and then differentiates the result with respect to `x2`, and so on. For example

```
diff(diff((x^2*y^2 + 4*x^2*y + 6*x*y^2), y), x)
```

$8x + 12y + 4xy$

is equivalent to

```
diff(x^2*y^2 + 4*x^2*y + 6*x*y^2, y, x)
```

$8x + 12y + 4xy$

---

**Note** To improve performance, MuPAD assumes that all mixed derivatives commute. For example,  $\frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y)$ .

---

This assumption suffices for most of engineering and scientific problems.

For further computations, delete `f`:

```
delete f:
```

### Derivatives of a Function

MuPAD provides two differentiation functions, `diff` and `D`. The `diff` function serves for differentiating mathematical expressions, such as `sin(x)`, `cos(2y)`, `exp(x^2)`, `x^2 + 1`, `f(y)`, and so on.

To differentiate a standard function, such as `sin`, `exp`, `heaviside`, or a custom function, such as `f := x -> x^2 + 1`, use the differential operator `D`:

`D(sin), D(exp), D(heaviside)`

`cos, exp, dirac`

`f := x -> x^2 + 1: D(f)`

`x → 2 x`

' is a shortcut for the differential operator D:

`sin', sin'(x), f'`

`cos, cos(x), (x → 2 x)`

The command `D(f)(x)` assumes that `f` is a univariate function, and represents the derivative of `f` at the point `x`. For example, the derivative of the sine function at the point  $x^2$  is:

`D(sin)(x^2)`

`cos(x2)`

Note that in this example you differentiate the `sin` function, not the function `f := x -> sin(x^2)`. Differentiating `f` returns this result:

`f := x -> sin(x^2): D(f)`

`x → 2 x cos(x2)`

For details about using the operator `D` for computing second- and higher-order derivatives of functions, see [Differentiating Functions](#).

## Integration

### Indefinite Integrals

To compute integrals use the `int` command. For example, you can compute indefinite integrals:

```
int((cos(x))^3, x)
```

$$\sin(x) - \frac{\sin(x)^3}{3}$$

The `int` command returns results without an integration constant.

### Definite Integrals

To find a definite integral, pass the upper and lower limits of the integration interval to the `int` function:

```
int((cos(x))^3, x = 0..PI/4)
```

$$\frac{5\sqrt{2}}{12}$$

You can use `infinity` as a limit when computing a definite integral:

```
int(sin(x)/x, x = -infinity..infinity)
```

$$\pi$$

### Numeric Approximation

If MuPAD cannot evaluate an expression in a closed form, it returns the expression. For example:

```
int(sin(x^2)^2, x = -1..1)
```

$$\int_{-1}^1 \sin(x^2)^2 dx$$

You can approximate the value of an integral numerically using the `float` command. For example:

```
float(int(sin(x^2)^2, (x = -1..1)))
```

```
0.3324031519
```

You also can use the `numeric::int` command to evaluate an integral numerically. For example:

```
numeric::int(sin(x^2)^2, x = -1..1)
```

```
0.3324031519
```

## Linear Algebra

### Create a Matrix

To create a matrix in MuPAD, use the `matrix` command:

```
A := matrix([[1, 2], [3, 4], [5, 6]]);
```

```
B := matrix([[1, 2, 3], [4, 5, 6]])
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

You also can create vectors using the `matrix` command:

```
V := matrix([1, 2, 3])
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

You can explicitly declare the matrix dimensions:

```
C := matrix(3, 3, [[-1, -2, -3], [-4, -5, -6], [-7, -8, -9]]);  
W := matrix(1, 3, [1, 2, 3])
```

$$\begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

$$(1 \ 2 \ 3)$$

If you declare matrix dimensions and enter rows or columns shorter than the declared dimensions, MuPAD pads the matrix with zero elements:

```
F := matrix(3, 3, [[1, -1, 0], [2, -2]])
```

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If you declare matrix dimensions and enter rows or columns longer than the declared dimensions, MuPAD returns the following error message:

```
matrix(3, 2, [[-1, -2, -3], [-4, -5, -6], [-7, -8, -9]])
```

```
Error: The number of columns does not match. [(Dom::Matrix(Dom::Expre
```

You also can create a diagonal matrix:

```
G := matrix(4, 4, [1, 2, 3, 4], Diagonal)
```



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

## Operate on Matrices

To add, subtract, multiply and divide matrices, use standard arithmetic operators. For example, to multiply two matrices, enter:

```
A := matrix([[1, 2], [3, 4], [5, 6]]);
B := matrix([[1, 2, 3], [4, 5, 6]]);
A*B
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}$$

If you add number  $x$  to a matrix  $A$ , MuPAD adds  $x$  times an identity matrix to  $A$ . For example:

```
C := matrix(3, 3, [[-1, -2, -3], [-4, -5, -6], [-7, -8, -9]]);
C + 10
```

$$\begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

$$\begin{pmatrix} 9 & -2 & -3 \\ -4 & 5 & -6 \\ -7 & -8 & 1 \end{pmatrix}$$

You can compute the determinant and the inverse of a square matrix:

```
G := matrix([[1, 2, 0], [2, 1, 2], [0, 2, 1]]); det(G); 1/G
```

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

-7

$$\begin{pmatrix} \frac{3}{7} & \frac{2}{7} & -\frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

### Linear Algebra Library

The MuPAD `linalg` library contains the functions for handling linear algebraic operations. Using this library, you can perform a wide variety of computations on matrices and vectors. For example, to find the eigenvalues of the square matrices  $G$ ,  $F$ , and  $(A*B)$ , use the `linalg::eigenvalue` command:

```
linalg::eigenvalues(G);  
linalg::eigenvalues(F);  
linalg::eigenvalues(A*B)
```

$\{1, 1 - 2\sqrt{2}, 2\sqrt{2} + 1\}$

$\{-1, 0\}$

$$\{0, 43 - 7\sqrt{37}, 7\sqrt{37} + 43\}$$

To see all the functions available in this library, enter `info(linalg)` in an input region. You can obtain detailed information about a specific function by entering `?functionname`. For example, to open the help page on the eigenvalue function, enter `?linalg::eigenvalues`.

## Solve Equations

### Solve Equations with One Variable

To solve a simple algebraic equation with one variable, use the `solve` command:

$$\text{solve}(x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 = 0, x)$$

$$\{-5, -3, -1, 2, 4\}$$

### Solving Equations with Parameters

You can solve an equation with symbolic parameters:

$$\text{solve}(a*x^2 + b*x + c = 0, x)$$

$$\left\{ \begin{array}{ll} \left\{ -\frac{b+\sigma_1}{2a}, -\frac{b-\sigma_1}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{array} \right.$$

where

$$\sigma_1 = \sqrt{b^2 - 4ac}$$

If you want to get the solution for particular values of the parameters, use the `assuming` command. For example, you can solve the following equation assuming that  $a$  is positive:

```
solve(a*x^2 + b*x + c = 0, x) assuming a > 0
```

$$\left\{ -\frac{b + \sqrt{b^2 - 4ac}}{2a}, -\frac{b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

For more information, see [Using Assumptions](#).

### Solve Systems of Equations

You can solve a system of equations:

```
solve([x^2 + x*y + y^2 = 1, x^2 - y^2 = 0], [x, y])
```

$$\left\{ \left[ x = -\frac{\sqrt{3}}{3}, y = -\frac{\sqrt{3}}{3} \right], \left[ x = \frac{\sqrt{3}}{3}, y = \frac{\sqrt{3}}{3} \right], [x = -1, y = 1], [x = 1, y = -1] \right\}$$

or you can solve a system of equations containing symbolic parameters:

```
solve([x^2 + y^2 = a, x^2 - y^2 = b], [x, y])
```

$$\left\{ [x = -\sigma_2, y = -\sigma_1], [x = -\sigma_2, y = \sigma_1], [x = \sigma_2, y = -\sigma_1], [x = \sigma_2, y = \sigma_1] \right\}$$

where

$$\sigma_1 = \frac{\sqrt{2} \sqrt{a-b}}{2}$$

$$\sigma_2 = \frac{\sqrt{2} \sqrt{a+b}}{2}$$

## Solve Ordinary Differential Equations

You can solve different types of ordinary differential equations:

```
o := ode(x^2*diff(y(x), x, x) + 2*x*diff(y(x), x) + x, y(x)):
solve(o)
```

$$\left\{ C_3 - \frac{x}{2} + \frac{C_2}{x} \right\}$$

## Solve Inequalities

Also, you can solve inequalities:

```
solve(x^4 >= 5, x)
```

$$\left( -\infty, -5^{1/4} \right] \cup \left[ 5^{1/4}, \infty \right) \cup \left\{ z i \mid z \in \left( -\infty, -5^{1/4} \right] \cup \left[ 5^{1/4}, \infty \right) \right\}$$

If you want to get the result over the field of real numbers only, assume that  $x$  is a real number:

```
assume(x in R_); solve(x^4 >= 5, x)
```

$$\left( -\infty, -5^{1/4} \right] \cup \left[ 5^{1/4}, \infty \right)$$

You can pick the solutions that are positive:

```
solve(x^4 >= 5, x) assuming x > 0
```

$$\left[ 5^{1/4}, \infty \right)$$

## Manipulate Expressions

### Transform and Simplify Polynomial Expressions

There are several ways to present a polynomial expression. The standard polynomial form is a sum of monomials. To get this form of a polynomial expression, use the `expand` command:

```
expand((x - 1)*(x + 1)*(x^2 + x + 1)*
(x^2 + 1)*(x^2 - x + 1)*(x^4 - x^2 + 1))
```

$$x^{12} - 1$$

You can factor this expression using the `factor` command:

```
factor(x^12 - 1)
```

$$(x - 1)(x + 1)(x^2 + x + 1)(x^2 + 1)(x^2 - x + 1)(x^4 - x^2 + 1)$$

For multivariable expressions, you can specify a variable and collect the terms with the same powers in this variable:

```
collect((x - a)^4 + a*x^3 + b^2*x + b*x + 10*a^4 +
(b + a*x)^2, x)
```

$$x^4 + (-3a)x^3 + (7a^2)x^2 + (-4a^3 + 2ab + b^2 + b)x + 11a^4 + b^2$$

For rational expressions, you can use the `partfrac` command to present the expression as a sum of fractions (partial fraction decomposition). For example:

```
partfrac((7*x^2 + 7*x + 6)/(x^3 + 2*x^2 + 2*x + 1))
```

$$\frac{6}{x+1} + \frac{x}{x^2+x+1}$$

MuPAD also provides two general simplification functions: `simplify` and `Simplify`. The `simplify` function is faster and it can handle most of the elementary expressions:

```
simplify((x - 1)*(x + 1)*(x^2 + x + 1)*(x^2 + 1)*
(x^2 - x + 1)*(x^4 - x^2 + 1))
```

$$x^{12} - 1$$

The `Simplify` function searches for simpler results deeper than the `simplify` function. The more extensive search makes this function slower than `simplify`. The `Simplify` function allows you to extend the simplification rule set with your own rules and serves better for transforming more complex expressions. For the elementary expressions it gives the same result as `simplify`:

```
Simplify((x - 1)*(x + 1)*(x^2 + x + 1)*(x^2 + 1)*
(x^2 - x + 1)*(x^4 - x^2 + 1))
```

$$x^{12} - 1$$

For the following expression the two simplification functions give different forms of the same mathematical expression:

```
f := exp(wrightOmega(-ln(3/5)))*exp(ln(5) - ln(3)):
simplify(f);
Simplify(f)
```

$$\frac{5 e^{\omega\left(-\ln\left(\frac{3}{5}\right)\right)}}{3}$$

$$e^{\ln\left(\frac{5}{3}\right) + W_0\left(\frac{5}{3}\right)}$$

Note that there is no universal simplification strategy, because the meaning of the simplest representation of a symbolic expression cannot be defined clearly. Different problems require different forms of the same mathematical

expression. You can use the general simplification functions `simplify` and `Simplify` to check if they give a simpler form of the expression you use.

### Transform and Simplify Trigonometric Expressions

You also can transform and simplify trigonometric expressions. The functions for manipulating trigonometric expressions are the same as for polynomial expressions. For example, to expand a trigonometric expression, use the `expand` command:

```
expand(sin(5*x))
```

$$16 \sin(x) \cos(x)^4 - 12 \sin(x) \cos(x)^2 + \sin(x)$$

To factor the trigonometric expression, use the `factor` command:

```
factor(cos(x)^4 + 4*cos(x)^3*sin(x) + 6*cos(x)^2*sin(x)^2 +  
4*cos(x)*sin(x)^3 + sin(x)^4)
```

$$(\cos(x) + \sin(x))^4$$

You can use the general simplification functions on trigonometric expressions:

```
simplify(cos(x)^2 + sin(x)^2)
```

1

```
simplify(cos(x)^4 + sin(x)^4 + sin(x)*cos(x))
```

$$-\frac{\sin(2x)^2}{2} + \frac{\sin(2x)}{2} + 1$$

```
Simplify(cos(x)^4 + sin(x)^4 + sin(x)*cos(x))
```

$$-\frac{(\sin(2x) + 1)(\sin(2x) - 2)}{2}$$



## Use Assumptions in Your Computations

### Solve Expressions with Assumptions

By default, all variables in MuPAD represent complex numbers. When solving equations or simplifying expressions, the software considers all possible cases for complex numbers. If you are solving an equation or simplifying an expression, this default assumption leads to the exact and complete set of results including complex solutions:

```
solve(x^(5/2) = 1, x)
```

$$\left\{ 1, -\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}i}}{4}, -\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}i}}{4} \right\}$$

To obtain real solutions only, pass the assumption to MuPAD using the `assuming` command:

```
solve(x^(5/2) = 1, x) assuming x in R_
```

```
{1}
```

You can make various assumptions on the values that a variable represents. For example, you can solve an equation assuming that the variable  $x$  represents only positive values:

```
solve(x^4 - 1 = 0, x) assuming x > 0
```

```
{1}
```

You can make multiple assumptions:

```
solve(x^4 - a = 0, x) assuming a = 16 and x in R_
```

```
{-2, 2}
```

## Integrate with Assumptions

You can use assumptions when integrating mathematical expressions. For example, without an assumption on the variable  $x$ , the following integral depends on the sign of the expression  $x^2 - 1$ :

```
int(1/abs(x^2 - 1), x)
```

$$-\frac{\operatorname{arctanh}(x)}{\operatorname{sign}(x^2 - 1)}$$

If you know that  $x > 1$ , you can pass the assumption to the integral:

```
int(1/abs(x^2 - 1), x) assuming x > 1
```

$$-\operatorname{arctanh}(x)$$

## Simplify Expressions with Assumptions

Using assumptions along with the simplification functions narrows down the possible values that variables represent and can provide much shorter results than the simplification functions alone. For example:

```
simplify(sqrt(x^2 + 2*x + 1) + sqrt(x^2 - 2*x + 1) +  
sqrt(x^2 + 4*x + 4) + sqrt(x^2 - 4*x + 4))
```

$$\sqrt{(x-1)^2} + \sqrt{(x+1)^2} + \sqrt{(x-2)^2} + \sqrt{(x+2)^2}$$

versus

```
simplify(sqrt(x^2 + 2*x + 1) + sqrt(x^2 - 2*x + 1) +  
sqrt(x^2 + 4*x + 4) + sqrt(x^2 - 4*x + 4)) assuming x > 2
```

$$4x$$

You can pass assumptions to the following functions: `expand`, `simplify`, `limit`, `solve`, and `int`. The `Simplify` function does not allow assumptions on variables.

## Use Graphics

In this section...
“Graphic Options Available in MuPAD” on page 1-42
“Basic Plotting” on page 1-43
“Format Plots” on page 1-53
“Present Graphics” on page 1-60
“Create Animated Graphics” on page 1-63

### Graphic Options Available in MuPAD

#### Basic Plotting Options

MuPAD presents many options for creating and working with graphics and animations. The simplest way to create a plot in MuPAD is to use the `plot` command. Using this command, you can:

- Create 2-D and 3-D function plots
- Specify plotting range
- Create plots for piecewise functions
- Create multiple function plots in one graph
- Create animated 2-D and 3-D function plots

You can format the plot interactively.

#### Advanced Plotting Options

The `plot` command provides a basic way to create function plots. For example, you can:

- Create a 2-D function plot using `plot::Function2d`.
- Create a 3-D function plot using `plot::Function3d`.
- Create animated plots.

- Create function plots in polar or spherical coordinates.
- Create turtle graphics and Lindenmayer systems.
- Choose colors, fonts, legends, axes appearance, grid lines, tick marks, line, and marker styles.
- Apply affine transformations to a plot. You can scale, rotate, reflect, or move a plot.
- Set cameras for a 3-D plot.
- See the MuPAD gallery of plots.

To see all functions available in the MuPAD graphics library, enter:

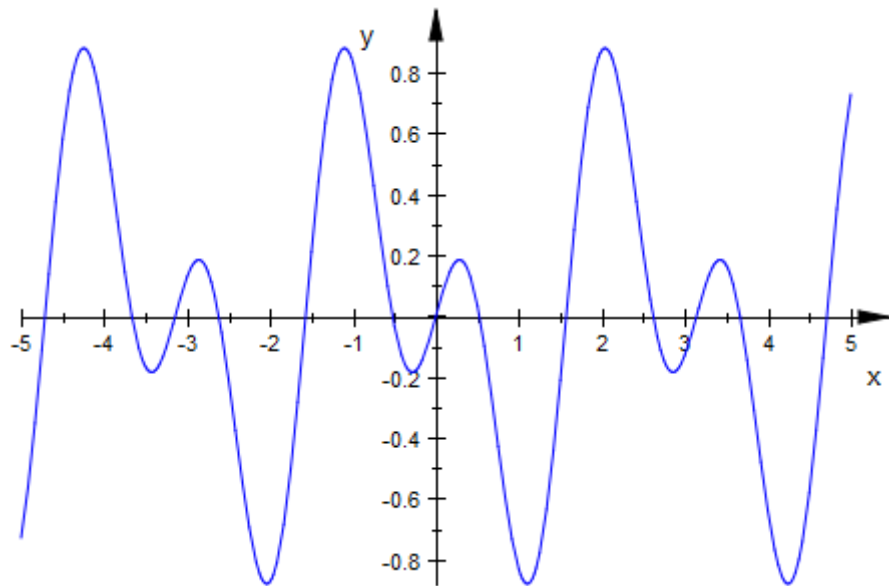
```
info(plot)
```

## Basic Plotting

### Create 2-D Plots

The simple way to create a 2-D plot of a function is to use the `plot` command:

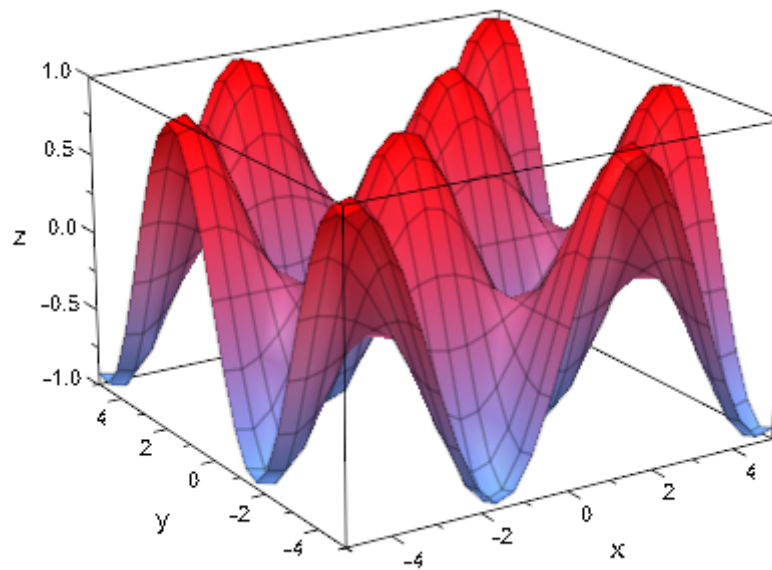
```
plot(sin(x)*cos(3*x))
```



### Create 3-D Plots

The simple way to create a 3-D plot of a function is to use the `plot` command with the option `#3D`:

```
plot(sin(x)*sin(y), #3D)
```



---

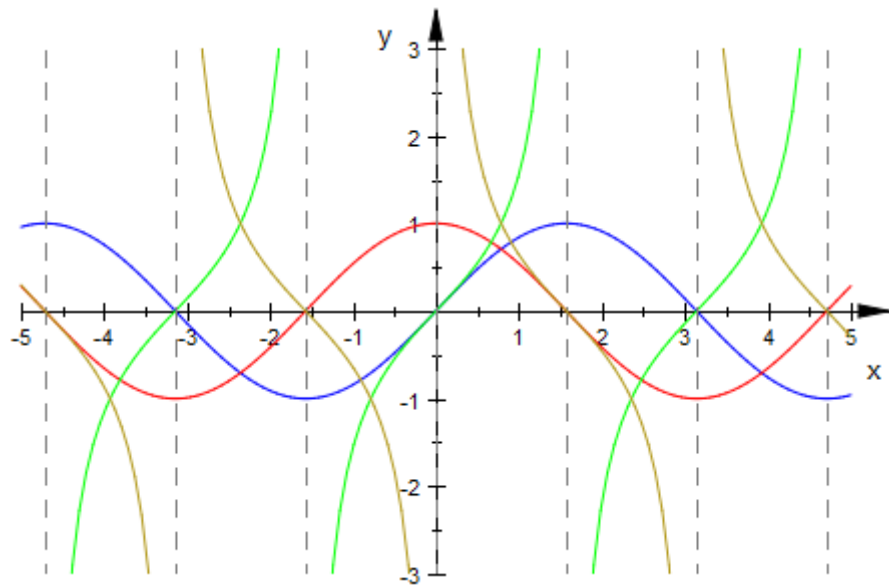
**Note** By default, for a function of two variables, the `plot` command creates a 2-D animation. Using the option `#3D` lets you create a 3-D plot instead of a 2-D animation.

---

### Plot Multiple Functions in One Graph

To plot several functions in one figure, list all the functions, separating them by commas. MuPAD uses different colors when plotting multiple functions:

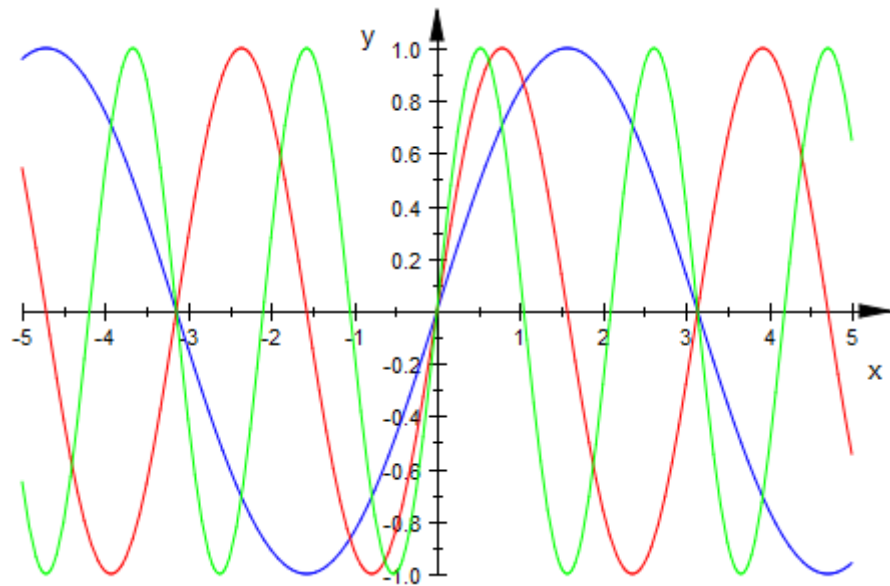
```
plot(sin(x), cos(x), tan(x), cot(x))
```



You can use the sequence generator `$` to create a sequence of functions:

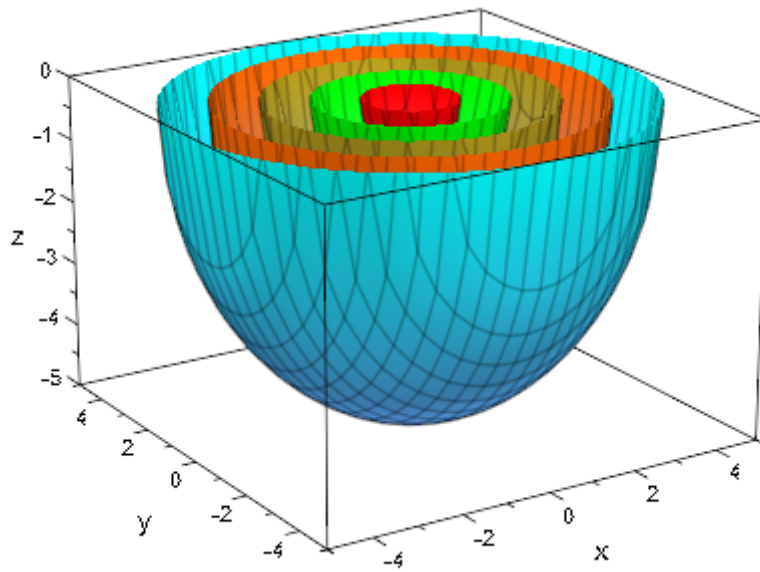
```
plot(sin(k*x) $ k = 1..3)
```





You also can plot multiple functions in one 3-D graph:

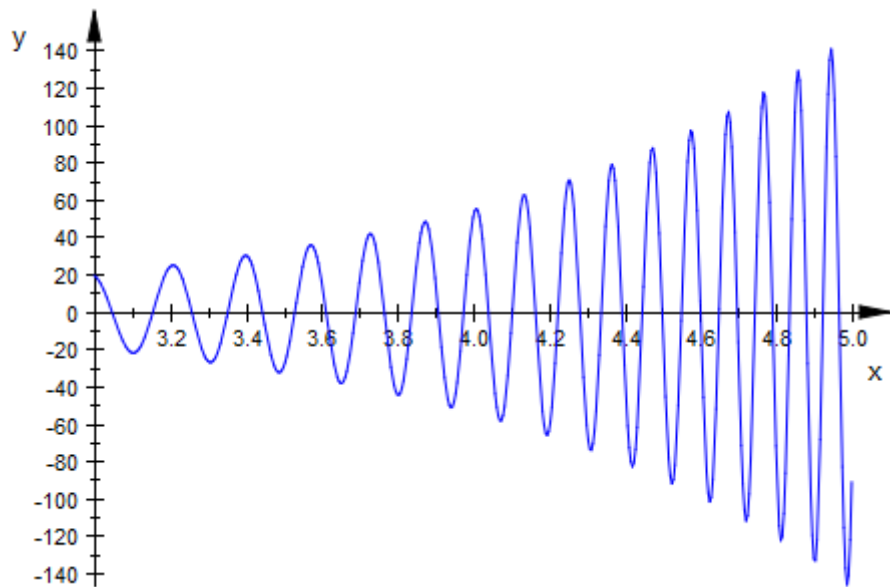
```
plot(-sqrt(r^2 - x^2 - y^2) $ r = 1.5, #3D)
```



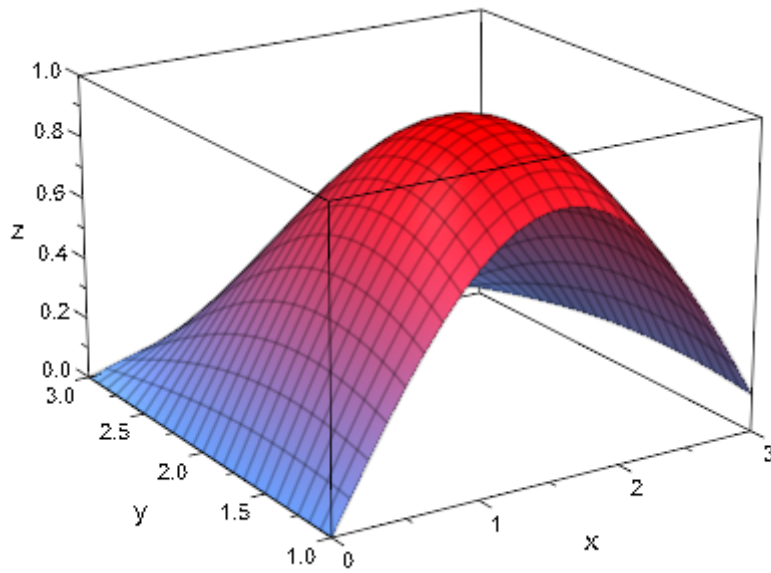
### Specify Plot Ranges

You can specify a range over which to plot a function:

```
plot(sin(x^3)*exp(x), x = 3..5)
```

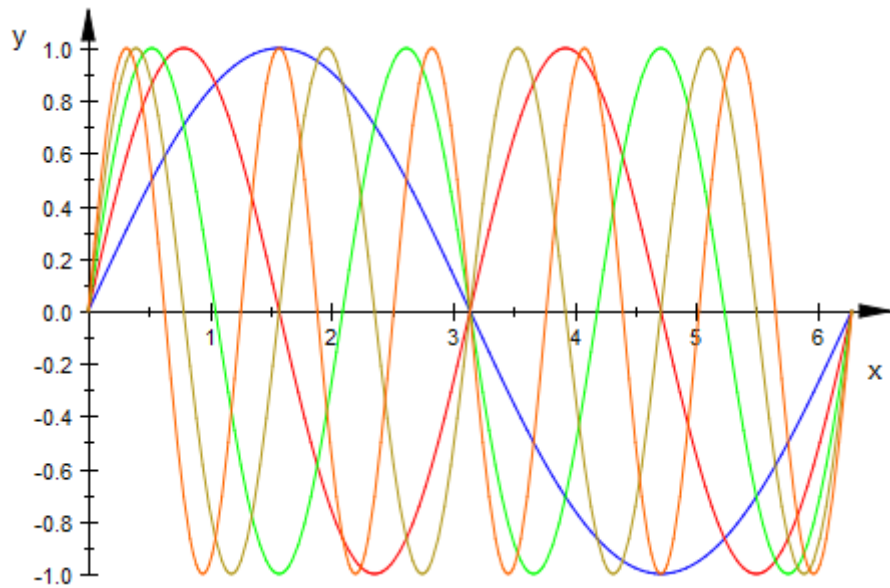


```
plot(sin(x)*sin(y), x = 0..3, y = 1..3, #3D)
```



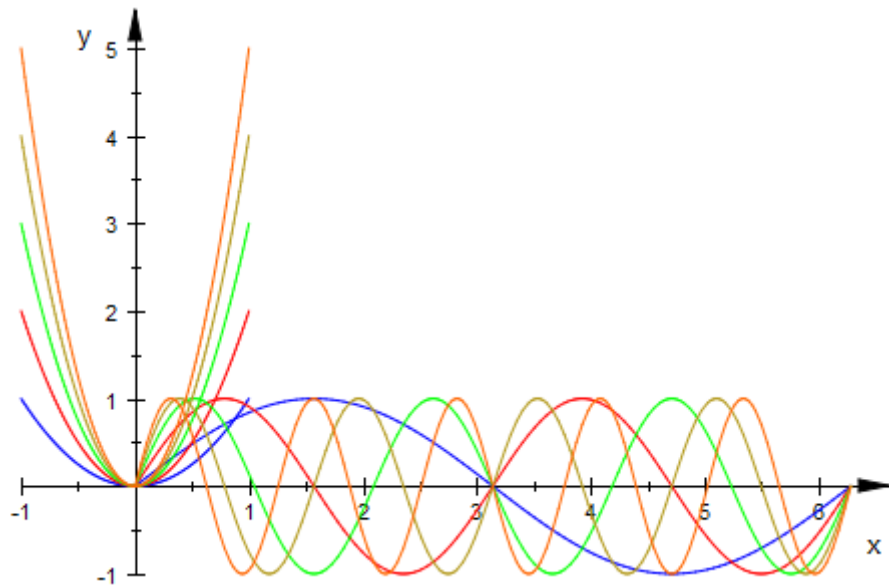
For multiple functions plotted in one graph, you can specify one range for all the functions:

```
plot(sin(k*x) $ k = 1..5, x = 0..2*PI)
```



To specify different ranges for multiple functions plotted in one graph, use different variables:

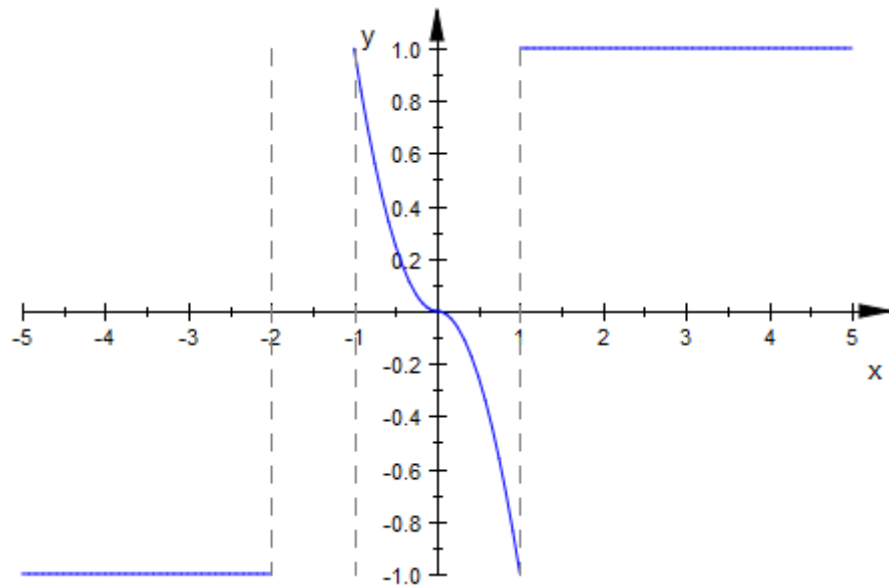
```
plot({sin(k*x), k*t^2} $ k = 1..5, x = 0..2*PI, t = -1..1)
```



### Plot Piecewise Functions

To specify a piecewise function, use the `piecewise` command. You can plot a piecewise function even if it is undefined at some points. For example, you can plot the following function although the function is not defined for  $-2 < x < -1$ :

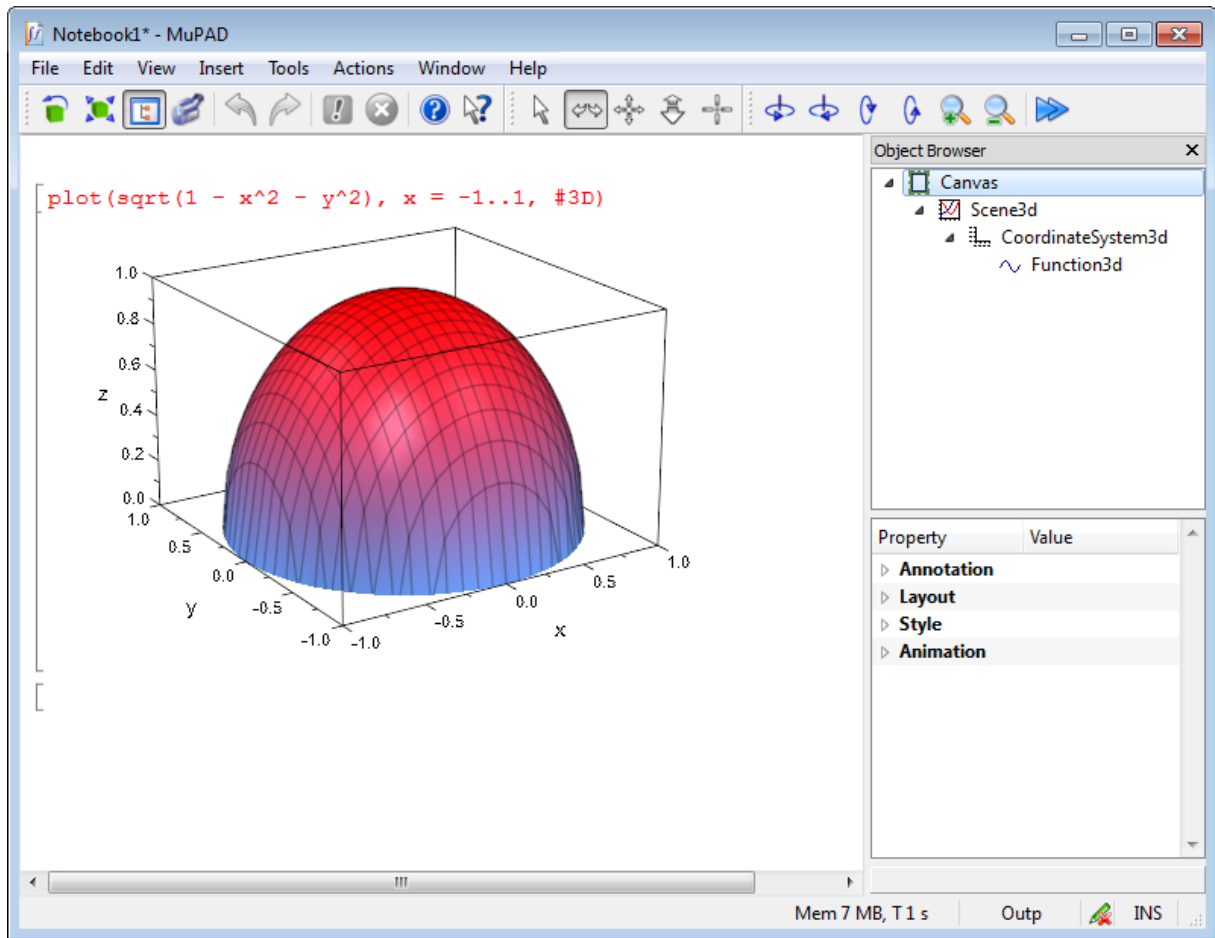
```
plot(piecewise([x < - 2, - 1], [-1 < x and x < 0, x^2],  
[0 < x and x < 1, -x^2], [x > 1, 1]))
```




## Format Plots

### Enable Plot Formatting Mode

In MuPAD, you can format your graphic results interactively when working in plot formatting mode. To switch to graphics formatting mode, click any place on a plot. In this mode, the **Object Browser** pane appears.

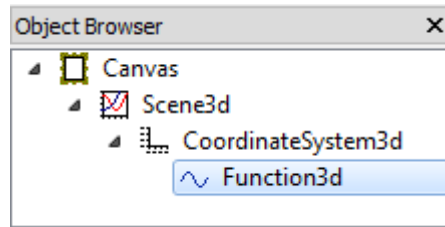


If you do not see the **Object Browser** and **Property** panes, select

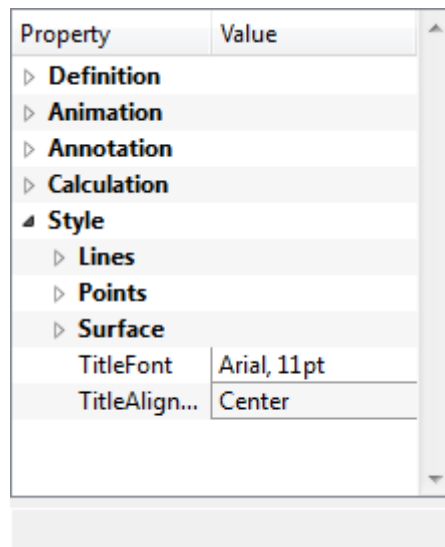
**View > Object Browser** or click  on the toolbar.

The top of the **Object Browser** pane displays the components of your graphics such as scene (background), coordinate system, and a function plot. For further information on the structure of graphics, see [The Full Picture: Graphical Trees](#).



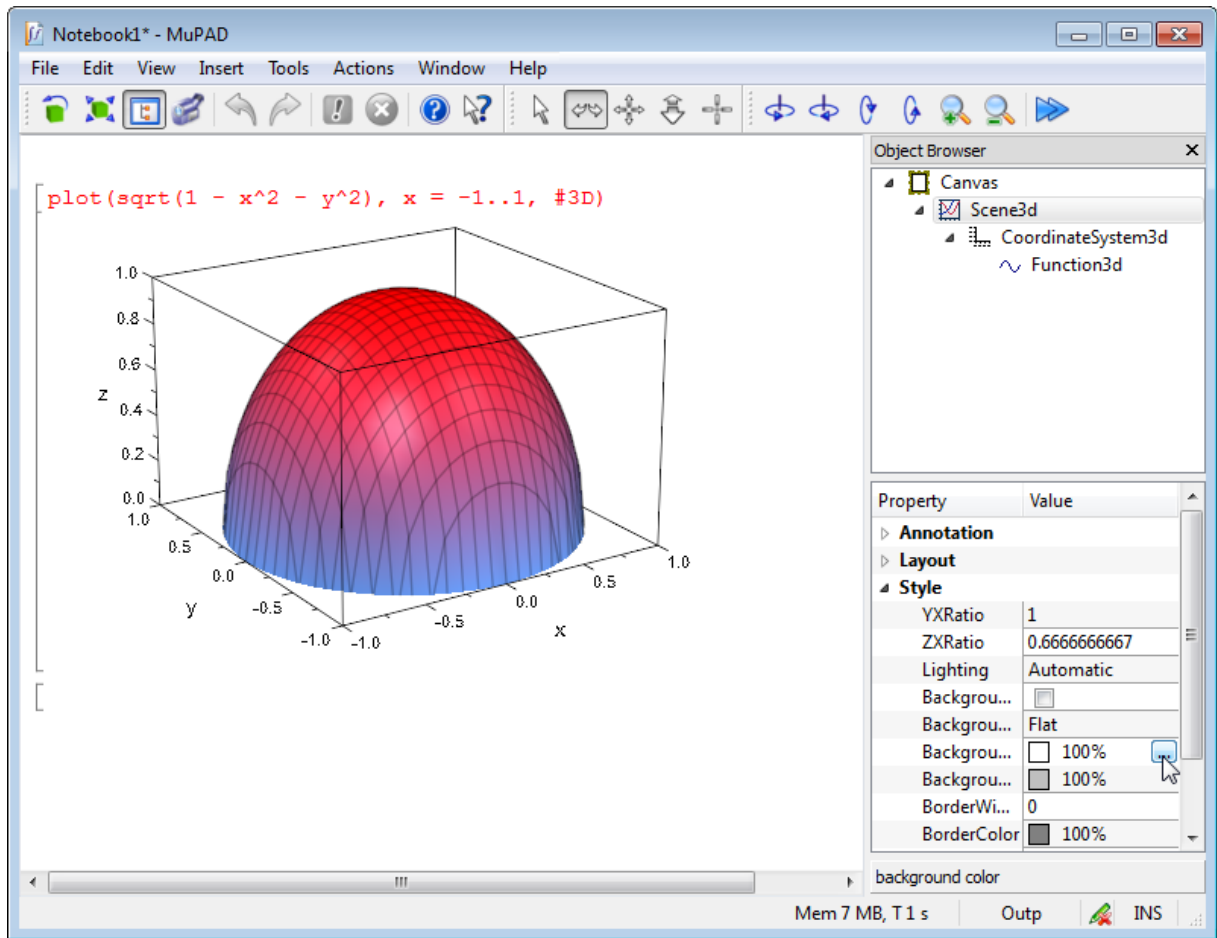


After you select a component in the **Object Browser** pane, the bottom of the pane displays the properties of this component.

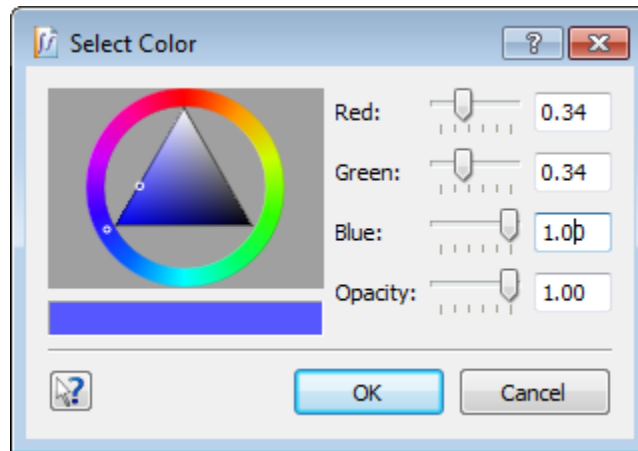


## Change Background Settings

To change background settings of your graphics, switch to plot formatting mode and select **Scene** at the top of the **Object Browser** pane. The bottom of the pane shows background properties that you can change. For example, you can change the background color. To choose the color, select **BackgroundColor** and click the ellipsis button.

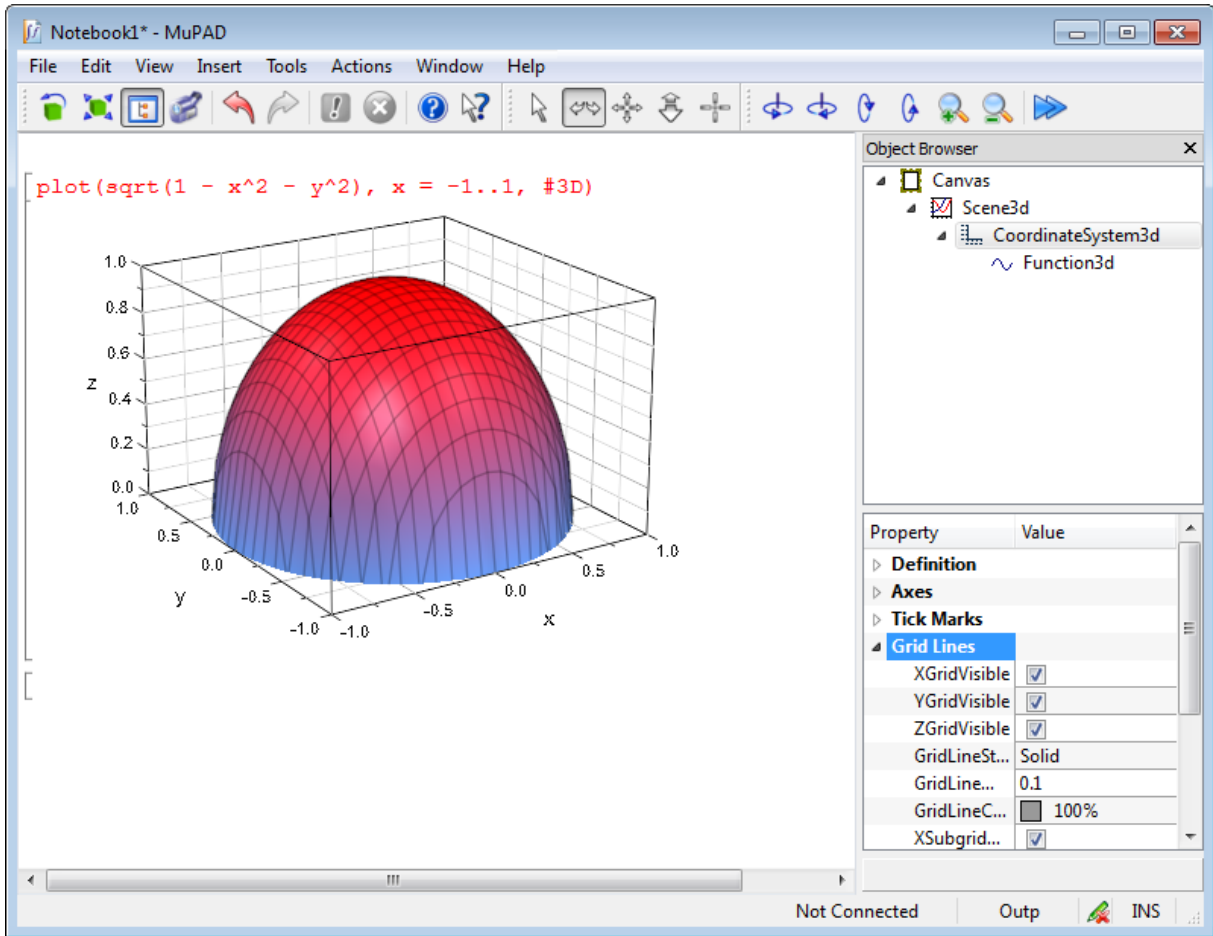


You can use predefined colors or select a color from a more extensive palette.



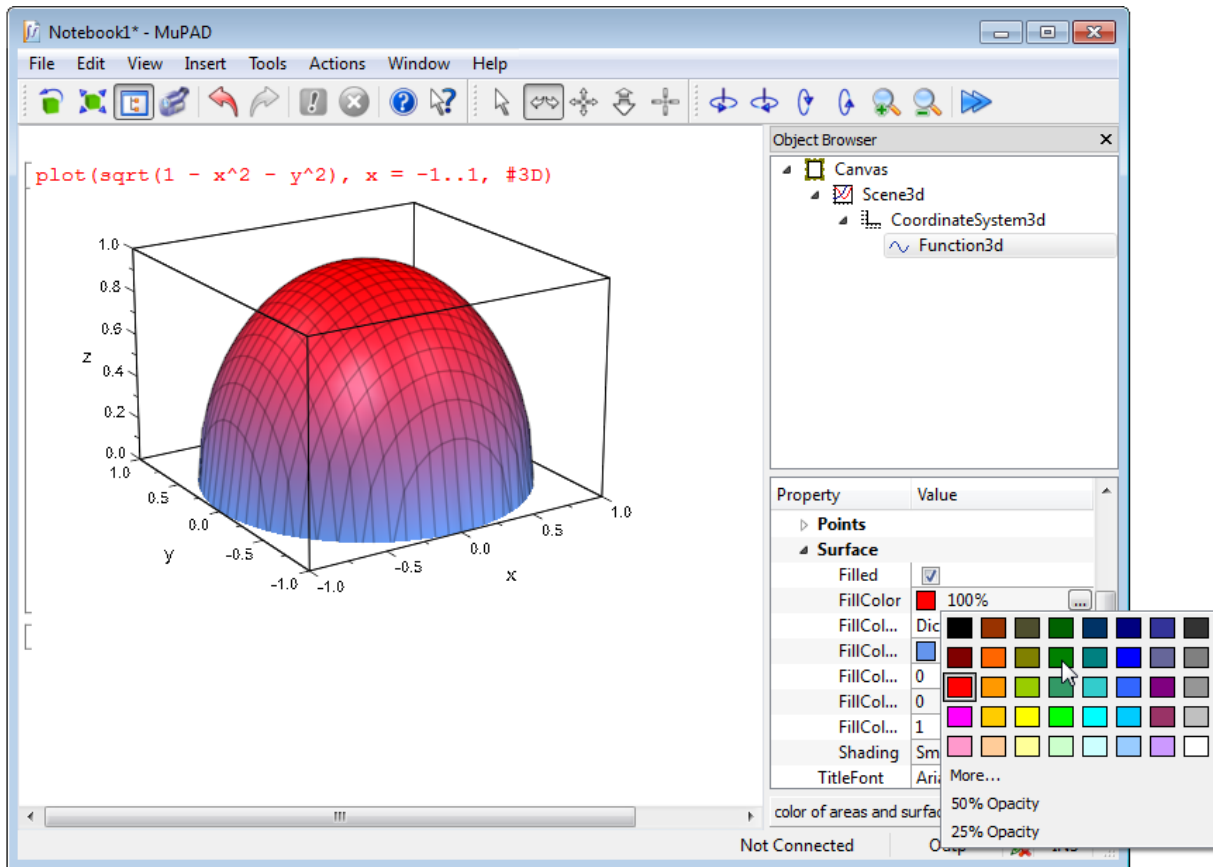
### Modify Axes

To format the axes of your graphics, switch to plot formatting mode and select **Coordinate System** at the top of the **Object Browser** pane. The bottom of the pane shows axes properties that you can change. For example, you can add grid lines.

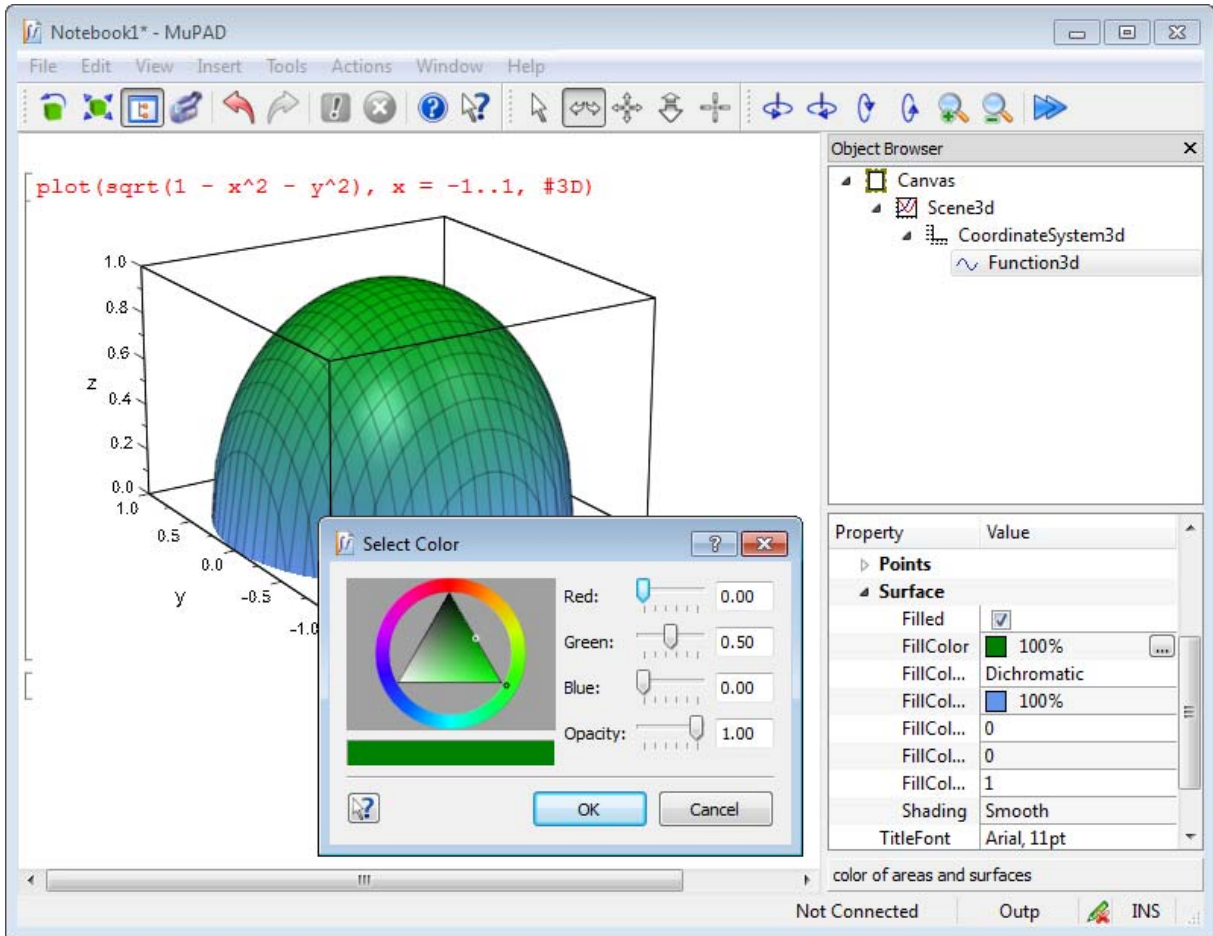


## Modify Function Plot

To format the function plot, switch to plot formatting mode and select **Function** at the top of the **Object Browser** pane. The bottom of the pane shows plot properties that you can change. For example, you can change the color of a function plot.

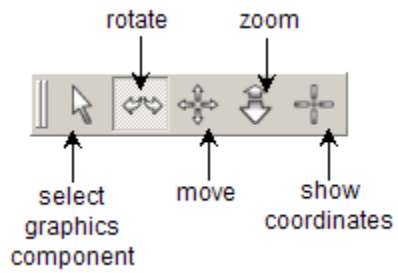


You can use predefined colors or select a color from a more extensive palette.

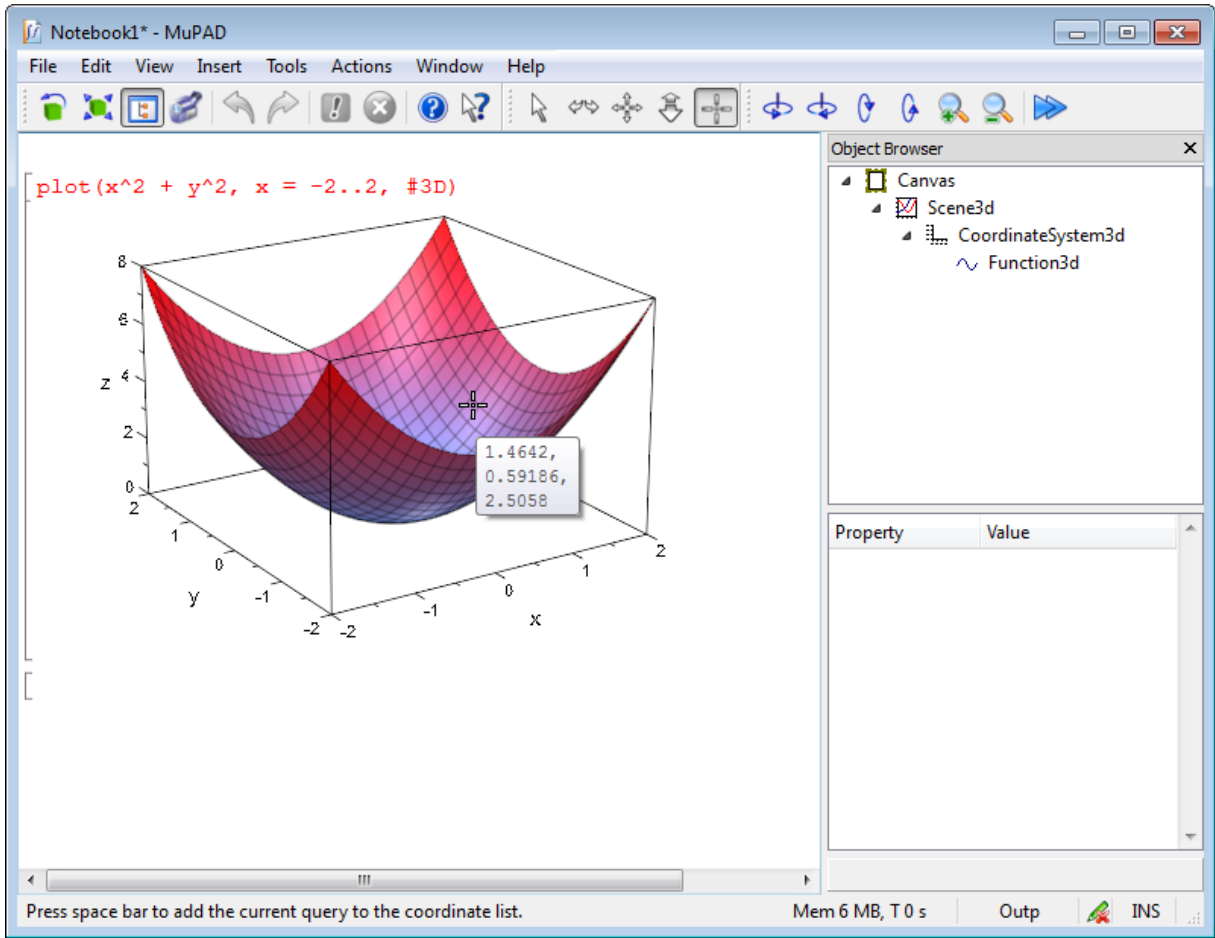


## Present Graphics

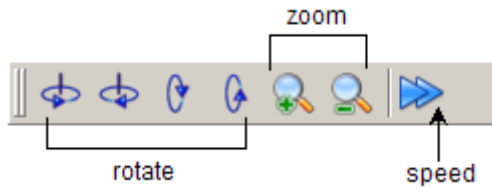
When you present graphic results in MuPAD, you can move, zoom, and rotate your graphics. You also can select different components of a plot. When presenting graphic results, switch to the plot formatting mode. You can use the following toolbar to manually rotate, move, zoom your plot, and show coordinates of any point on your plot:



To see the coordinates for a point on your plot, click the point and hold the mouse button. You can move the cursor while holding the mouse button and see the coordinates of all the points on the path of the cursor.



You can use the toolbar to rotate and zoom your plot automatically. You also can change the speed for rotation and zooming.



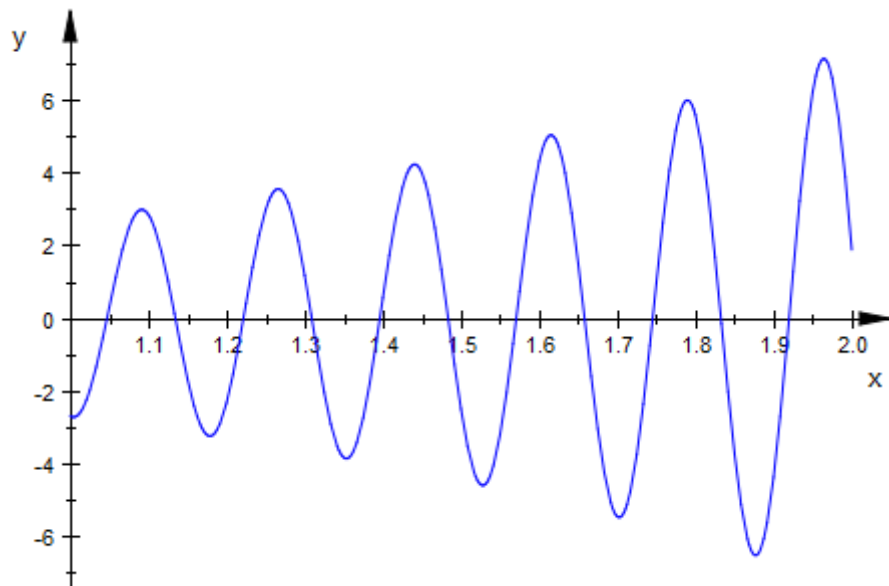


## Create Animated Graphics

### Creating Animated 2-D Plots

To create an animated plot, use an additional changing parameter for the function you want to plot. Specify the range for this parameter. The following example presents an animated plot of a function with the parameter  $a$  that gradually changes value from 2 to 6:

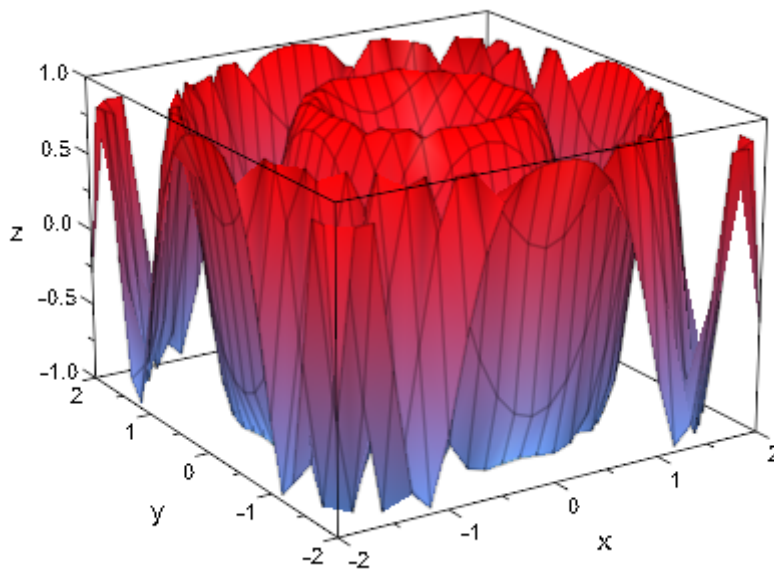
```
plot(exp(x)*sin(a^2*x), x = 1..2, a = 2..6)
```



### Create Animated 3-D Plots

To create an animated 3-D plot, use an additional changing parameter for the function you want to plot. Specify the range for this parameter and the option `#3D`. The following example presents an animated plot of a function with the parameter  $a$  that gradually changes value from 0.1 to 2:

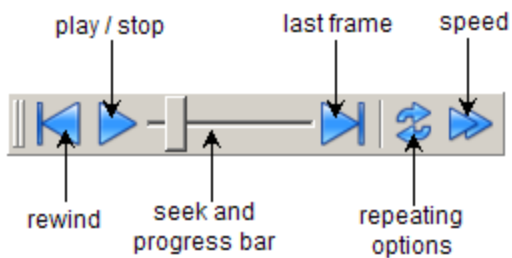
```
plot(sin(a*x^2 + a*y^2), x = -2..2, y = -2..2, a = 0.1..2, #3D)
```



## Play Animations

MuPAD displays the first frame of an animation as static picture. To play the animation, click the picture.

When MuPAD plays an animation, the **Animation** toolbar with the player controls appears:



## Count Backwards

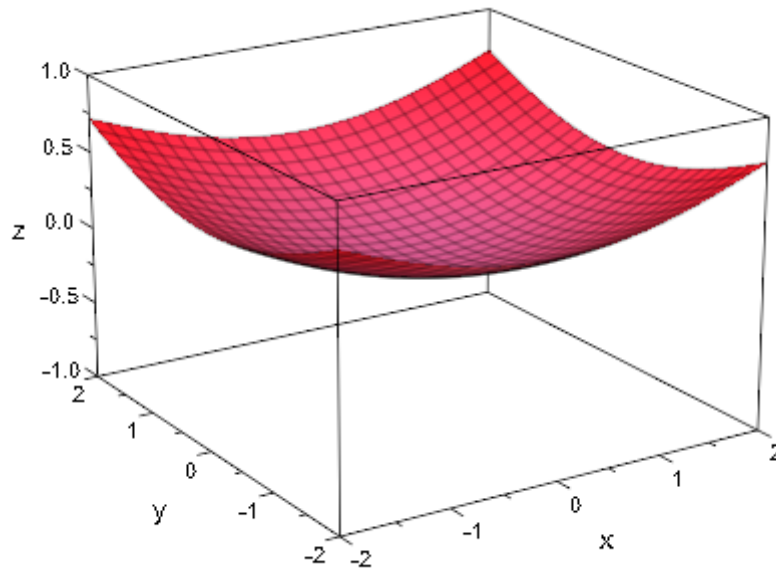
To play an animation forward and then backward, click the Repetition button



and select the option **Back and Forth**.

You also can specify the range for a parameter so that the initial value is greater than the final value. The following example creates an animated plot of the function using the parameter  $a$  that gradually changes value from 2 to 0.1:

```
plot(sin(a*x^2 + a*y^2), x = -2..2, y = -2..2, a = 2..0.1, #3D)
```



## Format and Export Documents and Graphics

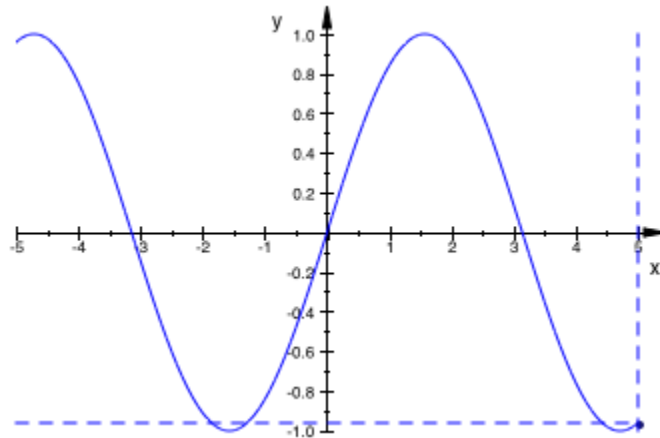
In this section...
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“Format Expressions in Input Regions” on page 1-74
“Change Default Format Settings” on page 1-77
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“Save and Export Graphics” on page 1-106

### Format Text

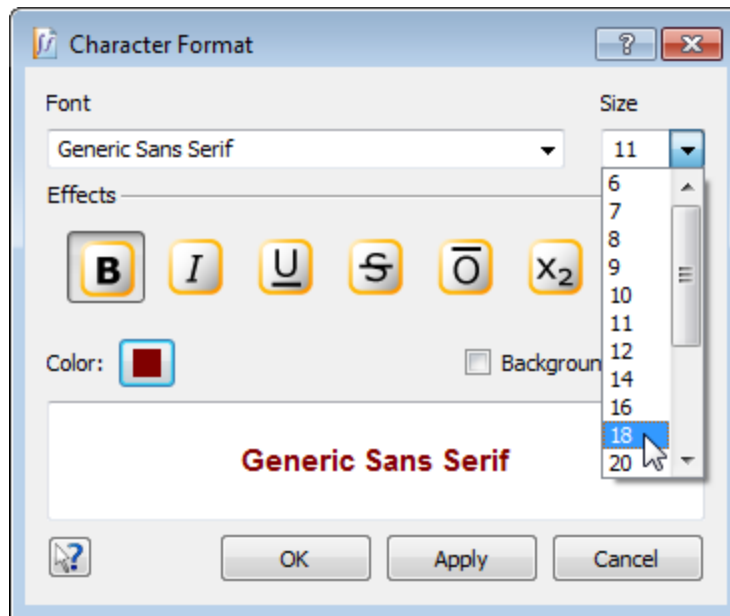
#### Choose Font Style, Size, and Colors

To change the font for a particular piece of text:

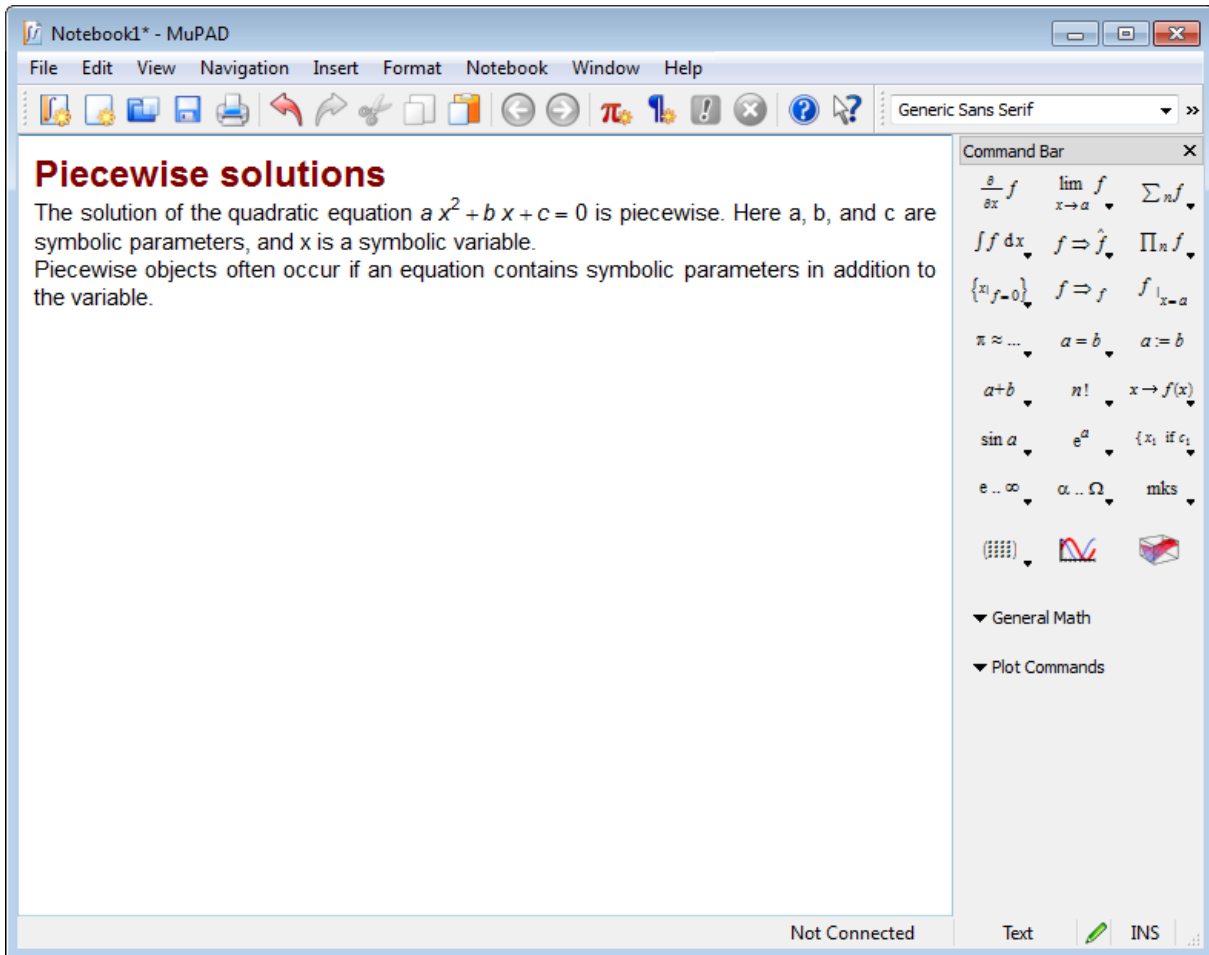
- 1 Select text that you want to format.
- 2 Select **Format>Characters** from the main menu or use context menu.



- 3 In the Character Format dialog box choose the font style, font size, font and background colors, and effects. The window at the bottom of the dialog box shows a preview of your changes.



If you want to experiment with different fonts, and see how the formatted text looks in your notebook, click the **Apply** button. This button applies formatting to the selected text and leaves the Character Format dialog box open. You can change font and color of your text several times without having to open this dialog box for each change. When you finish formatting, click **OK** to close the Character Format dialog box.



To format selected text, you also can use the **Format** toolbar. If you do not see the **Format** toolbar, select **View>Toolbars>Format** from the main menu.

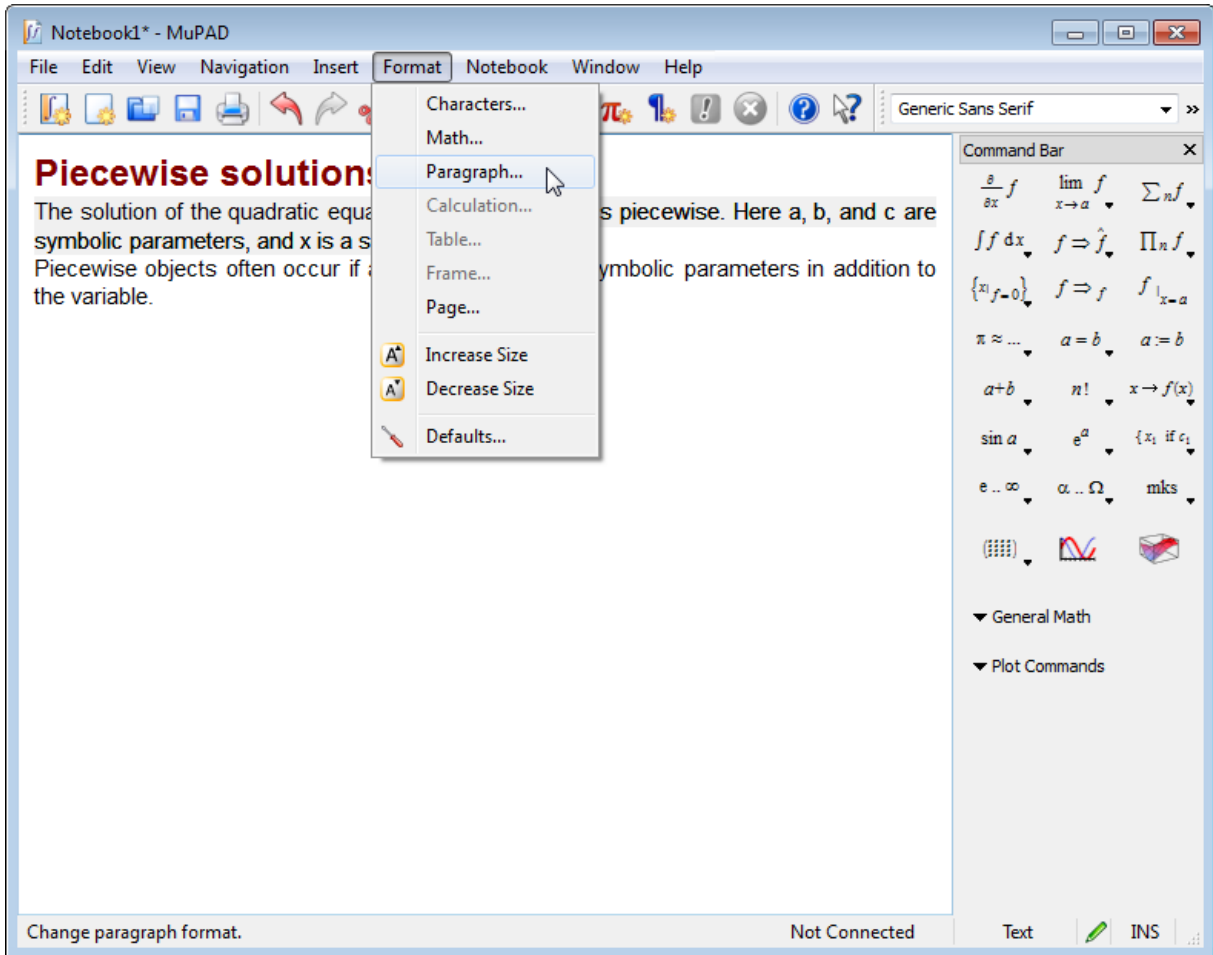


To change the font size quickly, you can use **Format>Increase Size** and **Format>Decrease Size** or the corresponding buttons on the **Format** toolbar.

### **Choose Indention, Spacing, and Alignment**

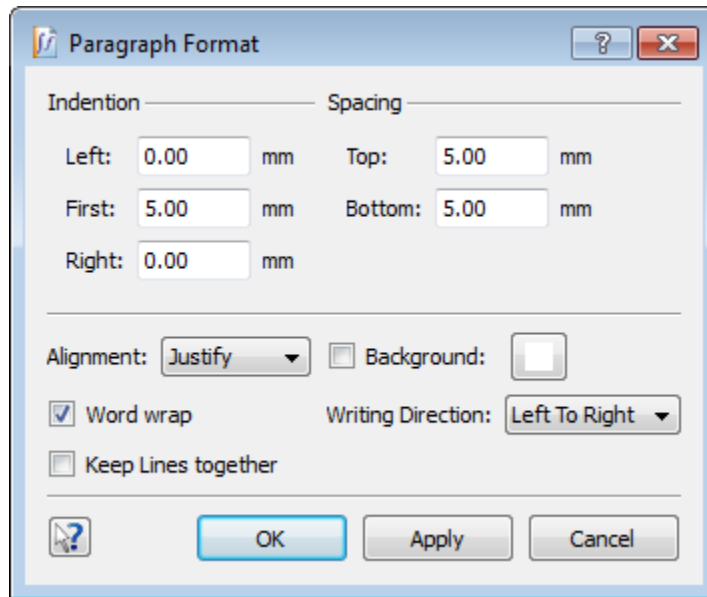
To change paragraphs settings such as indention, spacing, alignment, and writing direction:

- 1** Select the paragraphs you want to format.
- 2** Select **Format>Paragraph** from the main menu or use the context menu.

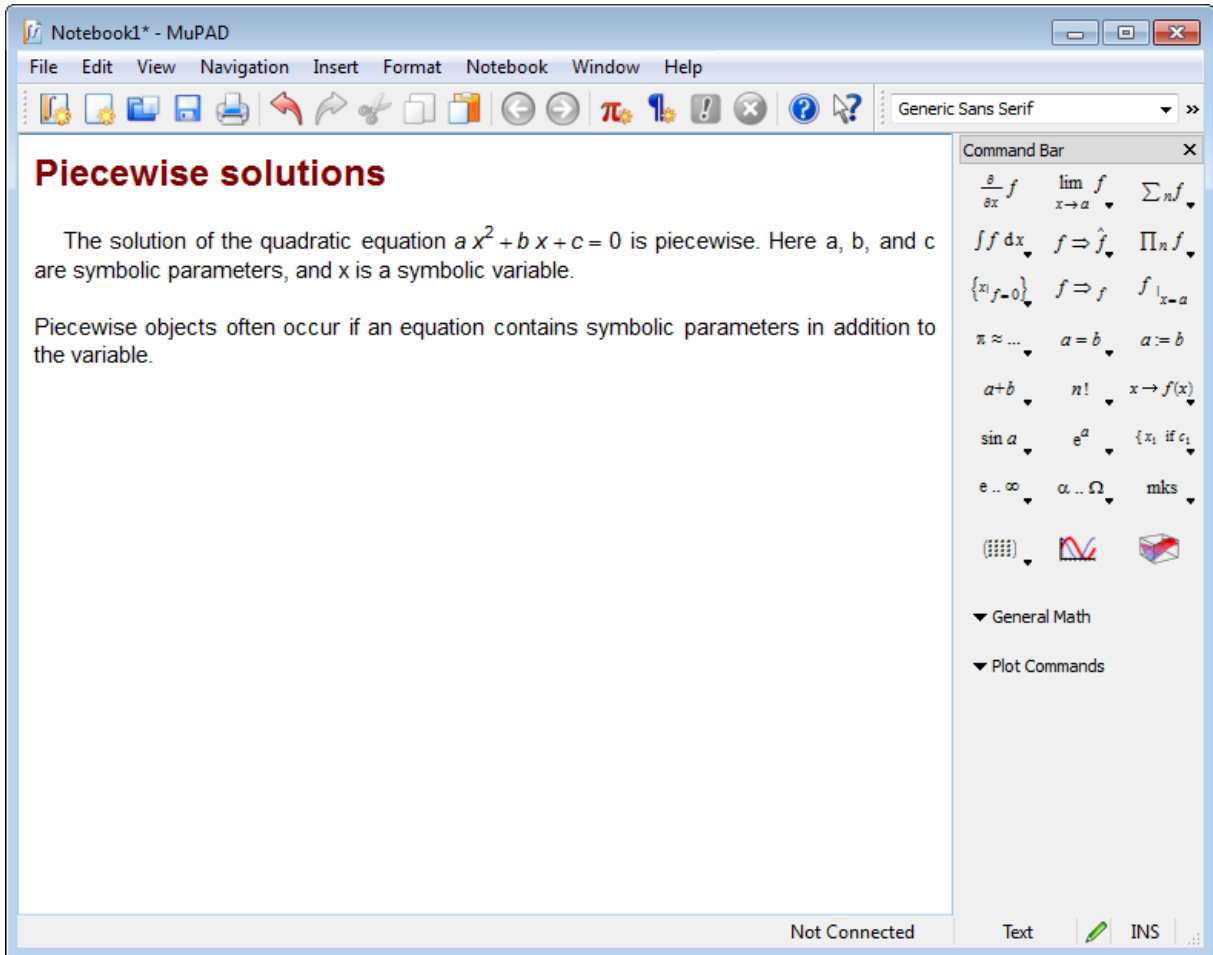


- 3 In the Paragraph Format dialog box choose indentation, spacing, alignment, background color, and writing direction of the text. The writing direction is a language-specific option that allows you to type from right to left.





If you want to experiment with different paragraph settings, and see how the formatted text looks in your notebook, click the **Apply** button. This button applies formatting to the selected text and leaves the Paragraph Format dialog box open. You can change paragraph settings several times without having to open this dialog box for each change. When you finish formatting, click **OK** to close the Paragraph Format dialog box:

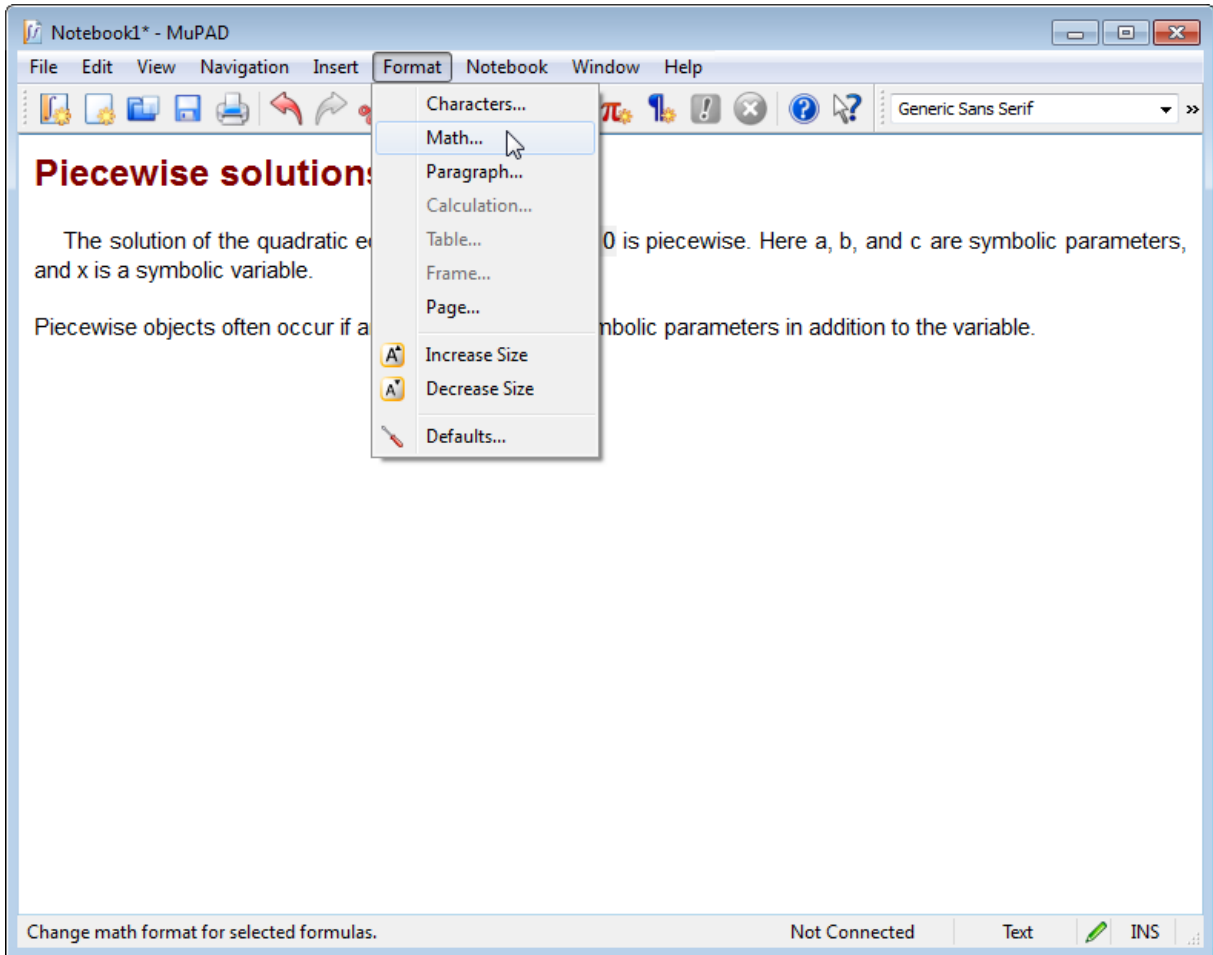


## Format Mathematical Expressions

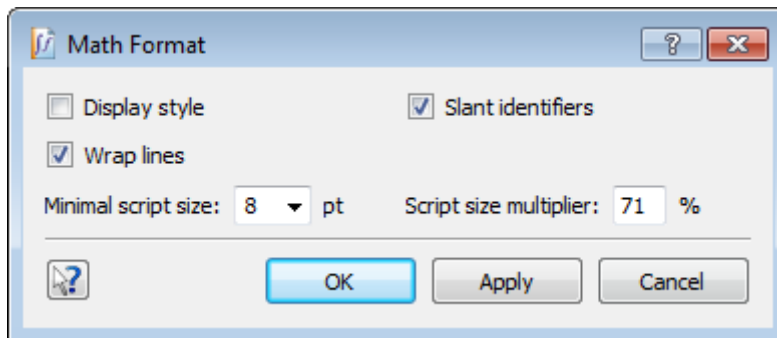
You can change font style, size, and color of mathematical expressions in text regions in the same way you format regular text. See Choosing Font Style, Size, and Colors for more details.

For additional formatting of mathematical expressions:

- 1 Select **Format > Math**.



- 2 Set your formatting preferences. You can define the script size, choose between inline (embedded in text) or displayed styles, and use the **Slant identifiers** check box to italicize variables. Also, you can specify whether you want to wrap long mathematical expressions to a notebook window size.



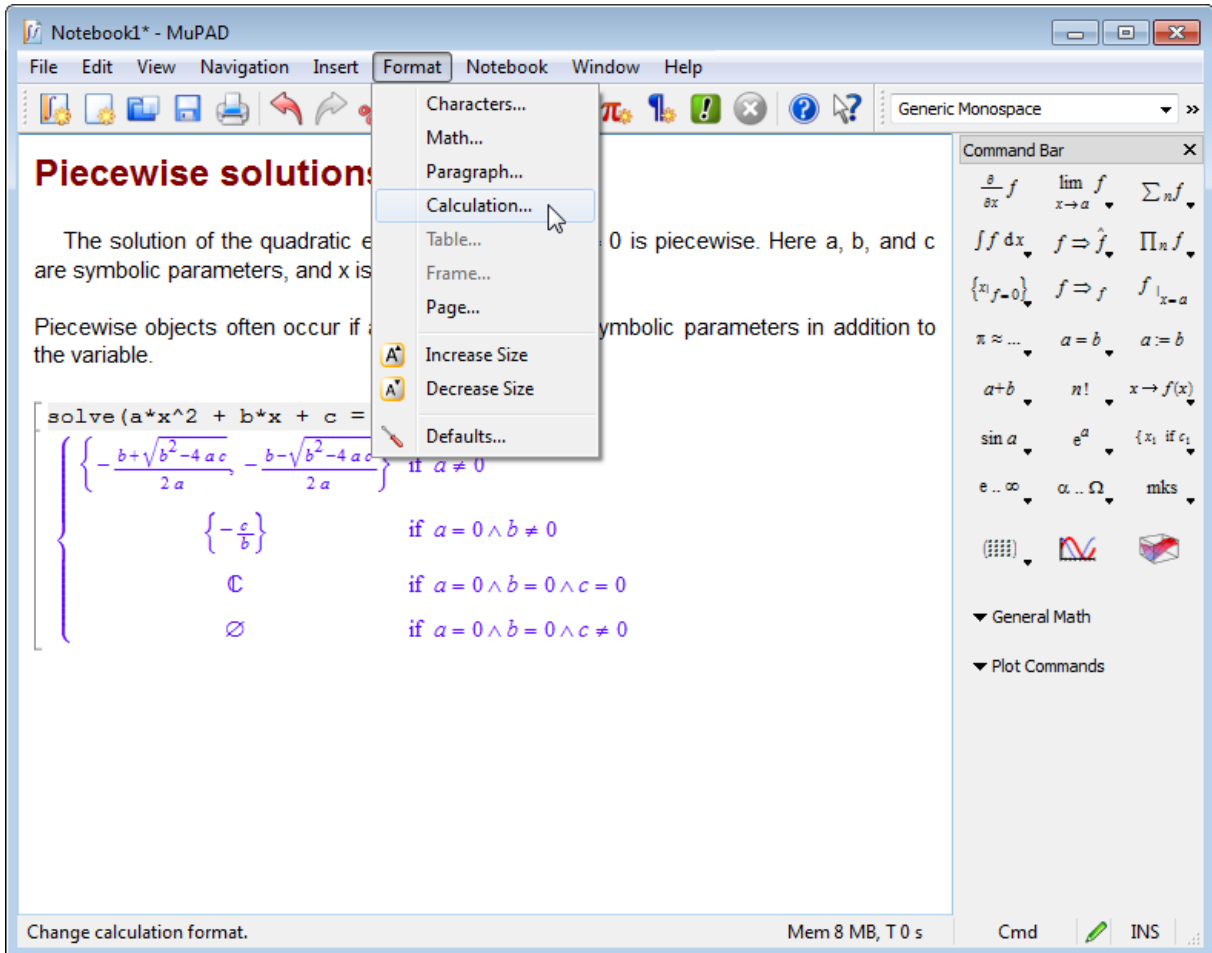
If you want to experiment with different settings for mathematical expressions and see how the formatted expression looks in your notebook, click **Apply**. This button applies formatting to the selected text and leaves the Math Format dialog box open. You can change settings several times without having to open this dialog box for each change. When you finish formatting, click **OK** to close the Math Format dialog box.

## Format Expressions in Input Regions

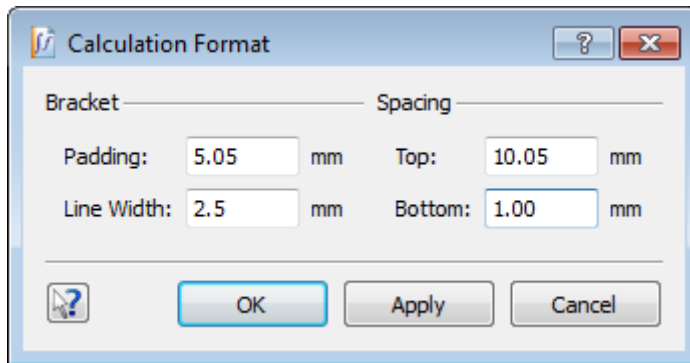
You can change font style, size, and color of mathematical expressions in Text regions in the same way you format regular text. See *Choosing Font Style, Size, and Colors* for more details.

For additional formatting of commands and expression in the input regions:

- 1 Select **Format>Calculation** in the main menu.



- 2 Set your formatting preferences. You can define indentation, spacing at the top and bottom of the calculation, and width of the left bracket that encloses the input region.



If you want to experiment with different settings for input regions, and see how the formatted expressions and commands look in your notebook, click the **Apply** button. This button applies formatting to the selected region and leaves the Calculation Format dialog box open. You can change settings several times without having to open this dialog box for each change. When you finish formatting, click **OK** to close the Calculation Format dialog box.

The screenshot shows the MuPAD Notebook1\* window. The title bar reads "Notebook1\* - MuPAD". The menu bar includes "File", "Edit", "View", "Navigation", "Insert", "Format", "Notebook", "Window", and "Help". The toolbar contains various icons for file operations, navigation, and mathematical symbols. The main workspace displays the following text:

**Piecewise solutions**

The solution of the quadratic equation  $ax^2 + bx + c = 0$  is piecewise. Here  $a$ ,  $b$ , and  $c$  are symbolic parameters, and  $x$  is a symbolic variable.

Piecewise objects often occur if an equation contains symbolic parameters in addition to the variable.

The command `solve(a*x^2 + b*x + c = 0, x)` is shown in red. The resulting piecewise solution is displayed in purple:

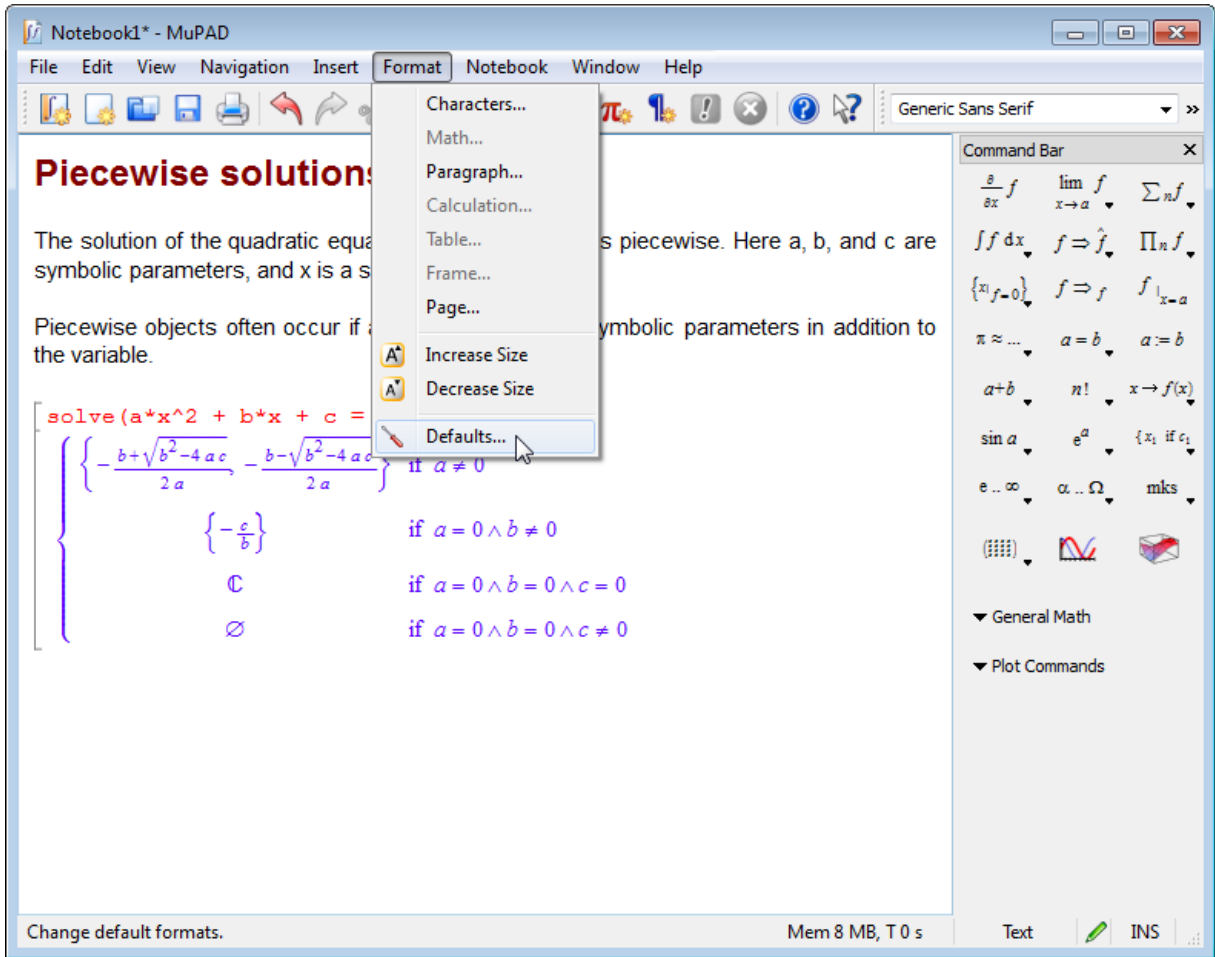
$$\left\{ \begin{array}{ll} \left\{ -\frac{b + \sqrt{b^2 - 4ac}}{2a}, -\frac{b - \sqrt{b^2 - 4ac}}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{array} \right.$$

The Command Bar on the right contains various mathematical symbols and functions, including  $\frac{\partial}{\partial x} f$ ,  $\lim_{x \rightarrow a} f$ ,  $\sum_n f$ ,  $\int f dx$ ,  $f \Rightarrow \hat{f}$ ,  $\prod_n f$ ,  $\{x\}_{f=0}$ ,  $f \Rightarrow f$ ,  $f|_{x=a}$ ,  $\pi \approx \dots$ ,  $a = b$ ,  $a := b$ ,  $a + b$ ,  $n!$ ,  $x \rightarrow f(x)$ ,  $\sin a$ ,  $e^a$ ,  $\{x\}$  if  $c_1$ ,  $e \dots \infty$ ,  $\alpha \dots \Omega$ , and  $mks$ . There are also icons for a grid, a graph, and a 3D box. The bottom status bar shows "Mem 8 MB, T 0 s" and "Text" mode.

## Change Default Format Settings

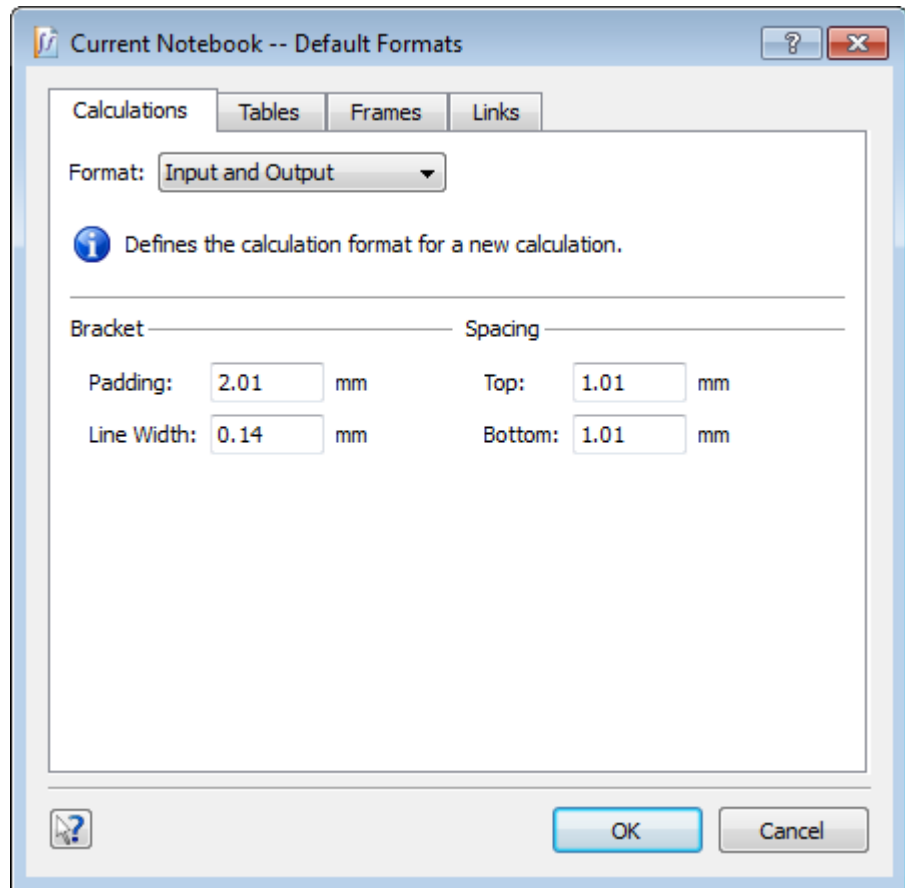
If you want to apply a specific format to the whole notebook, you can change the default settings for particular types of regions. For example, to change default setting for all commands and expressions in the input and output regions of a notebook:

- 1 Select **Format > Defaults** from the main menu.

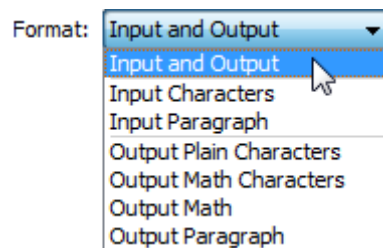


- 2 In the resulting dialog box, use tabs to select the required element. For input and output regions, select the **Calculations** tab.





- 3** From the drop-down menu **Format** select **Input and Output**.



- 4 In the appropriate fields, enter values for the size of indentation, the spacing at the top and bottom of the calculation, and the width of the left bracket that encloses each input and output region.
- 5 Click **OK** to apply the new default settings and close the Default Formats dialog box.

## Use Frames

If you want to format different parts of a notebook separately, use frames. Frames can include text, mathematical expressions, and commands. To insert a frame:

- 1 Select the place where you want to insert a frame.
- 2 Select **Insert>Frame**.

The screenshot shows the MuPAD Notebook1\* application window. The 'Insert' menu is open, and the 'Frame' option is highlighted with a mouse cursor. The document content includes:

**Piecewise solution**

The solution of the quadratic equation with symbolic parameters, and

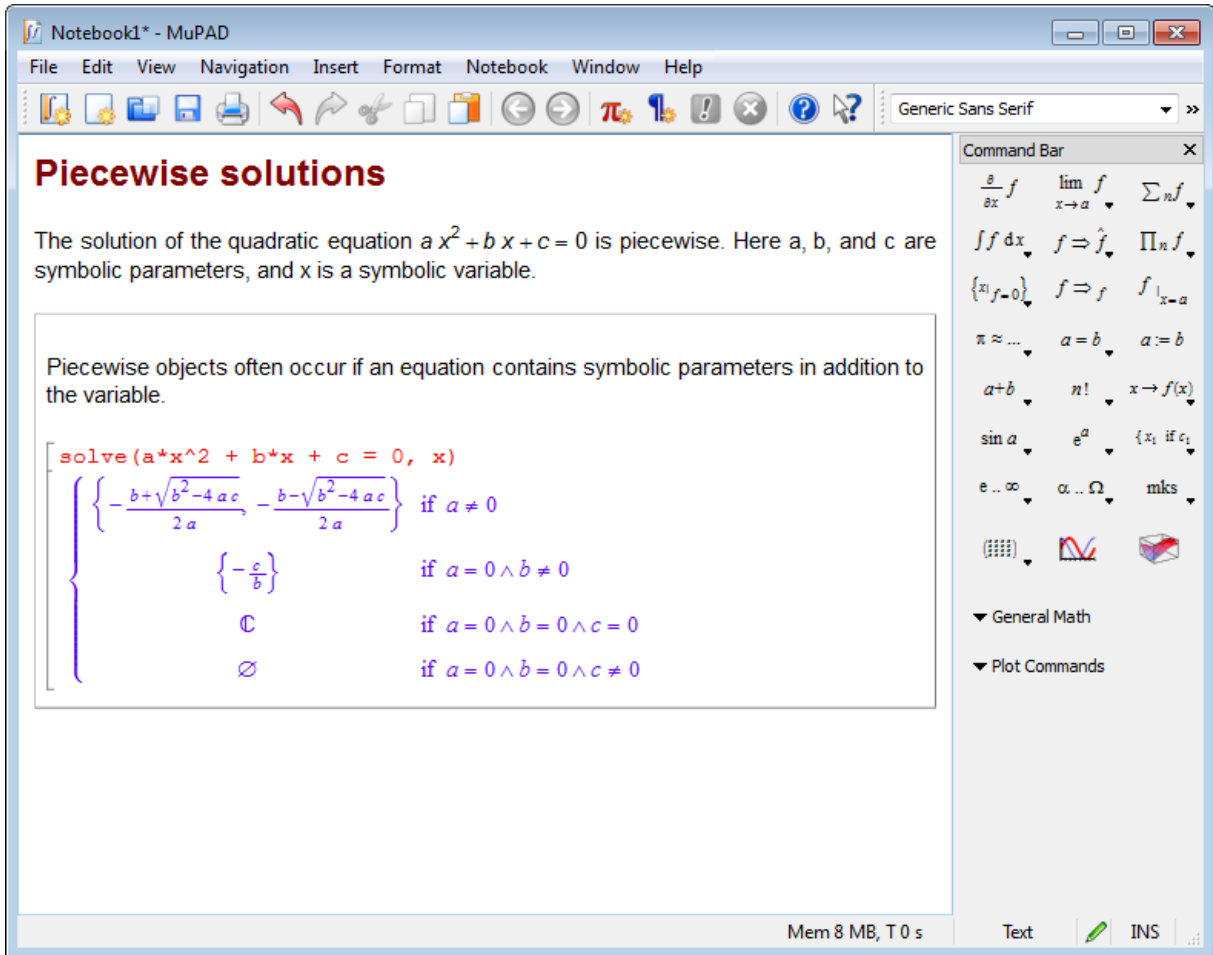
Piecewise objects often occur in the solution of the quadratic equation with parameters in addition to the variable.

```
solve(a*x^2 + b*x + c = 0, x)
```

$$\begin{cases} \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{cases}$$

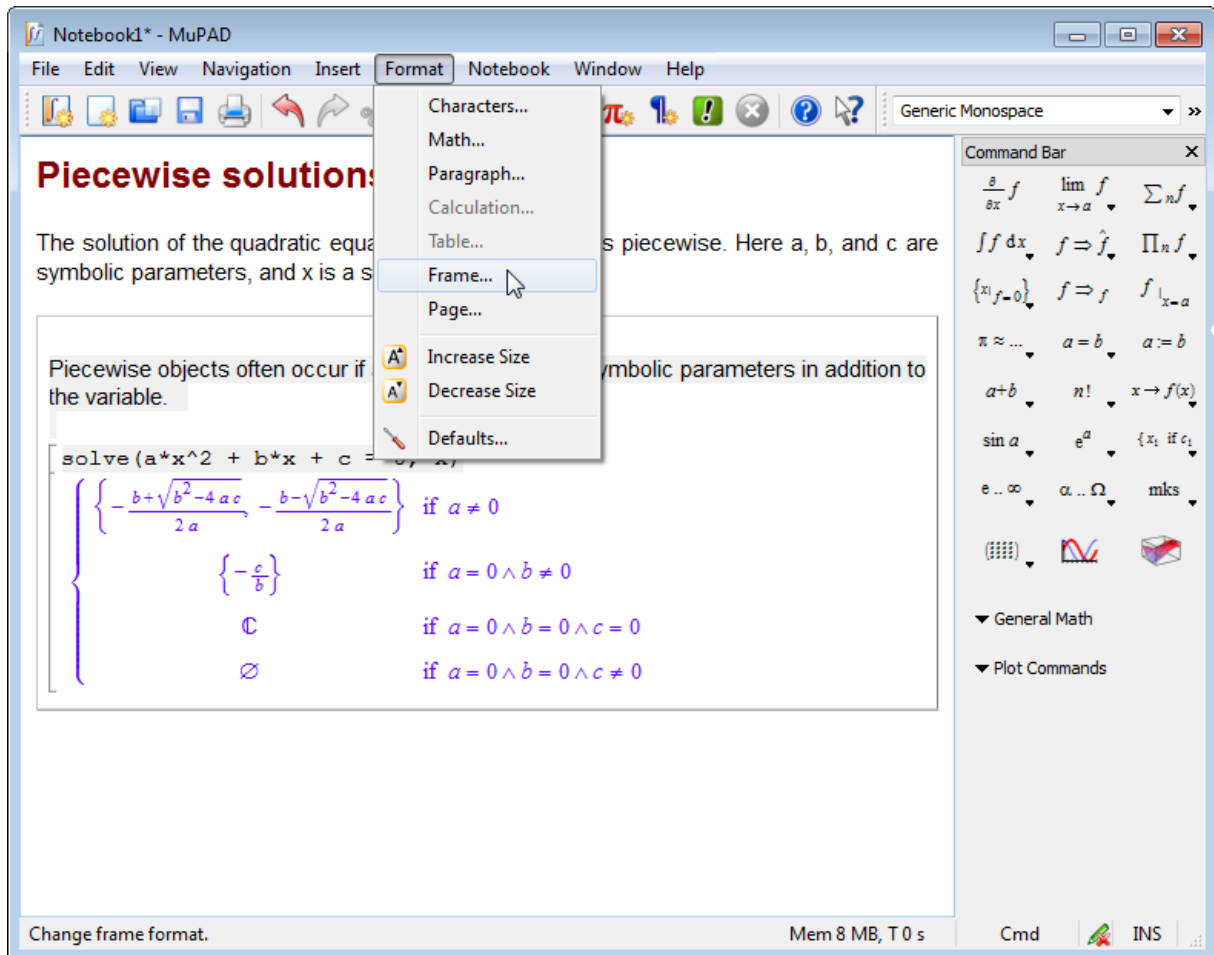
The 'Command Bar' on the right contains various mathematical symbols and operators. The status bar at the bottom indicates 'Insert frame at the current cursor position.', 'Mem 8 MB, T 0 s', and 'Text INS'.

**3** Drag the selected part into the frame.

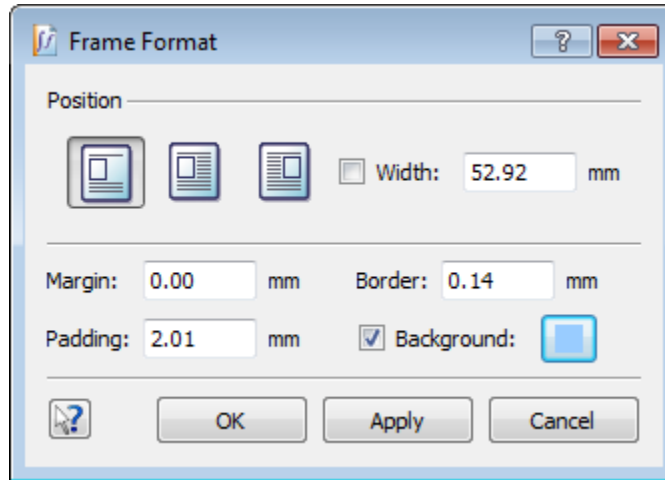


You can change the appearance of the frame. To format a frame:

- 1 Place cursor inside the frame you want to format.
- 2 Select **Format>Frame** from the main menu or right-click to use context menu.



- 3 In the appropriate fields of the Frame Format dialog box, type the size of left margin, frame border size, width of the left bracket that encloses the input region, and background color.



- 4 Click **OK** to apply the new frame settings and close the Frame Format dialog box.

**Piecewise solutions**

The solution of the quadratic equation  $ax^2 + bx + c = 0$  is piecewise. Here  $a$ ,  $b$ , and  $c$  are symbolic parameters, and  $x$  is a symbolic variable.

Piecewise objects often occur if an equation contains symbolic parameters in addition to the variable.

```

solve(a*x^2 + b*x + c = 0, x)
{
  { -b+sqrt(b^2-4ac)/2a, -b-sqrt(b^2-4ac)/2a } if a ≠ 0
  { -c/b } if a = 0 ∧ b ≠ 0
  ℂ if a = 0 ∧ b = 0 ∧ c = 0
  ∅ if a = 0 ∧ b = 0 ∧ c ≠ 0
}

```

Command Bar

General Math

Plot Commands

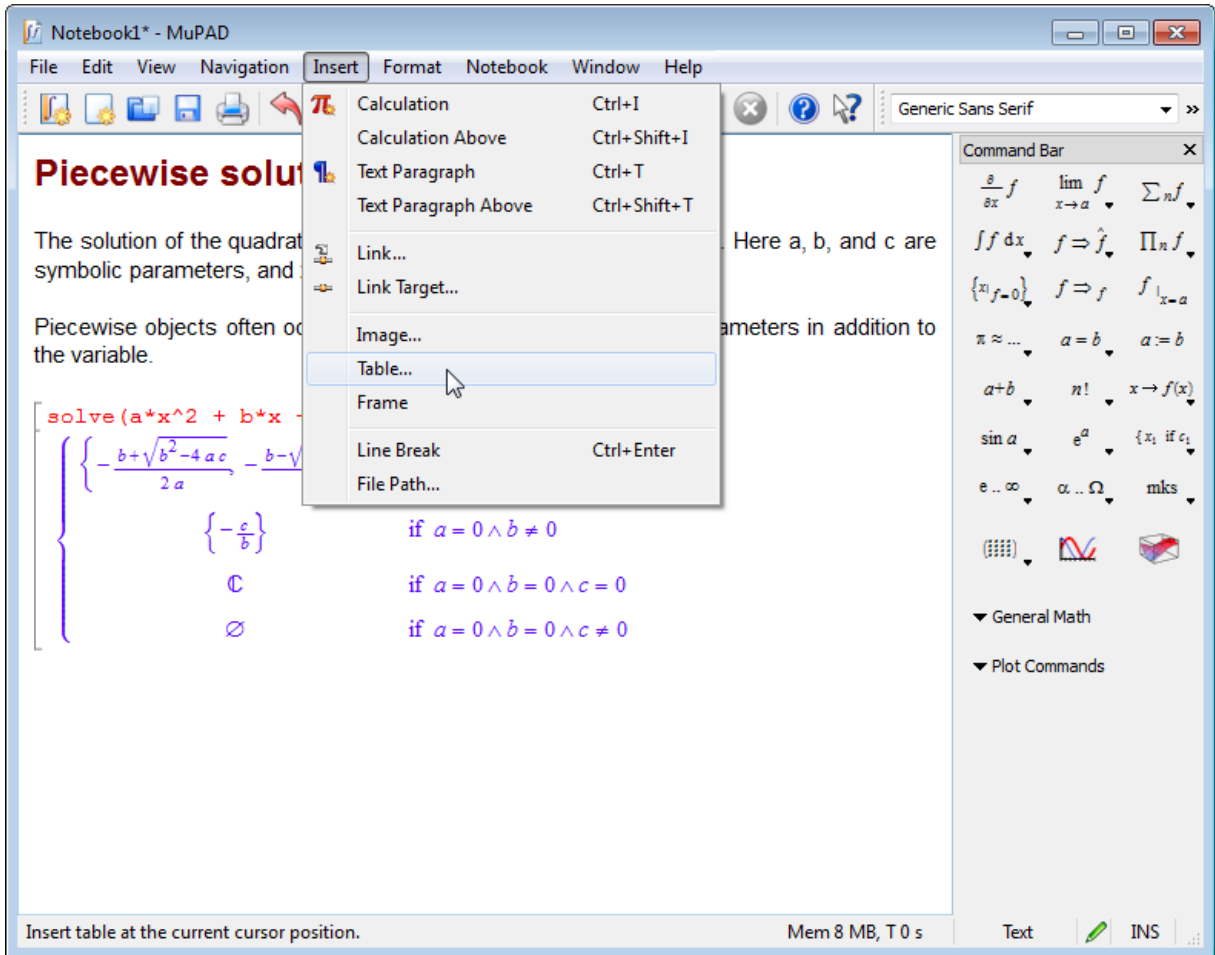
Mem 8 MB, T 0 s

## Use Tables

### Create Tables

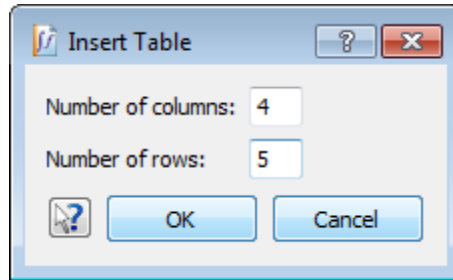
To insert a table in a notebook:

- 1 Select the place where you want the table to appear.
- 2 Select **Insert>Table**.



- 3 In the resulting dialog box, select the number of columns and rows and click **OK**.

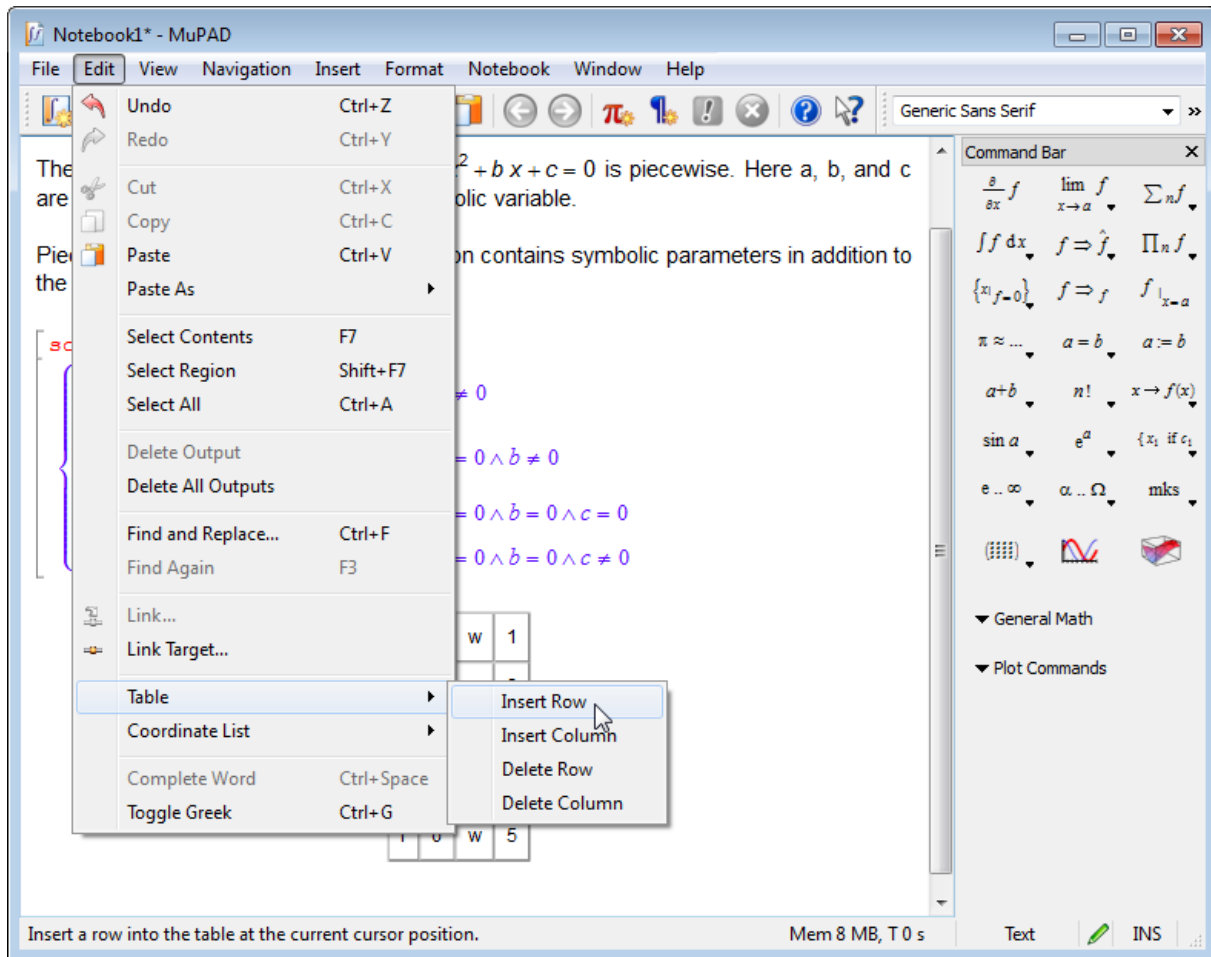




### **Add and Delete Rows and Columns**

To add a row or column to an existing table or delete an existing row or column:

- 1** Click a cell where you want to add or delete a row or column.
- 2** Select **Edit>Table** from the main menu and select the required action.



If you inserted a row, it appears above the row with the selected cell. If you inserted a column, it appears to the left from the selected cell.

The screenshot shows the MuPAD Notebook1\* interface. The main window contains the following text and code:

The solution of the quadratic equation  $ax^2 + bx + c = 0$  is piecewise. Here  $a$ ,  $b$ , and  $c$  are symbolic parameters, and  $x$  is a symbolic variable.

Piecewise objects often occur if an equation contains symbolic parameters in addition to the variable.

```
solve(a*x^2 + b*x + c = 0, x)
```

$$\left\{ \begin{array}{ll} \left\{ -\frac{b + \sqrt{b^2 - 4ac}}{2a}, -\frac{b - \sqrt{b^2 - 4ac}}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{array} \right.$$

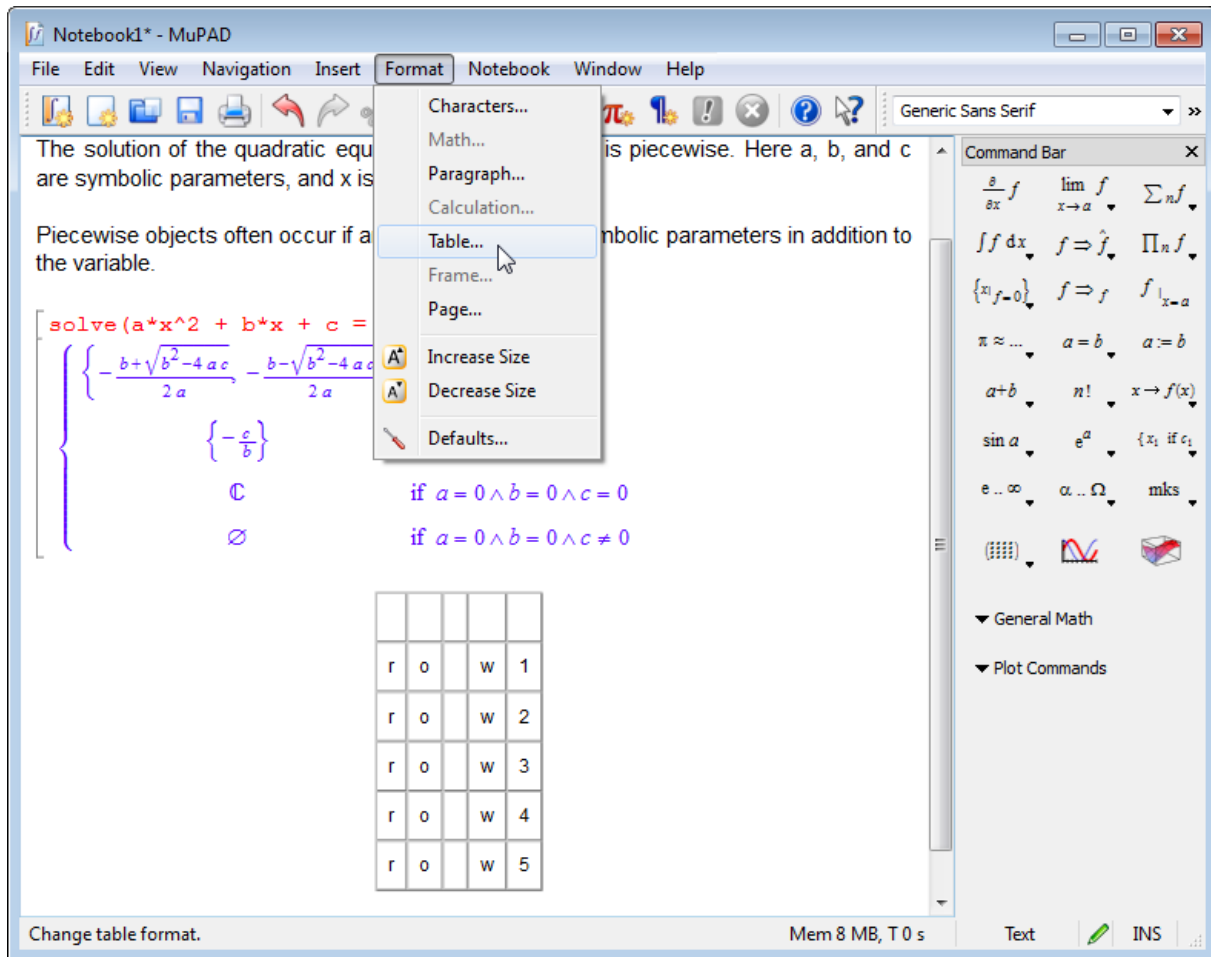
r	o	<b>w</b>	1	
r	o	w	2	
r	o	w	3	
r	o	w	4	
r	o	w	5	

The interface also shows a Command Bar on the right with various mathematical symbols and a status bar at the bottom indicating 'Mem 8 MB, T 0 s' and 'Text' mode.

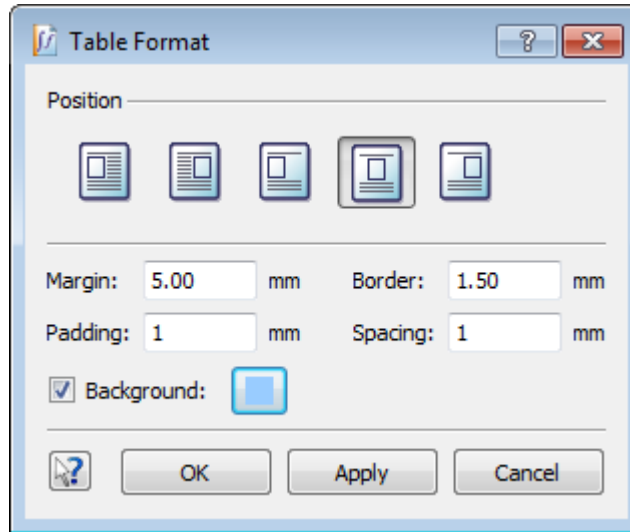
## Format Tables

You can change the appearance of a table in a MuPAD notebook. To format a table:

- 1 Click the table you want to format.
- 2 Select **Format>Table** from the main menu.



3 In the Table Format dialog box, select your settings:

**Position**

Text wrapping

**Margins**

Cell margins (all margins have the same size.)

**Padding**

Distance between the text and the border of a cell

**Border**

Width of the line used to draw cell borders

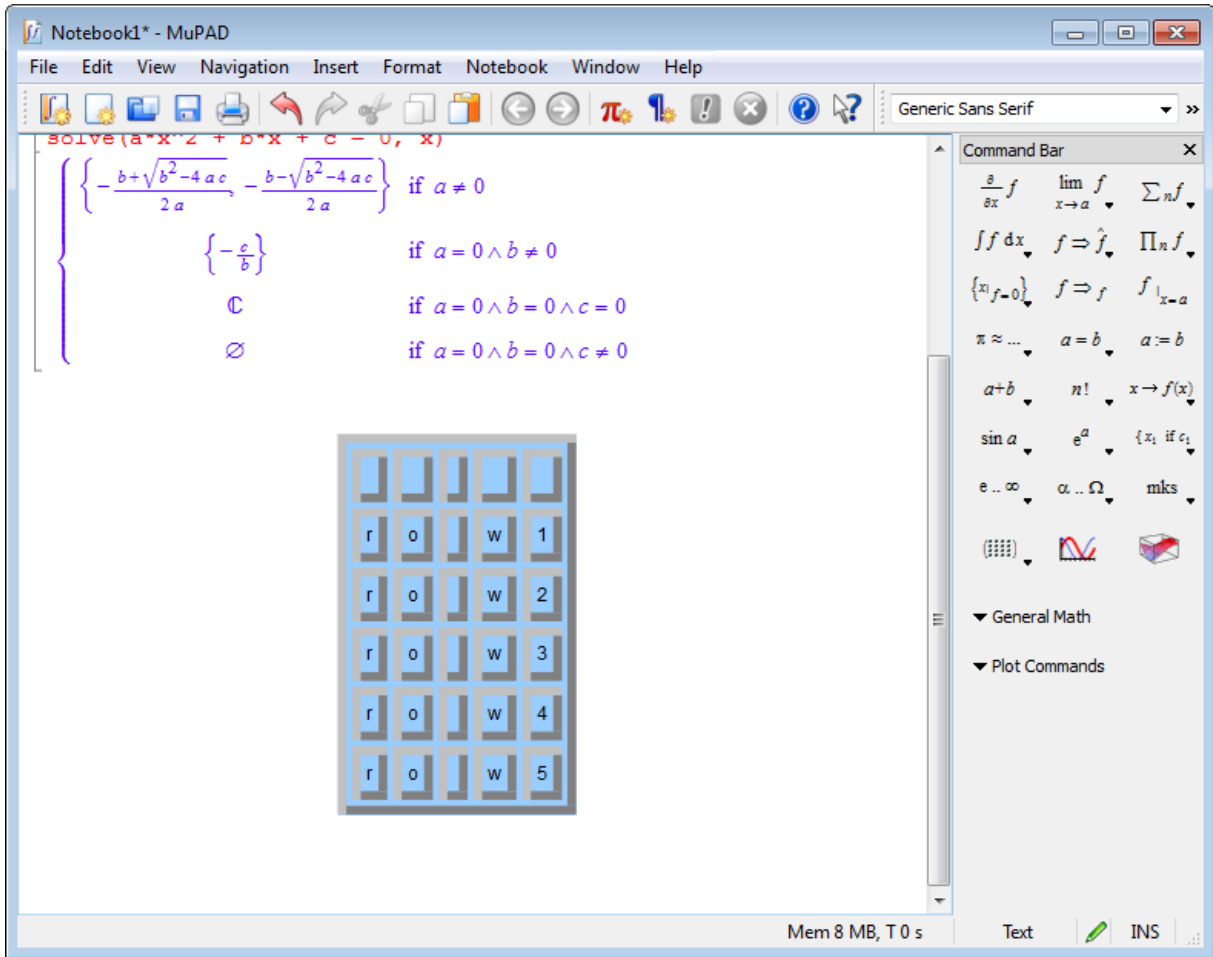
**Spacing**

Space between the cells

**Background**

Background color of the cells

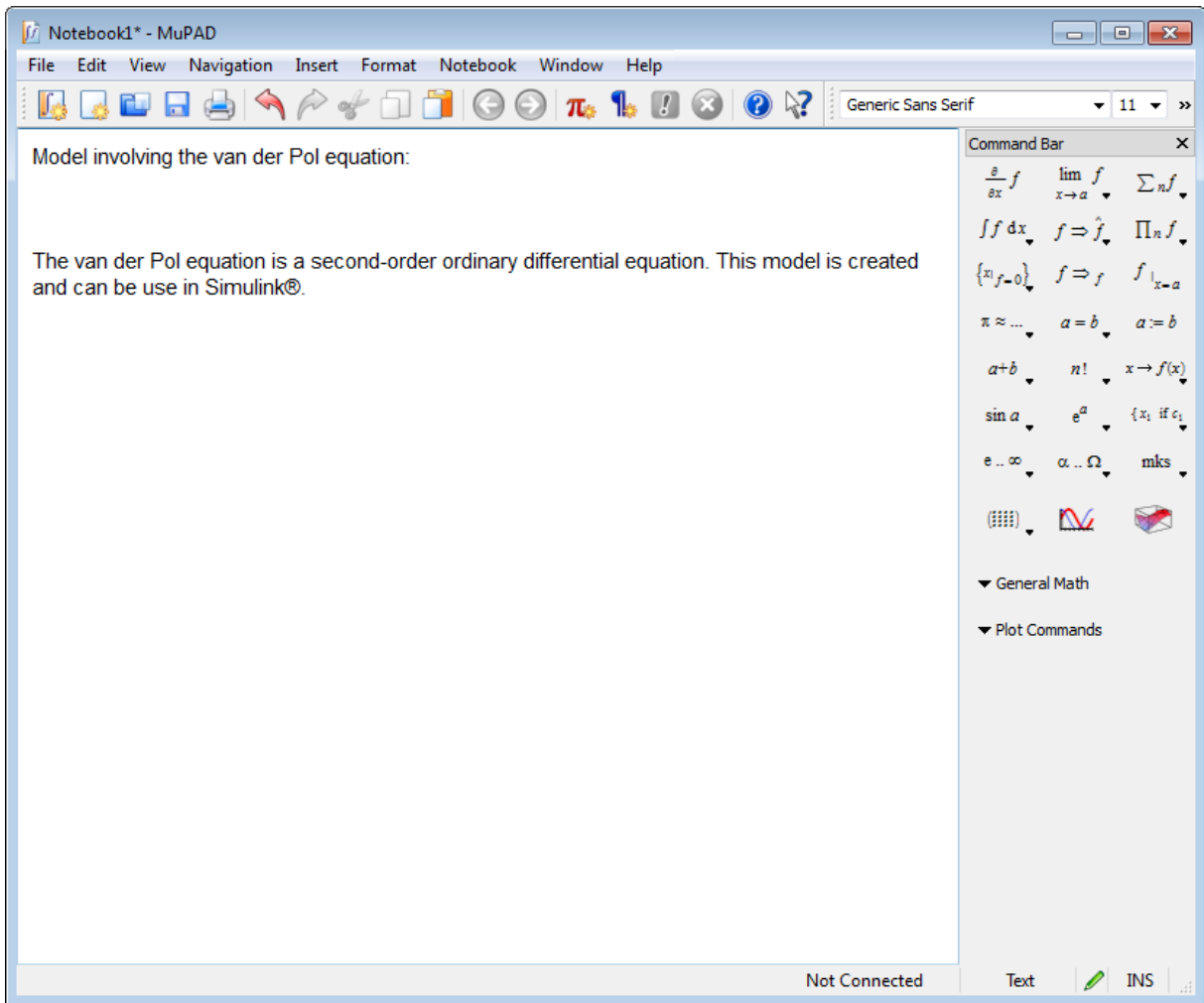
If you want to experiment with different settings, and see how the formatted table looks in your notebook, click the **Apply** button. This button applies formatting to the selected table and leaves the Table Format dialog box open. You can change settings several times without having to open this dialog box for each change. When you finish formatting, click **OK** to close the Table Format dialog box.



## Embed Graphics

To insert a picture into a text region:

- 1 Select the place in a text region where you want to insert a picture.
- 2 Select **Insert>Image** from the main menu and browse the image you want to insert in a notebook.



- 3 The selected image appears in the original size. You cannot format images in text regions of a notebook.

Notebook1\* - MuPAD

File Edit View Navigation Insert Format Notebook Window Help

Generic Sans Serif 11

Model involving the van der Pol equation:

The van der Pol equation is a second-order ordinary differential equation. This model is created and can be use in Simulink®.

Command Bar

$\frac{\partial}{\partial x} f$   $\lim_{x \rightarrow a} f$   $\sum_n f$

$\int f dx$   $f \Rightarrow \hat{f}$   $\prod_n f$

$\{x_i, f=0\}$   $f \Rightarrow f$   $f|_{x=a}$

$\pi \approx \dots$   $a = b$   $a := b$

$a + b$   $n!$   $x \rightarrow f(x)$

$\sin a$   $e^a$   $\{x_i \text{ if } c_i\}$

$e \dots \infty$   $\alpha \dots \Omega$   $\text{mks}$

General Math

Plot Commands

Not Connected Text INS

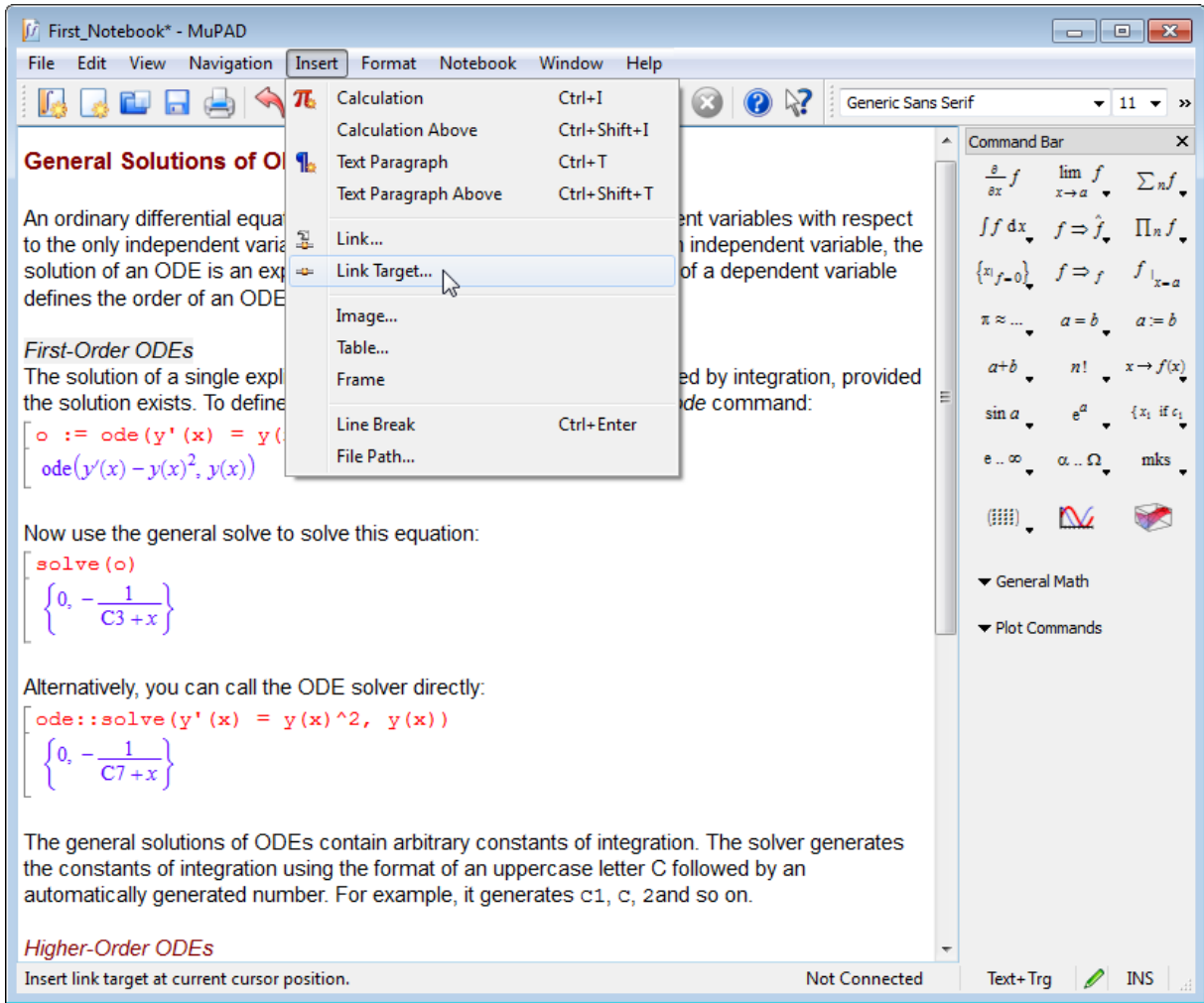


## Work with Links

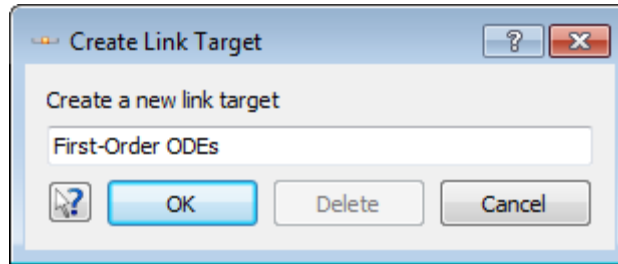
### Insert Links to Targets in Notebooks

**Create Link Targets.** You can insert a link to a particular location in the same notebook or in some other MuPAD notebook. This location is called a link target. First, use the following procedure to create a link target:

- 1** Open the notebook in which you want to create a link target.
- 2** Select the part of the notebook that you want to use as a link target. You can select any object in a notebook, except for output regions.
- 3** Select **Insert>Link Target** to declare the selected part of a notebook as a link target. Alternatively, use the context menu.



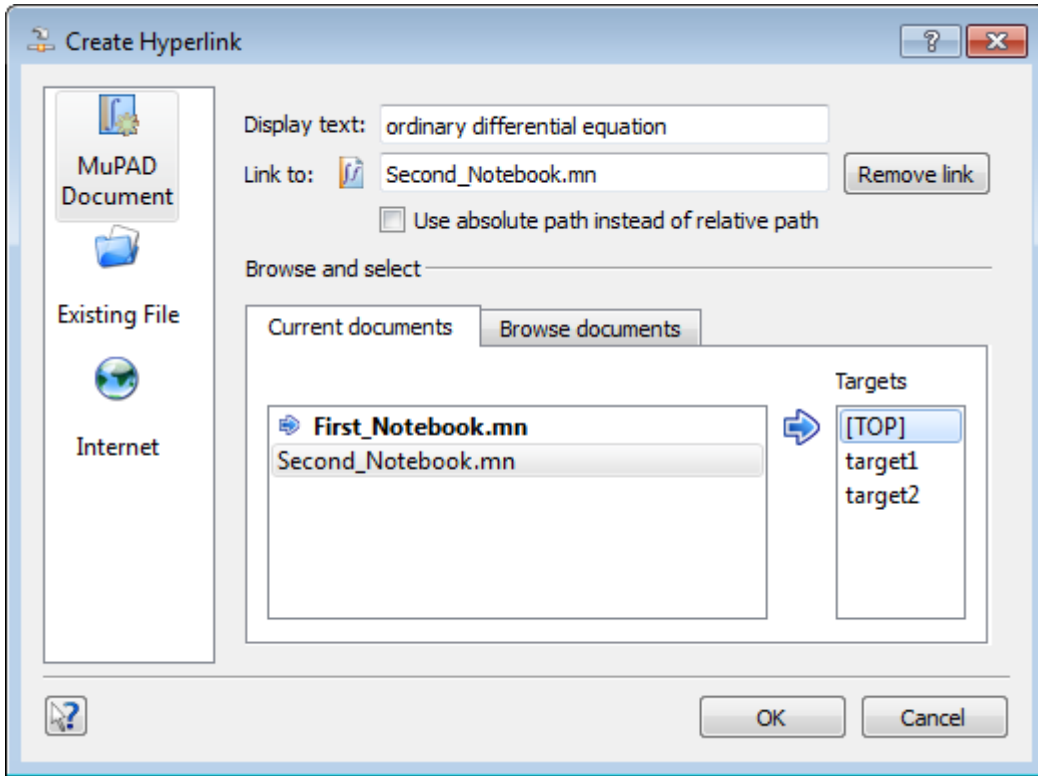
4 In the Create Link Target dialog box, type the name of the link target.



- 5 Save the notebook that contains the link target.

**Add Links.** To associate a link with a link target:

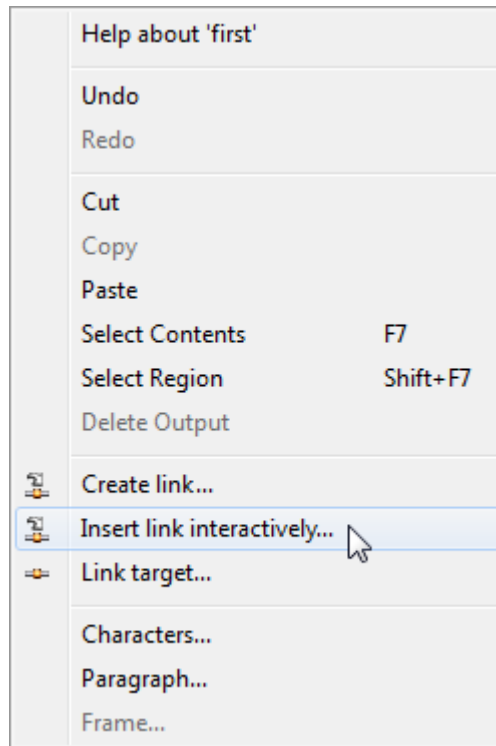
- 1 Open the notebook in which you want to insert a link. This notebook can be the same one where you defined the link target, or it can be any other notebook.
- 2 Select the part of a notebook where you want to create a link.
- 3 Select **Insert>Link** from the main menu or use the context menu.
- 4 In the Create Hyperlink dialog box, select **MuPAD Document** and the name of the notebook that you want to link to. The Targets list displays all link targets available in the selected notebook.
- 5 In the Targets list, select the link target that you want to use. If you want to create a link to the top of the notebook, select **TOP**.



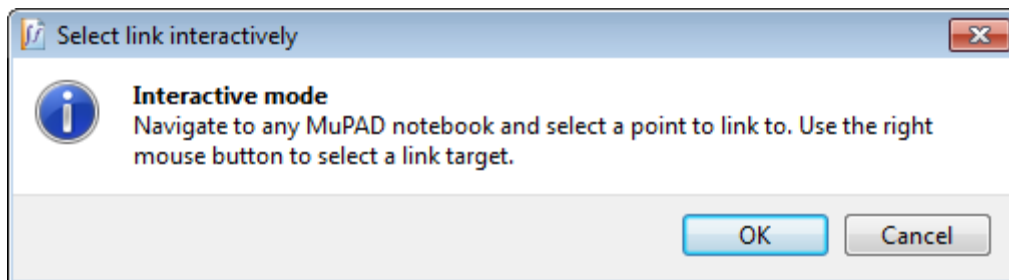
## Insert Links Interactively

To insert a link to a MuPAD notebook interactively (without choosing and naming the target in advance):

- 1 Open the notebook in which you want to insert a link.
- 2 Select the part of a notebook where you want to insert the link.
- 3 From the context menu select **Insert link interactively**.



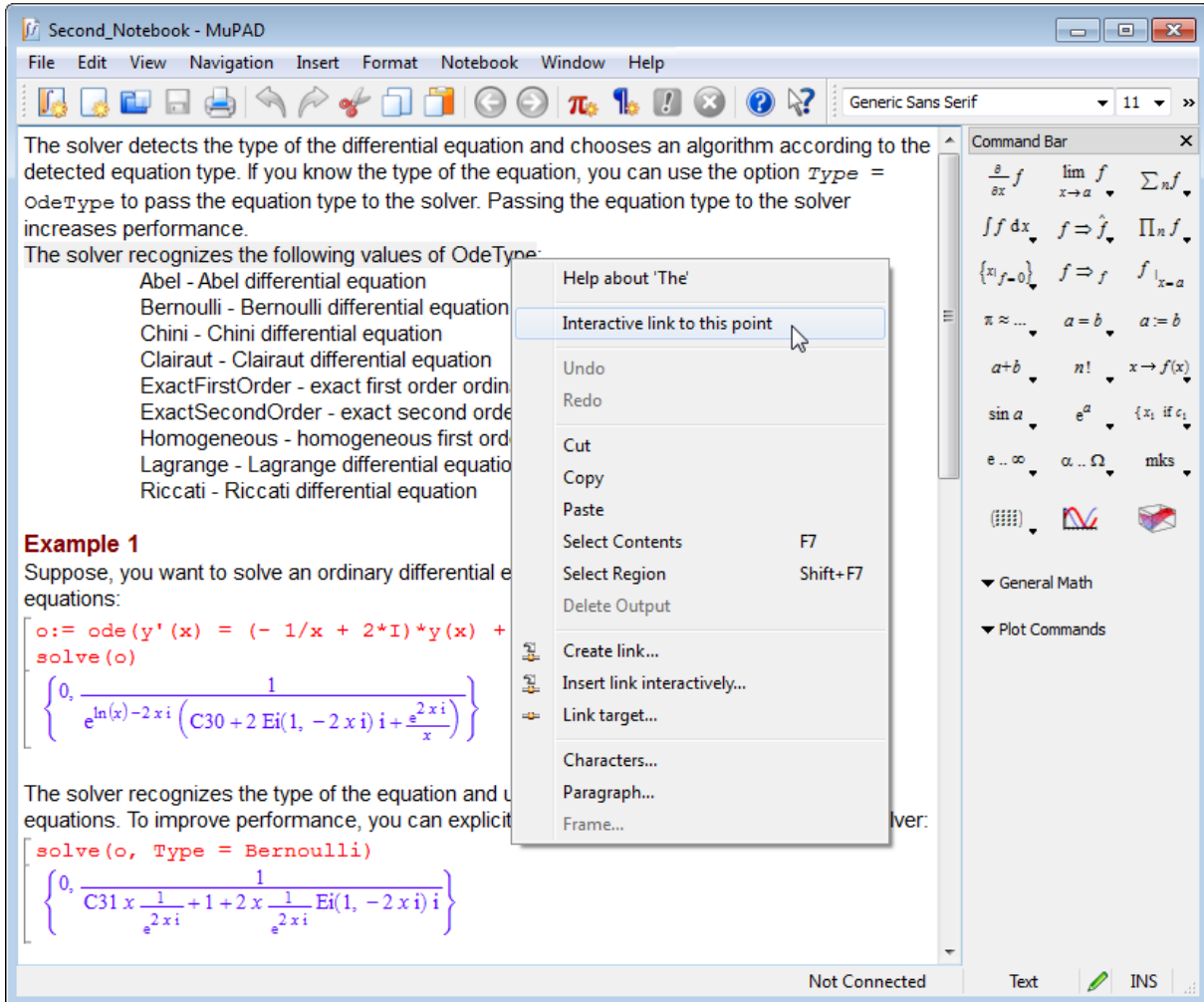
4 The following dialog box appears. To continue creating a link, click **OK**.



5 Open the notebook where you want to insert the link target. If you create a new notebook, save it before proceeding.


6 Select the part of the notebook to which you want to link.

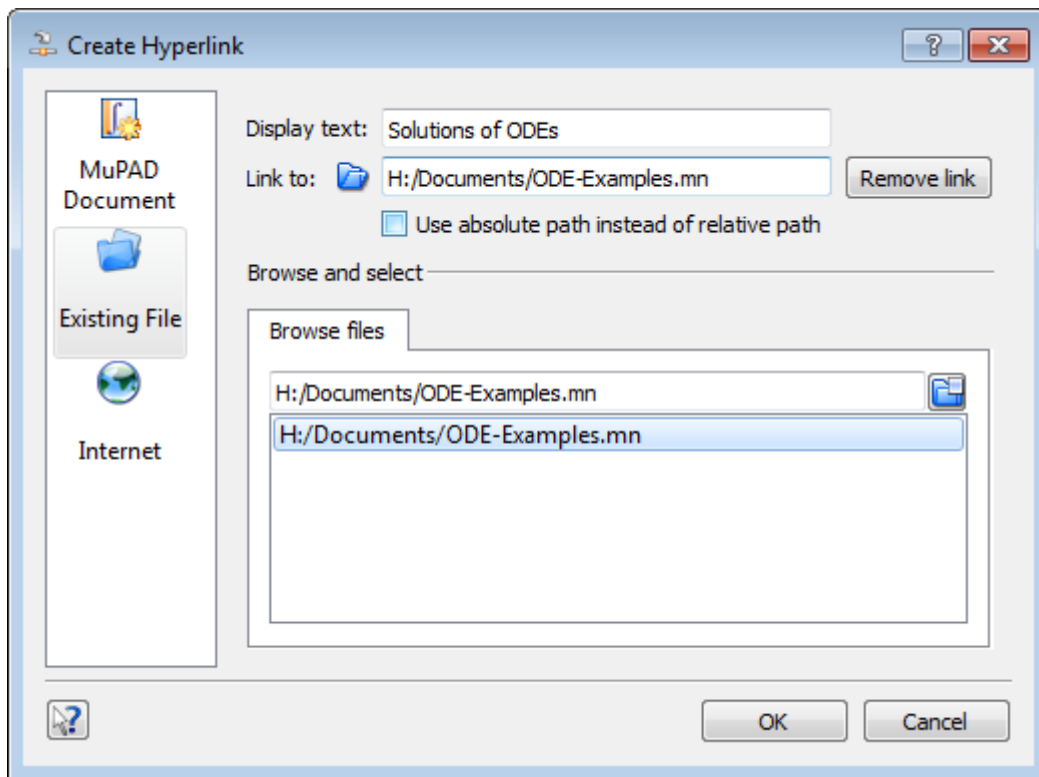
- 7 Right-click the selected part. From the context menu, select **Interactive link to this point**.



## Insert Links to Arbitrary Files

To insert a link to any file on your computer:

- 1 Select the part of a notebook where you want to insert a link.
- 2 Select **Insert>Link** from the main menu or use the context menu.
- 3 In the Create Hyperlink dialog box select **Existing File**.
- 4 Select the file you want to refer to. To select the file, enter the file name in the **Link to** field or choose the file from the history list. Alternatively, click the Open File button , and then browse for the file.

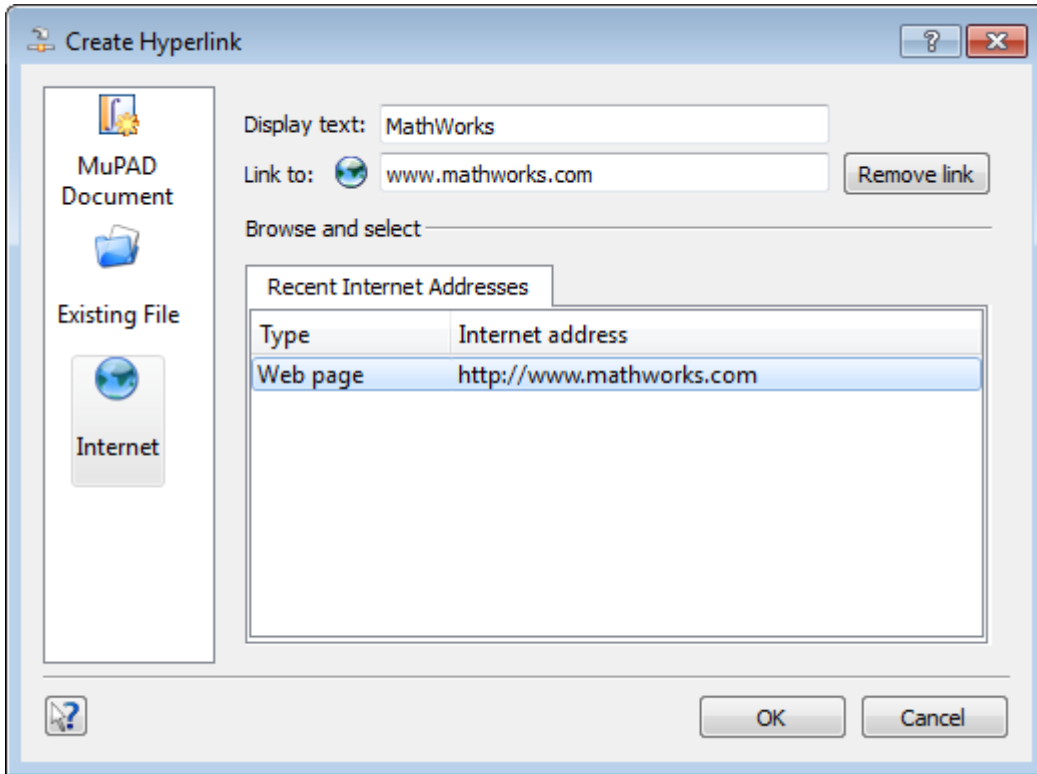


### Insert Links to Internet Addresses

To insert a link to an Internet address:

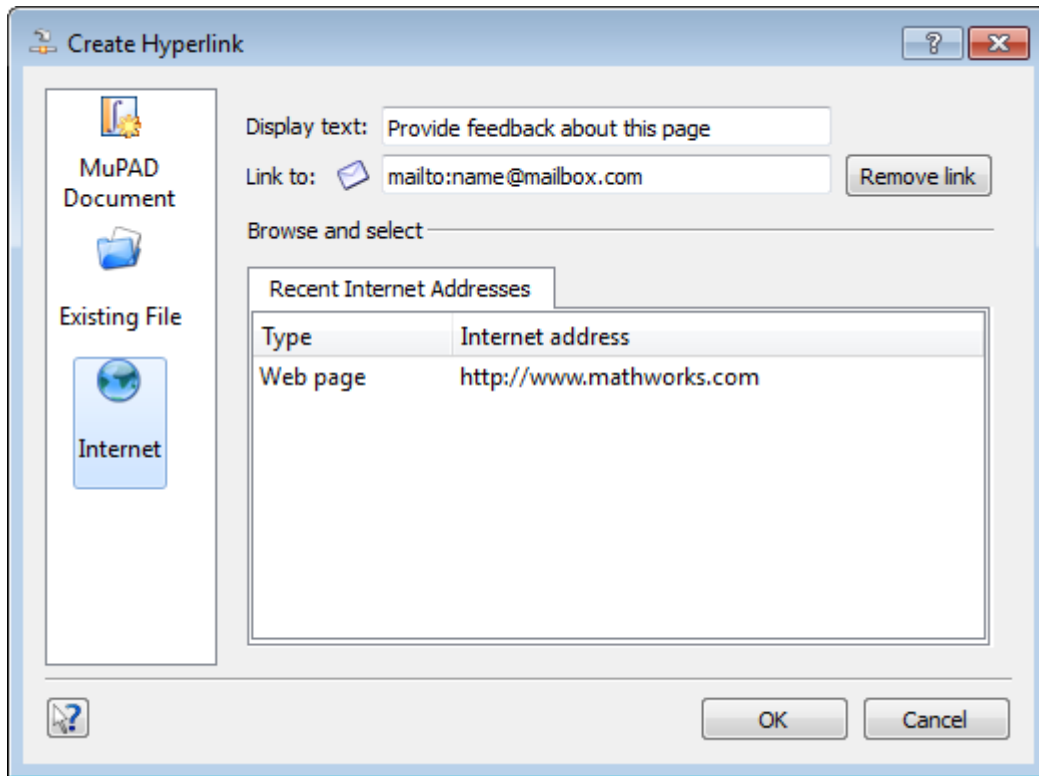
- 1 Select the part of the notebook where you want to insert a link.

- 2 Select **Insert>Link** from the main menu or use the context menu.
- 3 In the Create Hyperlink dialog box, select **Internet**.
- 4 Type an Internet address and click **OK**. For example, insert a link to the Web page.



To insert a link to an e-mail address, type `mailto:` before the address.

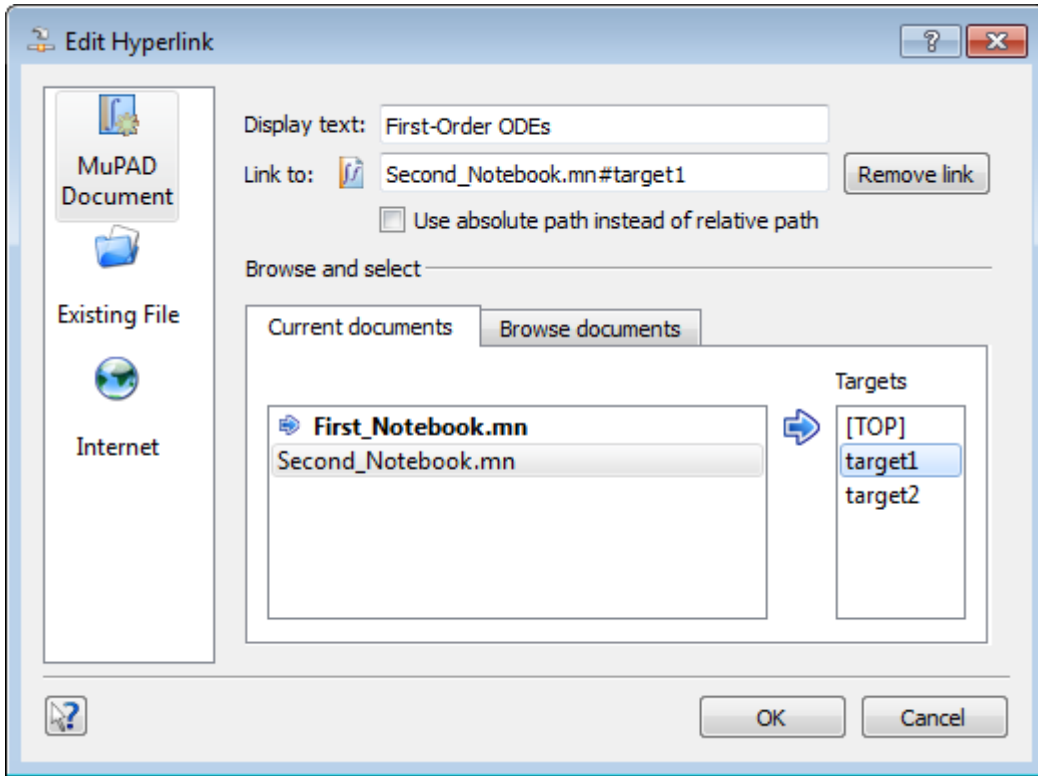




## Edit Existing Links

To edit a link:

- 1 Select the link that you want to edit.
- 2 Select **Edit>Link** from the main menu or use the context menu.
- 3 In the Edit Hyperlink dialog box, edit the link. You can change the display text or the link target or both.



## Delete Links

To delete a link:

- 1 Select the link that you want to delete.
- 2 Right-click to open the context menu.
- 3 Select **Remove Link**

Alternatively, you can use the Edit Hyperlink dialog box to delete a link. To delete a link by using the Edit Hyperlink dialog box:

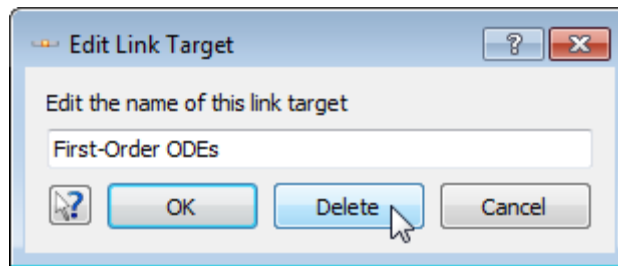
- 1 Select the link that you want to edit.

- 2 Select **Edit>Link** from the main menu or use the context menu.
- 3 In the Edit Hyperlink dialog box, click the **Remove Link** button.
- 4 Click **OK**.

## Delete Link Targets

If the list of link targets is very long, finding the correct link target can be difficult. To make this list shorter, delete link targets that you do not use. To delete a link target:

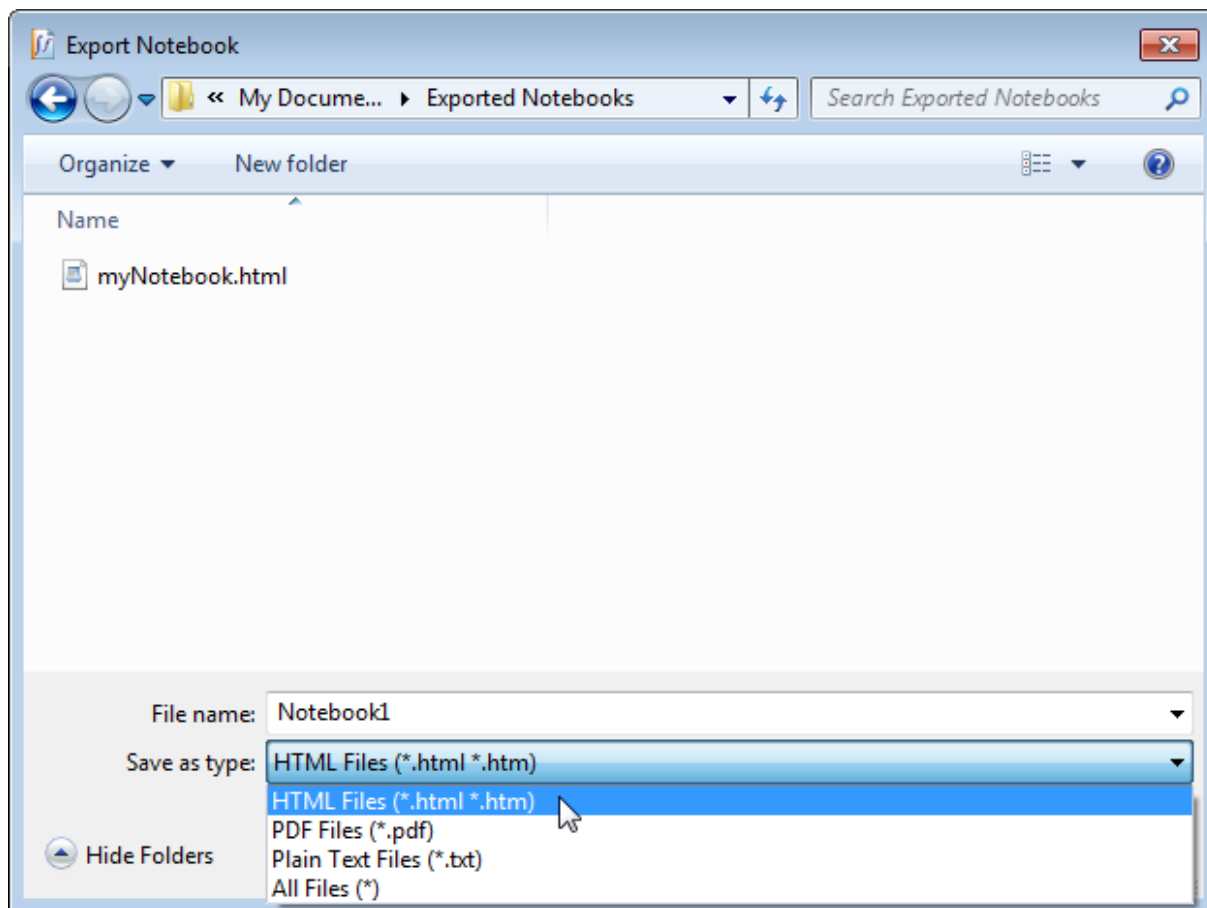
- 1 Find the link target in a notebook. To highlight all link targets in a notebook, select **View>Highlight Link Targets**.
- 2 Select the link target that you want to delete.
- 3 Select **Insert>Link Target** from the main menu or use the context menu.
- 4 In the Edit Link Target dialog box, click **Delete**.



## Export Notebooks to HTML, PDF, and Plain Text Formats

To export a notebook to HTML, PDF, or plain text format:

- 1 Select **File>Export** from the main menu.
- 2 From the drop-down menu, select the file format to which you want to export a notebook.



**3** In the Export Notebook dialog box, enter a name for the file and click **Save**.

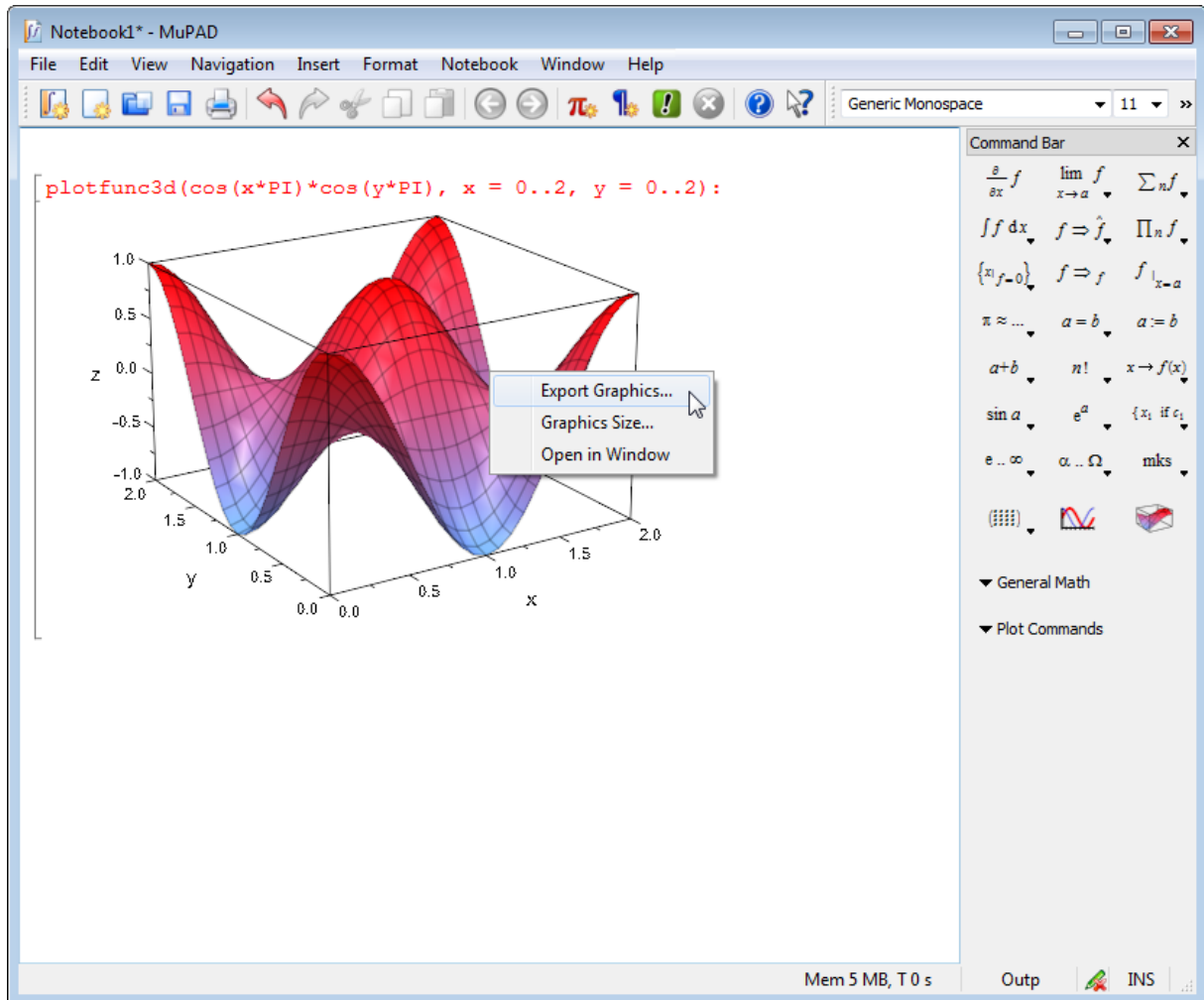
If you export a MuPAD document with links to PDF format, these links are replaced by regular text in the resulting PDF file.

## Save and Export Graphics

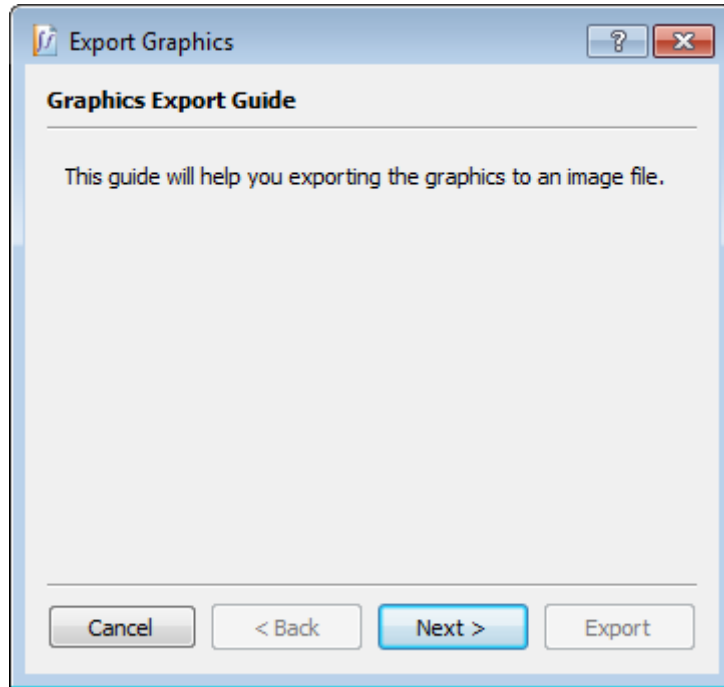
### Export Static Plots

To save the resulting plots separately from a notebook:

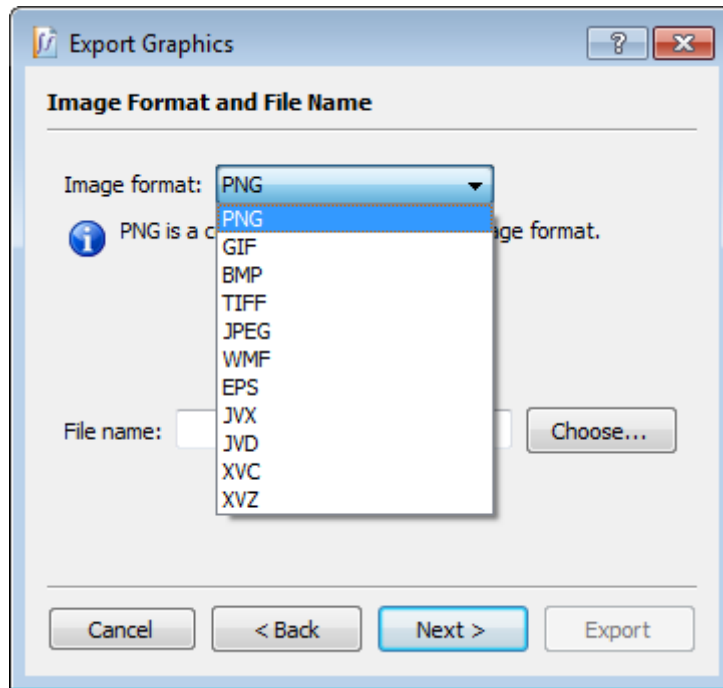
- 1 Double-click the plot you want to export. Select **File>Export Graphics** from the main menu or right-click to use the context menu.



- 2 Click **Next** in the Graphics Export Guide dialog box.



**3** Select the file format to which you want to export the image.



- 4 The next steps depend on the file format you selected. Follow the instructions in the Export Graphics dialog box.

### Choose the Export Format

The appearance of your exported graphics, its quality, and compatibility with other documents depend on the file format you select for saving graphics. The set of file formats available for exporting graphics from a MuPAD notebook could be limited by your operating system and by the type of image.

**Vector Formats.** You can save an image as a set of vector objects such as lines and geometric figures. Images in vector formats are resolution independent and scale almost without quality loss. These images can look different on different computers. MuPAD supports the following vector graphics formats:

- XVC/XVZ — the MuPAD native format. You can import the images saved in this format into a notebook and activate and manipulate them with the MuPAD notebook graphic tools.
- JVX/JVD — JavaView. Java® based format for 2-D and 3-D images that you can embed into HTML Web pages. Exporting an animation to JavaView, you can choose to export the current view or a sequence of images.
- EPS — Encapsulated PostScript®. Standard format for publishing images in scientific articles and books. This format serves best for 2-D images. 3-D images are exported as raster images embedded in EPS, and, therefore, can lose quality. You cannot export animations to this format.
- SVG — Scalable Vector Graphics. The format serves for including vector graphics and animations on Web pages. MuPAD does not support svg animations. You can only export the current view of an animation in this format. You cannot export 3-D images to this format.
- WMF — Windows® Metafile. You can use this file format on Windows operating platforms only.
- PDF — Portable Document Format. You can use this format to export non-animated 2-D images.

**Bitmap Formats.** Bitmap formats save an image pixel by pixel allowing the image to look identical on different computers. To save images in bitmap format, you need to select the image resolution. The quality of images saved in bitmap formats depend on the resolution and can be decreased if you scale an image after saving. MuPAD supports the following bitmap graphics formats:

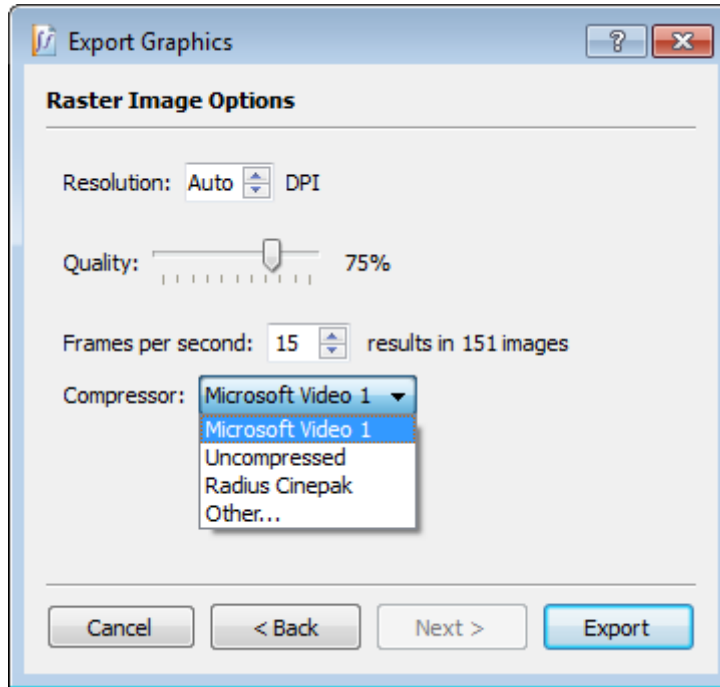
- PNG
- GIF
- BMP
- JPEG

### **Save Animations**

You can export an animation created in MuPAD into an animated GIF file. If you use a Windows systems, you can export an animation created in MuPAD into the special format AVI . To save an animation in this format, select a



compressor from the list of compressors available for your animation and operating system.

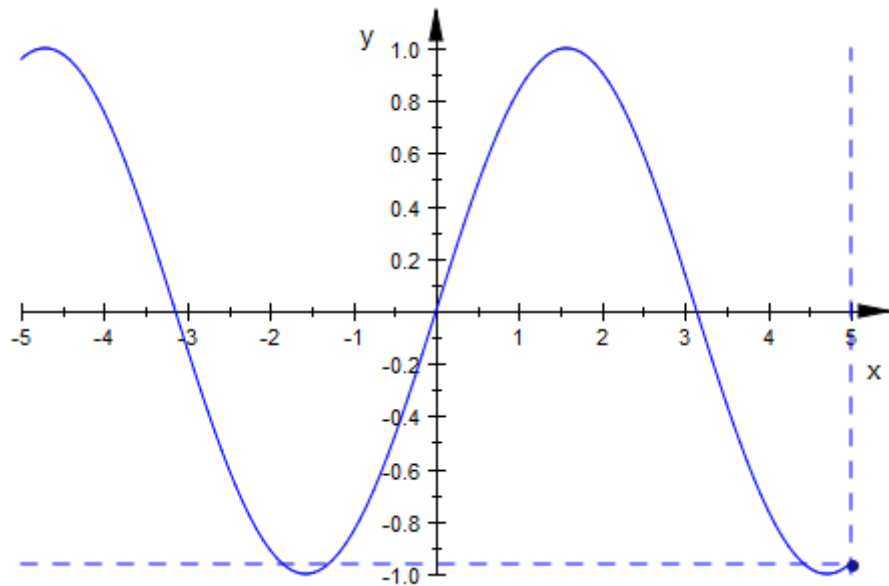


A compressor defines the size of the exported animation and the quality of images in the animation. Viewing the exported animation requires the installation of an appropriate compressor.

### Export Sequence of Static Images

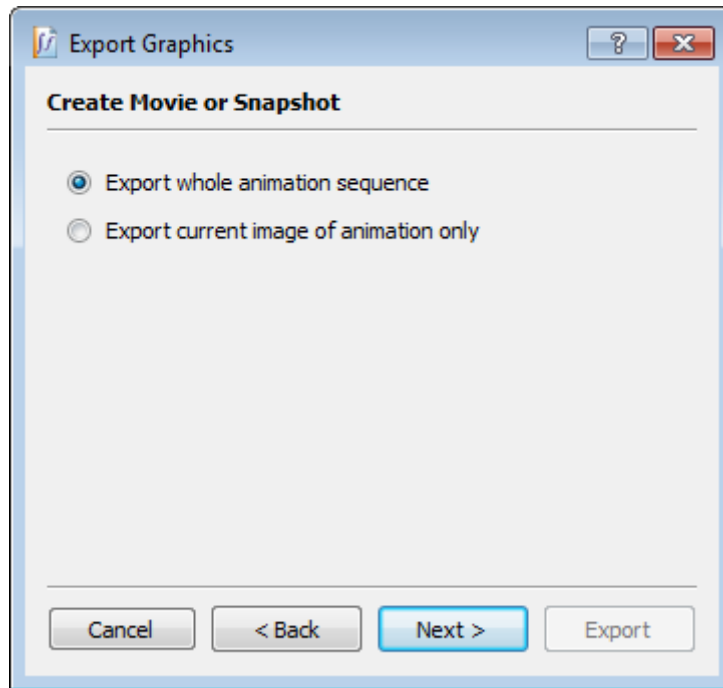
You can export an animation as a sequence of static images. For example, the following calculations result in an animated plot:

```
plot( plot::Function2d( sin(x) ),
      plot::Point2d( [x, sin(x)], x=-5..5 ),
      plot::Line2d( [x,-1], [x,1], x=-5..5, LineStyle=Dashed),
      plot::Line2d( [-5,sin(x)], [5,sin(x)],
                    x=-5..5, LineStyle=Dashed)
    )
```

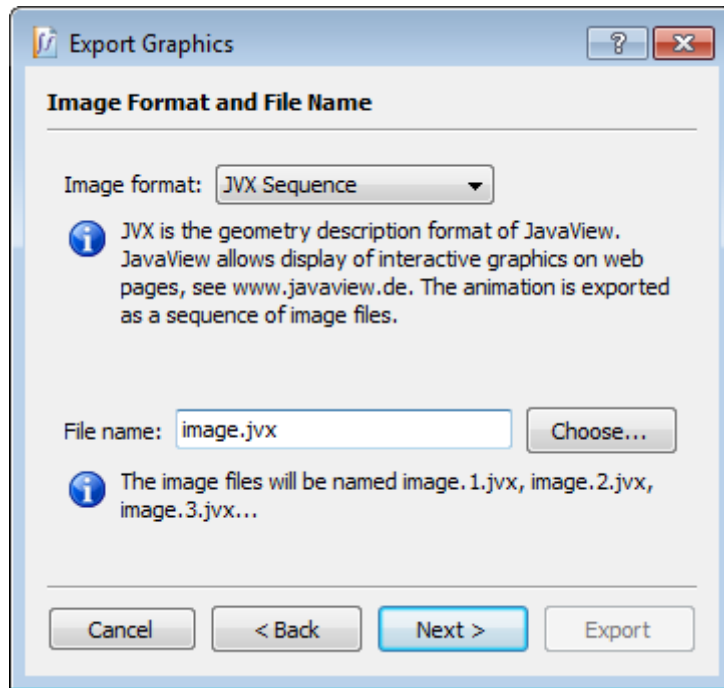


To export the resulting plot:

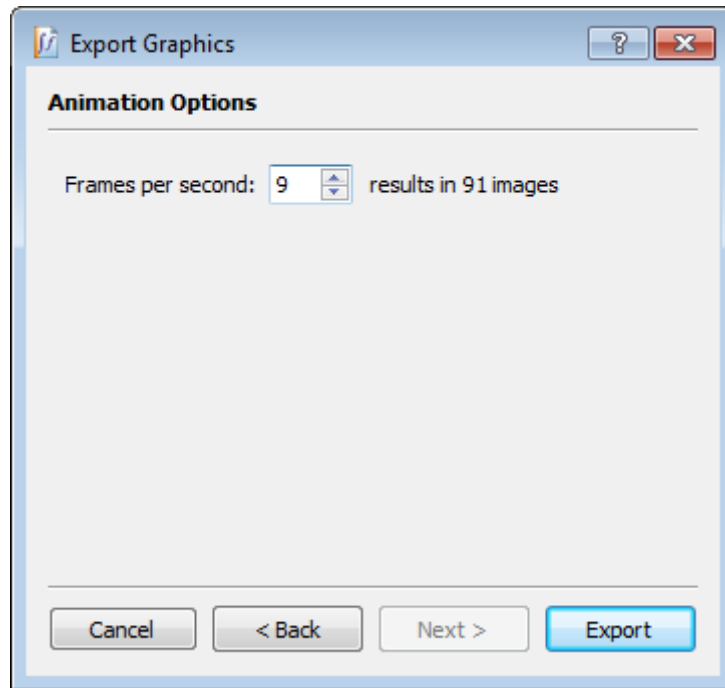
- 1 Select **File>Export Graphics** from the main menu or right-click to use the context menu.
- 2 Click **Next** in the Graphics Export Guide dialog box.
- 3 Select **Export whole animation sequence** and click **Next**.



- 4 Select the format for saving images of the sequence and the file name. A sequence of numbers automatically appends to the file name you enter. For example, when you enter `image.jvx`, you get the following sequence of files: `image.1.jvx`, `image.2.jvx`, and so on. This sequence of file names displays below the entry field for the file name.



- 5 Select the number of frames per second. This number defines the total number of image files you create.



## Use Data Structures

### In this section...

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### Mathematical Expressions

You can create mathematical expressions using MuPAD objects and operators. For example, the following equation presents a MuPAD expression:

$$x + y + 1 + 1/5 + \sin(5) = z$$

$$x + y + \sin(5) + \frac{6}{5} = z$$

Precedence levels determine the order in which MuPAD evaluates an expression. Operators in MuPAD expressions have precedences similar to the precedences of regular mathematical operators. For example, when you compute the expression  $a + b*c$ , you calculate  $b*c$ , and then add  $a$ .

To change the evaluation order, use parentheses:

$$1 + 2*3, (1 + 2)*3$$

$$7, 9$$

## Sequences

### Create Sequences

Sequences represent one of the basic data structures. Sequences can contain arbitrary MuPAD objects. For example, numbers, symbols, strings, or functions can be entries of a sequence. There are two methods for creating sequences in MuPAD:

- Separating MuPAD objects with commas
- Using the sequence generator

Separating MuPAD objects with commas creates a sequence of these objects:

```
sequence := a, b, c, d
```

*a, b, c, d*

As a shortcut for creating a sequence, use the sequence generator \$

```
x^2 $ x = -5..5
```

25, 16, 9, 4, 1, 0, 1, 4, 9, 16, 25

or the functional form of the sequence generator:

```
_seqgen(x^2, x, -5..5)
```

25, 16, 9, 4, 1, 0, 1, 4, 9, 16, 25

To create a sequence of identical objects, use the sequence generator:

```
x^2 $ 7
```

$x^2, x^2, x^2, x^2, x^2, x^2, x^2$

To create a new sequence using the entries of an existing sequence, use the sequence generator with the keyword `in` or the equivalent command `_seqin`. For example:

```
x^y $ y in (a, b, c, d);  
f(x) $ x in [a, b, c, d];  
_seqin(f(x), x, [a, b, c, d])
```

$x^a, x^b, x^c, x^d$

$f(a), f(b), f(c), f(d)$

$f(a), f(b), f(c), f(d)$

You cannot create nested sequences because MuPAD automatically flattens them:

```
sequence := (a, b, c, d);  
((a, b, 10), (1, 10, f))
```

$a, b, c, d$

$a, b, 10, 1, 10, f$

### Access Sequence Entries

To access particular entries of a sequence by their indices, use `_index` (you can use square brackets as a shortcut) or `op`:

```
sequence := a, b, c, d;  
sequence[2]; _index(sequence, 2..4);  
op(sequence, 2); op(sequence, 2..4)
```

$a, b, c, d$

$b$



*b, c, d*

*b*

*b, c, d*

---

**Note** `_index` uses the order in which the entries appear on the screen, and `op` uses the internal order of the entries. Although for sequences these orders are the same, for many other data structures they are different. For details, see the `_index` help page.

---

To access an entry counting numbers from the end of a sequence, use negative numbers:

```
sequence := a, b, c, d:
sequence[-2]
```

*c*

If you use an indexed assignment without creating a sequence, MuPAD generates a table instead of a sequence:

```
S[1] := x: S
```

$\overline{1|x}$

### Add, Replace, or Remove Sequence Entries

To add entries to a sequence, list the sequence and the new entries separating them with commas:

```
sequence := a, b, c:
sequence := sequence, d, e
```

*a, b, c, d, e*

To concatenate sequences, list the sequences separating them with commas:

```
sequence1 := a, b, c:  
sequence2 := t^3 $ t = 0..3:  
sequence3 := sequence1, sequence2
```

*a, b, c, 0, 1, 8, 27*

To replace an entry of a sequence by a MuPAD object, access the entry by its index, and assign the new value to the entry:

```
sequence := a, b, c, d:  
sequence[1] := NewEntry:  
sequence[2] := 1, 2, 3:  
sequence[-1] := matrix([[1, 2, 3], [5, 6, 7]]):  
sequence
```

*NewEntry, 1, 2, 3, c,  $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}$*

To remove an entry from a sequence, use the `delete` command:

```
sequence := a, b, c, d:  
delete sequence[2];  
sequence
```

*a, c, d*

## **Lists**

### **Create Lists**

Lists represent ordered data structures. Lists can contain arbitrary MuPAD objects. For example, numbers, symbols, strings, or functions can be entries of a list. To create a list, separate MuPAD objects with commas and enclose the structure in brackets:

```
list := [a, b, c, d]
```

```
[a, b, c, d]
```

Also, you can create a sequence, and convert it to a list. To convert a sequence to a list, enclose the sequence in brackets. As a shortcut for creating a sequence, use the sequence generator \$ or its functional form `_seqgen`. Enclose the sequence in brackets:

```
[i*(i-1) $ i = 1..10];  
[i $ 10]
```

```
[0, 2, 6, 12, 20, 30, 42, 56, 72, 90]
```

```
[i, i, i, i, i, i, i, i, i, i]
```

A list can contain lists as entries:

```
list1 := [1, list, 2]
```

```
[1, [a, b, c, d], 2]
```

A list can be empty:

```
empty_list := []
```

```
[]
```

MuPAD does not flatten lists like it flattens sequences. You can create nested lists:

```
list1 := [1, 2]:  
list2 := [5, 6]:  
list3 := [list1, 3, 4, list2]
```

```
[[1, 2], 3, 4, [5, 6]]
```

## Access List Entries

There are two ways to access particular entries of a list by their indices:

- If you want to use the order in which the entries appear on the screen, use brackets or `_index`.
- If you want to use the internal order of a list, use `op`.

In general, these two indices of an entry of a data structure can be different. For lists, the internal order is the same as what you see on the screen:

```
list := [a, b, c, d, e, f]:  
list[2]; _index(list, 3..5);  
op(list, 2); op(list, 3..5)
```

*b*

*[c, d, e]*

*b*

*c, d, e*

To access an entry counting numbers from the end of a list, use negative numbers:

```
list := [a, b, c, d, e, f]:  
list[-2]
```

*e*

If you use an indexed assignment without creating a list, MuPAD generates a table instead of a list:

```
L[1] := x: L
```

$$\overline{1|x}$$

## Operate on Lists

MuPAD lists support the following operations:

- Verifying that a list contains a particular object
- Using a list as a function in a function call
- Applying a function to all entries of a list
- Extracting entries of a list
- Dividing a list according to particular properties of its entries
- Arithmetical operations on lists

To check if an object belongs to a list, use the `contains` command. The command returns the position of the first occurrence of the object in the list. If the object does not belong to the list, `contains` returns 0:

```
list := [(i-5)/7 $ i = 10..20];
contains(list, 1);
contains(list, -1)
```

$$\left[ \frac{5}{7}, \frac{6}{7}, 1, \frac{8}{7}, \frac{9}{7}, \frac{10}{7}, \frac{11}{7}, \frac{12}{7}, \frac{13}{7}, 2, \frac{15}{7} \right]$$

3

0

If you use a list as the function in a function call, MuPAD returns the list of appropriate function calls:

```
[sin, cos, tan, f](x);
[sin, cos, tan, f](0.1)
```

$$[\sin(x), \cos(x), \tan(x), f(x)]$$

```
[0.09983341665, 0.9950041653, 0.1003346721, f(0.1)]
```

To apply a function to all entries of a list, use the function `map`:

```
map([x, 0.1, 1/5, PI], sin);  
map([x, 0.1, 1/5, PI], `+`, a, 1)
```

```
[sin(x), 0.09983341665, sin(1/5), 0]
```

```
[a+x+1, a+1.1, a+6/5, pi+a+1]
```

To extract entries with particular properties from a list, use the `select` command:

```
select([i $ i = 1..20], isprime)
```

```
[2, 3, 5, 7, 11, 13, 17, 19]
```

To divide a list into three lists according to particular properties, use the `split` command:

```
split([i $ i = 1..10], isprime)
```

```
[[2, 3, 5, 7], [1, 4, 6, 8, 9, 10], []]
```

The resulting three lists contain:

- Entries with the required properties
- Entries without the required properties
- Entries for which the required properties are unknown.

MuPAD supports the following arithmetical operations on lists: addition, subtraction, multiplication, division, and power. The lists you operate on

must contain an equal number of entries. MuPAD forms a new list containing the entries of the existing lists combined pairwise:

```
list1 := [a, b, c]:
list2 := [d, e, f]:
list1 + list2;
list1*list2;
list1^list2
```

$$[a+d, b+e, c+f]$$

$$[a d, b e, c f]$$

$$[a^d, b^e, c^f]$$

When you combine a list and a scalar, MuPAD combines a scalar with each entry of a list. For example:

```
list1 := [a, b, c]:
list1 + a;
list1^5;
list1*(a + 5)
```

$$[2 a, a+b, a+c]$$

$$[a^5, b^5, c^5]$$

$$[a(a+5), b(a+5), c(a+5)]$$


---

**Note** Combining a scalar and a list differs from combining a scalar and a matrix.

---

When you add a scalar to a matrix, MuPAD adds the scalar multiplied by an identity matrix to the original matrix. For example, define a matrix M as follows. Add the variable a the matrix M:

```
M := matrix([[1, 2, 3], [4, 5, 6], [7, 8, 9]]):  
M + a
```

$$\begin{pmatrix} a+1 & 2 & 3 \\ 4 & a+5 & 6 \\ 7 & 8 & a+9 \end{pmatrix}$$

Now define the rows of the matrix M by the following three lists. Add the variable a to each list. MuPAD adds the variable a to each entry of the three lists:

```
list1 := [1, 2, 3]:  
list2 := [4, 5, 6]:  
list3 := [7, 8, 9]:  
matrix([list1 + a, list2 + a, list3 + a]);
```

$$\begin{pmatrix} a+1 & a+2 & a+3 \\ a+4 & a+5 & a+6 \\ a+7 & a+8 & a+9 \end{pmatrix}$$

When you combine a scalar and an empty list, the result is an empty list:

```
[] + 2
```

```
[]
```

MuPAD lets you combine nested lists:

```
[[a, b], c, d] + 1;  
[[a, b], c, d] + [1, 2, 3]
```

```
[[a+1, b+1], c+1, d+1]
```

```
[[a+1, b+1], c+2, d+3]
```



To combine lists with unequal numbers of entries, use the `zip` command. By default, the resulting list has the same number of entries as the shortest list:

```
list1 := [a, b]:
list2 := [d, e, f]:
zip(list1, list2, _plus);
zip(list1, list2, _mult);
zip(list1, list2, _power)
```

$[a+d, b+e]$

$[a d, b e]$

$[a^d, b^e]$

To produce the list with the number of entries equal to the longer list, use a default value as additional parameter:

```
zip(list1, list2, _plus, 100)
```

$[a+d, b+e, f+100]$

### Add, Replace, or Remove List Entries

To add new entries to the end of a list, use the `append` command or the `.` (dot) operator:

```
list := [a, b, c]:
list := append(list, d, e, f);
list := list.[1, 2, 3, 4]
```

$[a, b, c, d, e, f]$

$[a, b, c, d, e, f, 1, 2, 3, 4]$

To concatenate lists, use the operator `'.'` (dot) or its functional form `_concat`:

```
list1 := [a, b, c]:  
list2 := [t^3 $ t = 0..3]:  
list3 := list1.list2;  
list4 := _concat(list2, list1)
```

```
[a, b, c, 0, 1, 8, 27]
```

```
[0, 1, 8, 27, a, b, c]
```

You can replace an entry of a list:

```
list := [a, b, c, d, e]:  
list[1] := newEntry:  
list[2] := [1, 2, 3]:  
list[-1] := matrix([[1, 2, 3], [5, 6, 7]]):  
list
```

```
[newEntry, [1, 2, 3], c, d,  $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}$ ]
```

To remove an entry from a list, use the `delete` command:

```
list := [a, b, c, d, e, f]:  
delete list[-1];  
list
```

```
[a, b, c, d, e]
```

## Sets

### Create Sets

Sets represent unordered mathematical structures. Sets can contain arbitrary MuPAD objects. For example, numbers, symbols, strings, or functions can be elements of a set. To create a set, separate MuPAD objects with commas and enclose the structure in braces:

```
set1 := {a, 3, b, c, d, 180, -15}
```

```
{-15, 3, 180, a, b, c, d}
```

Also, you can create a sequence, and convert it to a set. To convert a sequence to a set, enclose the sequence in braces. As a shortcut for creating a sequence, use the sequence generator \$ or its functional form `_seqgen`. Enclose the sequence in braces:

```
{i*(i-1) $ i = 1..10}
```

```
{0, 2, 6, 12, 20, 30, 42, 56, 72, 90}
```

The order of the elements in a set does not depend on the order in which you insert them. If an order of elements is important, use a list instead of a set:

```
[a, 3, b, c, d, 180, -15]
```

```
[a, 3, b, c, d, 180, -15]
```

MuPAD does not necessarily sort the elements of a set alphabetically:

```
set2 := {cos, tan, sin}
```

```
{tan, sin, cos}
```

A set cannot contain duplicate elements. When creating a set, MuPAD automatically removes duplicates:

```
set3 := {2, 6, 7, a, 6, 2, 2, a, b}
```

```
{2, 6, 7, a, b}
```

A set can be empty:

```
empty_set := {}
```

```
∅
```

## Access Set Elements

The position of an element of a set in an output region can differ from the internal position of the element in a set. To access an element in a set by its internal position, use the `op` command:

```
set2 := {[c,a,b], [b,c,a], [a,b,c]};  
op(set2, 1), op(set2, 2), op(set2, 3)
```

```
{[a, b, c], [b, c, a], [c, a, b]}
```

```
[c, a, b], [b, c, a], [a, b, c]
```

When using a notebook interactively, you can access an element of a set by its position in an output region. To access an element by the position as you see it on screen, use brackets or `_index`:

```
set2 := {[c,a,b], [b,c,a], [a,b,c]}:  
set2[1];  
_index(set2, 3)
```

```
[a, b, c]
```

```
[c, a, b]
```

You can access particular solutions from a set returned by the `solve` command. To use the order of elements of a set as they appear on screen, use brackets or `_index`:

```
solutions := solve(x^4 = 1, x);  
solutions[3];  
_index(solutions, 2..4)
```

```
{-1, 1, -i, i}
```

```
-i
```

```
{1, -i i}
```

If you use an indexed assignment without creating a set, MuPAD generates a table instead of a set:

```
set[1] := x: set
```

```
 $\overline{1|x}$ 
```

### Operate on Sets

MuPAD sets support the following operations:

- Defining an object as a member of a set
- Verifying that a set contains a particular object
- Using a set as a function in a function call
- Applying a function to all elements of a set
- Extracting entries of a set
- Computing the intersection of sets
- Dividing a set according to particular properties of its elements

To define an object as a member of a set, use the `in` command:

```
x in {1, 2, 3, a, d, 5}
```

```
 $x \in \{1, 2, 3, 5, a, d\}$ 
```

To check if an object belongs to a set, use the `contains` command:

```
set := {a, 3, b, c, d, 180, -15}:
contains(set, d);
contains(set, e);
```

```
TRUE
```

## FALSE

If you use a set as the function in a function call, MuPAD returns the set of appropriate function calls:

```
{sin, cos, tan, f}(x);  
{sin, cos, tan, f}(0.1)
```

```
{cos(x), f(x), sin(x), tan(x)}
```

```
{0.09983341665, 0.1003346721, 0.9950041653, f(0.1)}
```

To apply a function to all elements of a set, use the function map:

```
map({x, 0.1, 1/5, PI}, sin)
```

```
{0, 0.09983341665, sin( $\frac{1}{5}$ ), sin(x)}
```

To extract elements with particular properties from a set, use the `select` command:

```
select({{a, x, b}, {a}, {x, 1}}, contains, x)
```

```
{{1, x}, {a, b, x}}
```

To find the intersection of sets, use the `intersect` command:

```
S := {1,2,3};  
S intersect {2,3,4};
```

```
{2, 3}
```

To divide a set into three sets according to particular properties, use the `split` command:

```
split({{a, x, b}, {a}, {x, 1}}, contains, x)
```

```
[{{1, x}, {a, b, x}}, {{a}}, ∅]
```

The resulting three sets contain:

- Elements with the required properties
- Elements without the required properties
- Elements for which the required properties are unknown.

### Add, Replace, or Remove Set Elements

To add elements to a set:

- 1 Create a set containing the elements you want to add.
- 2 Combine the old and the new sets using the union command.

```
set := {a, b, c};
set := set union {d, e, f}
```

```
{a, b, c, d, e, f}
```

To replace an element of a set, use the `subs` command. The new element does not necessarily appear in place of the old one:

```
set4 := {a, b, 2, 6, 7};
subs(set4, a = 1)
```

```
{2, 6, 7, a, b}
```

```
{1, 2, 6, 7, b}
```

---

**Note** When you replace and delete elements of a set, the order of its elements can change even if you delete or replace the last element.

---

When replacing or deleting an element, always check that you access the element at the correct position:

```
set4;  
op(set4, 4)
```

```
{2, 6, 7, a, b}
```

```
6
```

The `subs` command does not modify the original set:

```
set4 := {a, b, 2, 6, 7};  
subs(set4, a = 1);  
set4
```

```
{1, 2, 6, 7, b}
```

```
{2, 6, 7, a, b}
```

To delete elements from a set, use the `minus` command. You can simultaneously delete several elements of a set:

```
set5 := {1, 2, 6, 7, b};  
set5 minus {1, b}
```

```
{2, 6, 7}
```

## Tables

### Create Tables

Tables associate arbitrary indices with arbitrary values. For example, you can use tables to represent collections of equations in the form `index = value`. To generate a table, use the `table` command:

```
T := table(a = b, c = d)
```



a	b
c	d

You can create tables from equations, existing tables, lists, or sets of equations:

```
table(s = t, T, [x = 6], {y = 13})
```

a	b
c	d
s	t
x	6
y	13

MuPAD inserts index-value pairs in a table in the same order as you enter them. Each new entry can override previous entries. The order of output does not reflect the order of input:

```
T1 := table([5 = a, 12 = c]):
T2 := table([a = 5, c = 12]):
T3 := table(5 = b, T1, T2, [a = 6], {c = 13})
```

5	a
12	c
a	6
c	13

### Access Table Elements

To access an entry of a table, use brackets or `_index`:

```
T := table(a = 11, c = 12):
T[a];
_index(T, c)
```

11

12

To access a value entry of a table by its index, also use brackets or `_index`:

```
T := table(a = 11, c = 12):
T[c]
```

12

If an index does not exist, you get:

```
T[b];
table(a = 11, c = 12)[b]
```

$T_b$

$a$	11
$c$	12

Before accessing a value entry of a table by its index, check that the index is available for the table:

```
contains(T, b);
contains(T, a);
T[a]
```

FALSE

TRUE

11

### Operate on Tables

MuPAD tables support the following operations:

- Extracting contents of a table as a collection of equations
- Listing indices and values separately

- Verifying that a table contains a particular object
- Searching for an object among the indices and the values a table

To extract the contents of a table as a collection of equations, use the `op` command:

```
op(T)
```

```
a = 11, c = 12
```

You can list indices and values of a table separately:

```
leftSide := lhs(T);  
rightSide := rhs(T)
```

```
[a, c]
```

```
[11, 12]
```

To check if an object belongs to the indices of a table, use the `contains` command:

```
T := table(a = 11, c = 12):  
contains(T, a);  
contains(T, 11)
```

```
TRUE
```

```
FALSE
```

If you want to search for an object among the indices and the values of a table, use the `has` command:

```
T := table(a = 11, c = 12):  
has(T, 11);  
has(T, c);
```

```
has(T, x)
```

```
TRUE
```

```
TRUE
```

```
FALSE
```

### Replace or Remove Table Entries

To replace an entry of a table, access the entry by its index, and assign the new value to the entry:

```
T := table(a = 11, c = 12):  
T[a] := 5:  
T
```

```
 a | 5  
---  
 c | 12
```

To remove an entry from a table, use the `delete` command:

```
delete(T[a]):  
T;
```

```
 c | 12
```

## Arrays

### Create Arrays

Arrays represent multidimensional data structures. You can use only integers for array indices. To generate an array, use the `array` command:

```
A := array(0..2, 0..3);  
B := array(0..2, 0..3, 0..4)
```

$$\begin{pmatrix} \text{NIL} & \text{NIL} & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & \text{NIL} \end{pmatrix}$$

```
array(0..2, 0..3, 0..4)
```

For two-dimensional arrays, the first range defines the number of rows, the second range defines the number of columns. Ranges for array indices do not necessarily start with 0 or 1:

```
A := array(3..5, 1..2)
```

$$\begin{pmatrix} \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} \end{pmatrix}$$

### Access Array Entries

To access an entry of an array, use brackets or `_index`:

```
A := array(0..1, 0..2, [[1, 2, 3], [a, b, c]]);
A[0, 2];
_index(A, 1, 1)
```

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

3

b

Create an array of random numbers:

```
A := array(1..3, 1..4, [frandom() $ i = 1..12]);
domtype(A)
```

$$\begin{pmatrix} 0.2703581656 & 0.8310371787 & 0.153156516 & 0.9948127808 \\ 0.2662729021 & 0.1801642277 & 0.452083055 & 0.6787819563 \\ 0.3549849261 & 0.6818588132 & 0.7219186551 & 0.4738297742 \end{pmatrix}$$

DOM\_ARRAY

To access a range of array entries, first convert the array to a matrix:

```
B := matrix(A);  
domtype(B)
```

$$\begin{pmatrix} 0.2703581656 & 0.8310371787 & 0.153156516 & 0.9948127808 \\ 0.2662729021 & 0.1801642277 & 0.452083055 & 0.6787819563 \\ 0.3549849261 & 0.6818588132 & 0.7219186551 & 0.4738297742 \end{pmatrix}$$

Dom::Matrix()

Use the ranges of indices to access the entries of the matrix. The result is also a matrix.

```
b := B[1..2,1..3];  
domtype(b)
```

$$\begin{pmatrix} 0.2703581656 & 0.8310371787 & 0.153156516 \\ 0.2662729021 & 0.1801642277 & 0.452083055 \end{pmatrix}$$

Dom::Matrix()

To convert matrix b to an array, use the coerce function:

```
a := coerce(b, DOM_ARRAY);  
domtype(a)
```

$$\begin{pmatrix} 0.2703581656 & 0.8310371787 & 0.153156516 \\ 0.2662729021 & 0.1801642277 & 0.452083055 \end{pmatrix}$$

## DOM\_ARRAY

Alternatively, you can index into an array by using this command:

```
array(1..2, 1..3,[A[i, j] $ j = 1..3 $ i = 1..2])
( 0.2703581656 0.8310371787 0.153156516
  0.2662729021 0.1801642277 0.452083055 )
```

For further computations, delete A, B, b, and a:

```
delete A, B, b, a
```

## Operate on Arrays

MuPAD arrays support the following operations:

- Assigning values to the entries of arrays
- Comparing arrays

MuPAD does not support arithmetical operations on arrays.

You can assign values to the entries of an array:

```
A := array(0..1, 0..2):
A[0, 0] := 1: A[0, 1] := 2: A[0, 2] := 3:
A[1, 0] := a: A[1, 1] := b: A[1, 2] := c:
A
```

```
( 1 2 3
  a b c )
```

You also can provide the values of the entries while creating an array:

```
A := array(0..1, 0..2, [[1, 2, 3], [a, b, c]]);
B := array(1..2, 1..3, 1..5, [[[[i, j, k] $ k=1..5 ]
                               $ j=1..3] $ i=1..2]):
B[2,3,4]
```

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$
$$[2, 3, 4]$$

MuPAD accepts nested and flat lists as array entries:

```
array([[1, 2, 3], [a, b, c]]);  
array(1..2, 1..3, [1, 2, 3, a, b, c]);
```

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

When comparing arrays, MuPAD compares both indices and values of the entries. By default, indices start with 1:

```
A1 := array([[1, 2, 3], [a, b, c]]):  
A2 := array(0..1, 0..2, [1, 2, 3, a, b, c]):  
A3 := array(1..2, 1..3, [1, 2, 3, a, b, c]):  
bool(A1 = A2);  
bool(A1 = A3)
```

FALSE

TRUE

You cannot use arithmetical operations on arrays:

```
A1 + A2
```

```
Error: The operand is invalid. [_plus]
```



To use arithmetical operations, convert arrays to matrices. For numeric data, you also can use Arrays with Hardware Floating-Point Numbers.

### Replace or Remove Array Entries

To replace an entry of an array, access the entry by its index, and assign the new value to the entry:

```
A := array(0..1, 0..2, [[1, 2, 3], [a, b, c]]):
A[0, 2] := NewValue:
A
```

$$\begin{pmatrix} 1 & 2 & \text{NewValue} \\ a & b & c \end{pmatrix}$$

To remove entries from an array, use the `delete` command. When you remove an entry of an array, the dimensions of an array do not change. MuPAD changes the entry value you remove to `NIL`:

```
A := array(0..1, 0..2, [[1, 2, 3], [a, b, c]]):
delete(A[0, 2]):
A
```

$$\begin{pmatrix} 1 & 2 & \text{NIL} \\ a & b & c \end{pmatrix}$$

### Arrays with Hardware Floating-Point Numbers

To create an array of hardware floating-point numbers, use the `hfarray` command. An array can contain complex floating-point numbers:

```
A := hfarray(0..1, 0..2, [[1, 2/3, I/3], [I, exp(1), PI]])
```

$$\begin{pmatrix} 1.0 & 0.6666666667 & 0.3333333333 i \\ 1.0 i & 2.718281828 & 3.141592654 \end{pmatrix}$$

Arrays of hardware floating-point numbers use less memory than regular arrays and matrices. You can use basic arithmetical operations on these arrays:

```
A + 2*A
```

$$\begin{pmatrix} 3.0 & 2.0 & 1.0i \\ 3.0i & 8.154845485 & 9.424777961 \end{pmatrix}$$

## Vectors and Matrices

### Create Matrices

The simplest way to create a matrix in MuPAD is to use the `matrix` command:

```
A := matrix([[a, b, c], [1, 2, 3]])
```

$$\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \end{pmatrix}$$

If you declare matrix dimensions and enter rows or columns shorter than the declared dimensions, MuPAD pads the matrix with zero elements:

```
A := matrix(2, 4, [[a, b, c], [1, 2, 3]])
```

$$\begin{pmatrix} a & b & c & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

If you declare matrix dimensions and enter rows or columns longer than the declared dimensions, MuPAD returns the following error message:

```
A := matrix(2, 1, [[a, b, c], [1, 2, 3]])
```

```
Error: The number of columns does not match. [(Dom::Matrix(Dom::Expres
```

As a shortcut for providing elements of a matrix, you can use the `->` command:

```
A := matrix(5, 5, (i, j) -> i*j)
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 16 & 20 \\ 5 & 10 & 15 & 20 & 25 \end{pmatrix}$$

### Create Vectors

To create a vector, also use the `matrix` command. The command `matrix([[x], [y], [z]])` creates a column vector. As a shortcut for creating a column vector, use:

```
a := matrix([x, y, z])
```

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

To create a row vector, declare the vector dimensions or use double brackets:

```
b1 := matrix(1, 3, [x, y, z]);
b2 := matrix([[x, y, z]])
```

$$(x \ y \ z)$$

$$(x \ y \ z)$$

### Combine Vectors into a Matrix

To create a matrix, you also can combine vectors by using the concatenation operator `(.)`:

```
v := matrix([1,2,3]);
w := matrix([4,5,6]);
A := v.w;
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

## Matrices Versus Arrays

Matrices and arrays are different data types:

Matrices	Arrays
Data containers with defined standard mathematical operations	Data containers for storing only
Slow access to data	Fast access to data
One- or two-dimensional	Multidimensional

## Convert Matrices and Arrays

To create a matrix from an array, use the `matrix` command:

```
A := array([[1, 2, 3], [x, y, z]]):  
B := matrix(A):  
type(A);  
type(B)
```

`DOM_ARRAY`

`Dom::Matrix()`

To convert a matrix into an array, use the `expr` command:

```
C := expr(B):  
type(C)
```

## DOM\_ARRAY

To convert a matrix or an array to a sequence, use the op command:

```
op(B);  
op(C)
```

```
1, 2, 3, x, y, z
```

```
1, 2, 3, x, y, z
```

To convert a matrix or an array to a list or a set:

- 1** Convert a matrix or an array to a sequence using the op command.
- 2** Create a list or a set from the sequence.

## Use the MuPAD Libraries

### In this section...

“Overview of Libraries” on page 1-148

“Standard Library” on page 1-150

“Find Information About a Library” on page 1-150

“Avoid Name Conflicts Between MuPAD Objects and Library Functions” on page 1-152

### Overview of Libraries

Libraries contain most of the MuPAD functionality. Each library includes a collection of functions for solving particular types of mathematical problems:

<code>combinat</code>	supports combinatorics operations
<code>solvelib</code>	contains various methods used by the function <code>solve</code>
<code>export</code>	supports exporting MuPAD data to external formats
<code>output</code>	supports formatted output of the MuPAD data
<code>fp</code>	supports functional programming methods
<code>generate</code>	supports conversion of the MuPAD expressions to C, FORTRAN, MATLAB®, and TeX codes
<code>groebner</code>	supports operating on ideals of multivariate polynomial rings over a field
<code>import</code>	supports importing external data to MuPAD
<code>transform</code>	supports integral transformations and the discrete Z-transform

<code>intlib</code>	supports manipulating and solving integrals
<code>linalg</code>	supports linear algebra operations
<code>linopt</code>	provides algorithms for linear and integer programming
<code>listlib</code>	supports manipulating lists
<code>polylib</code>	supports manipulating polynomials
<code>stringlib</code>	supports manipulating strings
<code>numlib</code>	supports number theory operations
<code>numeric</code>	provides algorithms for numeric mathematics
<code>ode</code>	supports manipulating and solving ordinary differential equations
<code>orthpoly</code>	provides a set of standard orthogonal polynomials
<code>Pref</code>	supports setting and restoring user preferences
<code>prog</code>	provides programming utilities for analyzing functions and tracing errors
<code>stats</code>	provides methods for statistical analysis
<code>Type</code>	supports checking types of MuPAD objects
<code>Symbol</code>	supports typesetting symbols

Functions included in libraries are written in the MuPAD language. The calling syntax for functions from all the libraries (except for the standard library) includes both the library name and the function name: `library::function`.

## Standard Library

The standard library presents the set of most frequently used functions including `diff`, `int`, `simplify`, `solve`, and other functions. For example:

```
diff(x^2,x)
```

```
2x
```

## Find Information About a Library

You can get information about the libraries using the `info` and `help` commands. The `info` command gives a list of functions of a particular library. For example, the `numlib` library presents a collection of functions for number theory operations:

```
info(numlib)
```

```
Library 'numlib': the package for elementary number theory
```

```
-- Interface:
```

```
numlib::Lambda,          numlib::Omega,          numlib::checkPrime  
numlib::contfrac,       numlib::contfracPeriodic, numlib::cornacchia  
numlib::decimal,       numlib::divisors,       numlib::factorGaus  
numlib::fibonacci,     numlib::fromAscii,      numlib::g_adic,  
numlib::ichrem,        numlib::igcdmult,       numlib::invphi,  
numlib::ispower,       numlib::isquadres,      numlib::issqr,  
numlib::jacobi,        numlib::lambda,         numlib::legendre,
```



```

numlib::lincongruence,  numlib::mersenne,      numlib::moebius,
numlib::mroots,        numlib::msqrts,      numlib::numdivisor
numlib::numprimedivisors, numlib::omega,      numlib::order,
numlib::phi,           numlib::pi,          numlib::primedivis
numlib::primroot,     numlib::proveprime,  numlib::reconstruc
numlib::sigma,        numlib::sqrt2cfrac,  numlib::sqrtmodp,
numlib::sumOfDigits,  numlib::sumdivisors, numlib::tau,
numlib::toAscii,

```

To see brief information about a particular library function, use the mouse pointer to hover the cursor over the function name.

For more information about the library and for information about the library functions, enter:

```
?numlib
```

To see the implementation of a library function, use the `expose` command:

```
expose(numlib::tau)
```

```

proc(a)
  name numlib::tau;
begin
  if args(0) <> 1 then
    error(message("symbolic:numlib:IncorrectNumberOfArguments"))
  else
    if not testtype(a, Type::Numeric) then
      return(procname(args()))
    else
      if domtype(a) <> DOM_INT then
        error(message("symbolic:numlib:ArgumentInteger"))
      end if
    end if
  end if
end

```

```
        end_if
    end_if
end_if;
numlib::numdivisors(a)
end_proc
```

## Avoid Name Conflicts Between MuPAD Objects and Library Functions

You can call any library function (except for the standard library functions) using the following syntax: `library::function`. If you frequently use some functions that do not belong to the standard library, it is possible to call them without specifying the library name. The `use` command exports functions of the MuPAD libraries to the global namespace allowing you to call them without using the library names. For example, you can export the function that computes the decimal expansion of a rational number:

```
use(numlib,decimal): decimal(1/3)
```

```
0, [3]
```

After exporting the decimal function, you can use it without using the library name `numlib`:

```
decimal(1/200)
```

```
0, 0, 0, 5
```

To call the `info`, `help`, or `?` commands, use the full name of an exported function including the name of a library:

```
?numlib::decimal
```

You cannot export a library function with the same name you use for another object:

```
hilbert := x: use(linalg, hilbert)
```

Warning: Identifier 'hilbert' already has a value. It is not exported

After clearing the variable `decimal`, you can export a function:

```
delete hilbert: use(linalg, hilbert): hilbert(3)
```

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

You also can export several functions from the same library simultaneously. For example, you can export the functions for finding the sum of digits and the set of positive divisors of an integer:

```
use(numlib, sumOfDigits, divisors): divisors(21); sumOfDigits(21)
```

```
[1, 3, 7, 21]
```

```
3
```

To export all functions of a library, pass the library name to the `use` command. If some of the library functions have name conflicts with other objects, the `use` command issues a warning for each name conflict:

```
use(numeric)
```

Warning: Identifier 'product' already has a value. It is not exported

Warning: Identifier 'int' already has a value. It is not exported. [u

Warning: Identifier 'indets' already has a value. It is not exported.

Warning: Identifier 'det' already has a value. It is not exported. [u

Warning: Identifier 'linsolve' already has a value. It is not exported

Warning: Identifier 'rationalize' already has a value. It is not expo

Warning: Identifier 'inverse' already has a value. It is not exported

Warning: Identifier 'solve' already has a value. It is not exported.

Warning: Identifier 'sum' already has a value. It is not exported. [u

Warning: Identifier 'sort' already has a value. It is not exported. [

These library functions have the same names as the standard library functions. You cannot delete standard library functions and resolve the name conflicts. Use the full function names such as `numeric::product` to call these functions.

# Programming Basics

## In this section...

“Conditional Control” on page 1-155

“Loops” on page 1-161

“Procedures” on page 1-171

“Functions” on page 1-179

“Shortcut for Closing Statements” on page 1-181

## Conditional Control

### Use if Statements

You can execute different groups of statements depending on particular conditions. Use `if` to define a condition, and use `then` to define the group of statements you want to execute when the condition is true:

```
x := -3:
if x < 0 then
  y := x + 2;
  x := -x;
  print(x, y)
end_if:
```

3, -1

You also can define the group of statements that you want to execute when the condition is false:

```
x := 3:
if x < 0 then
  y := x + 2;
  x := -x;
  print(x, y)
else
  y := x + 2;
```

```
x := x;  
print(x, y)  
end_if
```

3, 5

MuPAD does not require parentheses around conditions:

```
x := 10:  
if testtype(x, Type::Positive) = TRUE and  
   type(x) = DOM_INT then  
   print(Unquoted, "x = ".x." is a positive integer");  
end_if
```

x = 10 is a positive integer

### **Apply Multiple Conditions**

You can use multiple conditions in conditional statements. Combine multiple conditions by the logical operators:

```
x := 5:  
y := 6:  
if x > 0 and 1 - y > 0 then  
   print(Unquoted, "the condition is true")  
else  
   print(Unquoted, "the condition is false")  
end_if:
```

the condition is false

```
x := 5:  
y := 6:  
if x > 0 or 1 - y > 0 then  
   print(Unquoted, "the condition is true")
```

```
else
  print(Unquoted, "the condition is false")
end_if:
```

```
the condition is true
```

## Use Nested Conditional Statements

MuPAD supports the use of nested conditional statements. For example:

```
x := 5:
if testtype(x, DOM_COMPLEX) = TRUE then
  print("The Heaviside function is undefined for complex numbers")
else
  if x = 0 then
    heavisideX := 1/2
  else
    if x < 0 then
      heavisideX := 0
    else
      heavisideX := 1
    end_if:
  end_if;
end_if
```

```
1
```

For nested if ... else if, use the elif command as a shortcut:

```
x := 0:
if (testtype(x, DOM_COMPLEX) = TRUE) then
  print("The Heaviside function is undefined for complex numbers")
elif x = 0 then
  heavisideX := 1/2
elif x < 0 then
  heavisideX := 0
else
```

```
    heavisideX := 1;  
end_if
```

$\frac{1}{2}$

## Use case and otherwise Statements

To choose between several cases, use the `case` command. The `case` statement looks for the first valid condition. After that, it executes all the statements between this condition and the keyword `end_if`, without checking the conditions:

```
x := 4:  
case x  
  of 1 do  
  of 2 do  
  of 3 do print("three or less")  
  of 4 do print("four")  
  of 5 do print("five")  
  otherwise print("6 or more")  
end_case:
```

"four"

"five"

"6 or more"

To exit the `case` statement after executing the statements written under the first valid condition, use the `break` command. See [Exiting a Conditional Statement](#) for more details.



---

**Note** MuPAD executes the `case` statements differently from MATLAB. MATLAB executes only the first matching `case` statement and skips the following `case` statements. In MuPAD, you *must* use the `break` command to stop execution of the following statements.

---

### Exit a Conditional Statement

To exit a conditional statement after executing the statements written under the first valid condition, use the `break` command. For example, select the meaning of the traffic light signals:

```
trafficLight := yellow:
case trafficLight
  of red do print(Stop); break;
  of yellow do print(Caution); break;
  of green do print(Go); break;
end_case
```

Caution

### Return Value of a Conditional Statement

All MuPAD commands produce some return values. Conditional statements return the result of the last executed command:

```
mapColor := blue:
if mapColor = blue then
  "water"
else
  "land"
end_if
```

"water"

Use the return value of a conditional statement like any other value. For example, assign the return value to a variable. By default, MuPAD does not allow conditional statements in assignments. To create a valid assignment,

enclose conditional statements in parentheses. Suppress the output of the return value of a conditional statement with a colon:

```
mapColor := blue:
terrain := (if mapColor = blue then "water" else "land" end_if):
```

Write a sentence by concatenating the following string and the variable terrain:

```
print(Unquoted, "Blue color on maps usually shows ".terrain)
```

```
Blue color on maps usually shows water
```

The following case statement also returns the result of the last assignment:

```
palette := color:
case palette
  of monochrome do
    [color1, color2] := [black, white];
    break;
  of color do
    [color1, color2, color3] := [red, green, blue];
    break;
end_case
```

```
[red, green, blue]
```

### **Display Intermediate Results**

By default, MuPAD does not display intermediate results obtained inside a conditional statement even if you use semicolons after statements. To see intermediate results, use the print command inside a conditional statement:

```
Omega := 2:
if Omega > PI/2 and Omega < PI then
  signSinOmega := 1;
  signCosOmega := -1;
```

```
print(signSinOmega, signCosOmega)
end_if:
```

```
1, -1
```

## Loops

### Use Loops with a Fixed Number of Iterations (for Loops)

The for loop repeats a group of statements for a fixed number of iterations. The loop starts with the command `for` and ends with `end_for` or just `end`. MuPAD executes all statements between these two words as many times as you specify. For example, compute the factorial of an integer using the loop:

```
x := 1:
for i from 1 to 5 do
  x := x * i;
end_for
```

```
120
```

**More Efficient Alternative to for Loops.** You often can speed up the execution of MuPAD code by replacing for loops with the sequence generator `$`. For example, instead of the loop

```
x := 1:
for i from 1 to 10 do
  x := x * i;
end_for
```

```
3628800
```

use the statement:

```
`*`(i $ i = 1..10)
```

```
3628800
```

**Control Step Size and Count Up and Down.** By default, the loop increments the value of a counter by 1 in each step. You can change the step size:

```
for i from 1 to 3 step 1/2 do
  print(i)
end_for
```

1

$\frac{3}{2}$

2

$\frac{5}{2}$

3

To count backwards decreasing the value of a counter with each step, instead of to, use the keyword `downto`:

```
for i from 3 downto 1 do
  print(i)
end_for
```

3

2

1

**Use Mathematical Structures in for Loops.** MuPAD supports the use of structures such as lists and matrices in for loops:

```
for i in [1, -4, a, b] do
  print(i^2)
end_for
```

1

16

$a^2$

$b^2$

```
s := 0;
for i in linalg::hilbert(3) do
  s := s + i^2
end_for
```

$\frac{1199}{600}$

## Use Loops with Conditions (while and repeat Loops)

**Condition at the Beginning (while Loops).** MuPAD supports the use of loops with logical conditions. The `while` loop continues while the execution conditions are valid. The loop starts with `while` and ends with `end_while` or just `end`. MuPAD executes all the statements between these commands repeatedly as long as the execution conditions are true. In the `while` loop, MuPAD evaluates the conditions before each iteration. When the condition becomes false, the loop terminates without executing the statements of the current iteration:

```
i := 2;
while i < 16 do
  i := i^2;
  print(i)
end_while:
```

2

4

16

**Termination Condition at the End (repeat Loops).** The repeat loop continues until the termination condition becomes valid. The loop starts with `repeat` and ends with `end_repeat`. MuPAD executes all the statements between these commands repeatedly while the conditions are false. The `repeat` loop tests a termination condition at the end of each iteration. When the condition becomes true, the loop terminates after executing the statements of the current iteration:

```
i := 2;
repeat
  i := i^2;
  print(i)
until i >= 16 end_repeat:
```

2

4

16

**Avoid Infinite Loops: Set a Counter.** The `while` and `repeat` loops do not operate for a fixed number of steps. If the execution or termination conditions of such loops never become true, MuPAD can execute the statements within these loops infinitely. For example, if the termination condition is not valid during the first iteration, and it does not change inside the loop, the loop is infinite:

```
i := 1;
repeat
  print(i)
```

```
until i > 3 end_repeat:
```

To avoid this infinite loop, use the additional statement to change it in each iteration:

```
i := 1;
repeat
  i := i + 1;
  print(i)
until i > 3 end_repeat:
```

1

2

3

4

**Use Multiple Conditions.** You can use multiple conditions combining the expressions by and, or, xor, or other logical operators:

```
i := 2;
j := 3;
repeat
  i := i*j;
  j := j^2;
  print(i, j)
until i > 100 and j > 10 end_repeat:
```

6, 9

54, 81

4374, 6561

```
i := 2:
j := 3:
repeat
  i := i*j;
  j := j^2;
  print(i, j)
until i > 100 or j > 10 end_repeat:
```

6, 9

54, 81

### Use Nested Loops

You can place one or several loops inside another loop. Internal loops can be of the same or different types:

```
s := 0:
for i from 1 to 3 do
  for j from 1 to 2 do
    s := s + i + j;
  end_for
end_for:
print(s)
```

21

```
s := 0:
for i from 1 to 3 do
  j := 1:
  while j <= 2 do
    s := s + i + j;
    j := j + 1;
  end_while
end_for:
print(s)
```

21



## Exit a Loop

To add a possibility to exit a loop, use the `break` command. Suppose you want to exit a loop if some condition is true:

```
for i from 1 to 3 do
  for j from 1 to 2 do
    if i = j then
      print(Unquoted, "i = j = ".expr2text(i));
      break
    end_if
  end_for
end_for:
```

```
i = j = 1
```

```
i = j = 2
```

The `break` command lets you exit the loop in which you place this command. If you create nested loops and use `break` inside an inner loop, MuPAD continues to execute the statements in the outer loops:

```
for i from 1 to 3 do
  for j from 1 to 2 do
    if i = j then
      print(Unquoted, "break with i = j = ".i);
      break
    end_if;
    print(Unquoted, "i = ".i.", j = ".j);
  end_for
end_for:
```

```
break with i = j = 1
```

```
i = 2, j = 1
```

```
break with i = j = 2
```

```
i = 3, j = 1
```

```
i = 3, j = 2
```

Suppose you want to stop executing the statements and exit the nested loops as soon as the condition `i = j` is true. Use the additional variable for the break state of the inner loop. Use this variable to exit the outer loop:

```
breakAll := FALSE:
for i from 1 to 3 do
  for j from 1 to 2 do
    if i = j then
      print(Unquoted, "break with i = j = ".i);
      breakAll := TRUE;
      break
    end_if;
    print(Unquoted, "i = ".i.", j = ".j);
  end_for;
  if breakAll then
    break
  end_if;
end_for:
```

```
break with i = j = 1
```

## Skip Part of Iteration

To skip the commands from a particular point to the end of a loop and start the next iteration, use the `next` command:

```
for i from 1 to 3 do
  for j from 1 to 2 do
    if i = j then
      print(Unquoted, "i = j = ".expr2text(i));
      next
    end_if;
    print(Unquoted, "i = ".expr2text(i), "j = ".expr2text(j))
  end_for
end_for:
```

```
i = j = 1
```

```
i = 1, j = 2
```

```
i = 2, j = 1
```

```
i = j = 2
```

```
i = 3, j = 1
```

```
i = 3, j = 2
```

## Return Value of a Loop

All MuPAD commands produce some return values. Loops in MuPAD return the result of the last executed statement:

```
for i from 1 to 3 do
  x := 2*i;
  y := 3*i
end_for
```

9

Suppress the output of the return value with a colon:

```
for i from 1 to 3 do
  x := 2*i;
  y := 3*i
end_for:
```

## Display Intermediate Results

By default, MuPAD does not display intermediate results obtained inside a loop even if you use semicolons after commands. To see intermediate results, use the `print` command inside a loop:

```
i := 1;
while i < 3 do
  i := i + 1;
  print(i)
end_while:
```

1

2

3

To display results of each iteration in a `for` loop, also use the `print` command:

```
for i from 1 to 5 do
  x := i!;
  print(Unquoted, expr2text(i)."! = ".expr2text(x))
end_for
```

1! = 1

2! = 2

3! = 6

4! = 24

5! = 120

## Procedures

### Create a Procedure

If you want to execute a piece of code repeatedly, create and use a procedure. Define a procedure with the `proc` command. Enclose your code in the `begin` and `end_proc` commands:

```
myProc:= proc(n)
begin
  if n = 1 or n = 0 then
    1
  else
    n * myProc(n - 1)
  end_if;
```

```
end_proc:
```

Use `end` as a shortcut for `end_proc`.

### **Call a Procedure**

Now call the procedure:

```
myProc(5)
```

```
120
```

### **Control Return Values**

By default, a procedure returns the result of the last executed command. If you want to return other results, use the `return` command. For example, create a procedure that computes the factorials of integer numbers:

```
myProcReturn := proc(n)
begin
  if n = 0 or n = 1 then
    return(1);
  end_if;
  n * myProcReturn(n - 1);
end_proc:
```

Call the procedure:

```
myProcReturn(5)
```

```
120
```

To display the results on your screen without returning them, use the `print` command:

```
myProcPrint:= proc(n)
begin
  print(n);
  if n = 0 or n = 1 then
```

```
        return(1);
    end_if;
    n * myProcPrint(n - 1);
end_proc:
```

Call the procedure:

```
myProcPrint(5);
```

5

4

3

2

1

120

### **Return Multiple Results**

To return several results from a procedure, use such structures as lists or matrices as return values:

```
myProcSort:= proc(a, b)
begin
    if a < b then
        [a, b]
    else
        [b, a]
    end_if;
end_proc:
```

```
myProcSort(4/5, 5/7)
```

$$\begin{bmatrix} 5 & 4 \\ 7 & 5 \end{bmatrix}$$

## Return Symbolic Calls

Many built-in MuPAD procedures can return symbolic calls to themselves when they cannot compute results as exact values. For example, when you compute  $\sin(\text{PI}/2)$ , the `sin` function returns the exact value 1. At the same time, when you compute  $\sin(x/2)$ , the `sin` function returns the symbolic call to itself:

```
sin(x/2)
```

$$\sin\left(\frac{x}{2}\right)$$

To enable your custom procedure to return symbolic calls, use the special syntax `procname(args())`. For example, create the procedure that computes a factorial of its argument:

```
f := proc(x)
begin
  if testtype(x, Type::PosInt) then
    return(x!)
  else
    return(procname(args()))
  end_if:
end_proc:
```

If its argument is a positive integer, this procedure returns an exact number:

```
f(5), f(10)
```

```
120, 3628800
```

Otherwise, it does not error, but returns a symbolic call to itself:

```
f(1/3), f(1.1), f(x), f(x + 1)
```



$f\left(\frac{1}{3}\right), f(1.1), f(x), f(x+1)$ 

## Use Global and Local Variables

Inside a procedure, all variables fall into two categories: global and local. Global variables are accessible from everywhere inside a notebook. Local variables are accessible only from within a procedure.

**Global Variables.** Suppose you want to create a procedure `gProc` and use the global variable `gVar` inside the procedure:

```
gProc := proc(x)
begin
  gVar := gVar^2 + x^2 + 1
end_proc:
```

When you call this procedure, the value of the variable `gVar` changes:

```
gVar := 10;
gProc(5);
gVar
```

10

126

Multiple calls change the value of the global variable further:

```
gProc(5);
gVar
```

15902

---

**Note** Avoid using unnecessary global variables.

---

Global variables reduce code readability, occupy the global namespace, and often lead to errors. When you use global variables inside a procedure, always verify that each call to the procedure changes global variables as intended.

**Local Variables.** You can access and modify local variables only inside a procedure. Suppose, you use a variable `lVar` in your notebook:

```
lVar := 10
```

```
10
```

To declare a local variable, use the `local` command inside a procedure:

```
lProc := proc(x)
  local lVar;
begin
  lVar := 10;
  lVar := lVar^2 + x^2 + 1
end_proc:
```

When you call this procedure, the value of the variable `lVar` changes only inside a procedure. Outside the procedure, the variable does not change its value:

```
lProc(5):
lVar
```

```
10
```

If you declare a local variable, it does not inherit the value of a global variable with the same name. Local variables are not identifiers of type `DOM_IDENT`. They belong to a special domain type `DOM_VAR`. Therefore, you cannot use a local variable as a symbolic variable. Before performing computations with a local variable, you must assign a value to that variable. For example, without the assignment `lVar:= 10`, the procedure call `lProc` returns an error message:

```
lProc := proc(x)
local lVar;
```

```
begin
  lVar := lVar^2 + x^2 + 1
end_proc:

lProc(5)
```

```
Warning: Uninitialized variable 'lVar' is used.
Evaluating: lProc
```

```
Error: The operand is invalid. [_power]
Evaluating: lProc
```

Local variables cannot have assumptions.

### **Restore Values and Properties of Global Variables Modified in Procedures**

When you use global variables inside a procedure, you can save their original values and properties, and recover them after the procedure. Suppose, you want to use more decimal digits for calculations with floating-point numbers inside a procedure. By default, the number of digits is 10:

**DIGITS**

10

To save this default value, use the `save` command at the beginning of the procedure:

```
myProcSave := proc(newDigits, x)
  save DIGITS;
begin
  DIGITS := newDigits;
  print(float(x));
end_proc:
```

After you call the procedure `myProcSave`, MuPAD restores the value of the global variable `DIGITS`:

```
myProcSave(20, PI);  
DIGITS
```

```
3.1415926535897932385
```

```
10
```

The combination of `save` and `delete` lets you temporarily free the variable for the use inside a procedure. For example, the procedure cannot use the variable `x` because the variable has a value assigned to it:

```
x := 10:  
proc()  
begin  
  solve(x^2 + x = 1, x)  
end_proc();
```

```
Error: Invalid variable to solve for. [solve]
```

Use the `save` command to save the original value 10. Then, free the variable `x` using the `delete` command:

```
x := 10:  
proc()  
  save x;  
begin  
  delete x;  
  solve(x^2 + x = 1, x)  
end_proc()
```

```
 $\left\{-\frac{\sqrt{5}}{2}-\frac{1}{2}, \frac{\sqrt{5}}{2}-\frac{1}{2}\right\}$ 
```

After the procedure call, MuPAD restores the original value of  $x$ :

```
x
```

```
10
```

The `save` and `delete` combination is helpful when you want to use a symbolic variable (without any value assigned to it) inside a procedure. You cannot use local variables for that purpose because a local variable in MuPAD is not an identifier. A local variable must have a value assigned to it. Also, you cannot specify assumptions on local variables, and you cannot integrate with respect to local variables.

## Functions

### Call Existing Functions

If you want to execute the same code repeatedly, create a procedure and use it. As a shortcut for simple procedures, create and use functions. Compared to procedures, functions require less complicated syntax. Like procedures, functions let you use the same code for different arguments as many times as you need. For example, you can always calculate sine and cosine of a particular value:

```
sin(30.0), sin(-1.0), sin(0.5);  
cos(10.0), cos(-0.8), cos(3.0)
```

```
-0.9880316241, -0.8414709848, 0.4794255386
```

```
-0.8390715291, 0.6967067093, -0.9899924966
```

### Create Functions

To define your own functions in MuPAD, use the arrow operator:

```
f := x -> x^2
```

$$x \rightarrow x^2$$

After defining a function, call it in the same way you call system functions:

$$f(1), f(x), f(\sin(x))$$

$$1, x^2, \sin(x)^2$$

The arrow operator also can create a multivariate function:

$$g := (x, y) \rightarrow x^2 + y^3$$

$$(x, y) \rightarrow x^2 + y^3$$

Call the multivariate function with numeric or symbolic parameters:

$$g(5, 2); g(x, 2*x); g(a, b)$$

$$33$$

$$8x^3 + x^2$$

$$a^2 + b^3$$

### Evaluate Expressions While Creating Functions

If you use an arrow operator to define a function, MuPAD does not evaluate the right-side expression:

$$f1 := x \rightarrow \text{int}(x^2, x)$$

$$x \rightarrow \int x^2 dx$$

To evaluate the right-side expression when defining a function, use the double arrow operator:

```
f2 := x --> int(x^2, x)
```

$$x \rightarrow \frac{x^3}{3}$$

## Use Functions with Parameters

Besides symbolic variables, functions can contain symbolic parameters. To evaluate such a function for particular values of symbolic parameters, use `evalAt` or the vertical bar `|` as a shortcut:

```
fa := x -> x^2 + a:
fa(2);
fa(2) | a = 10
```

$$a+4$$

$$14$$

Functions with symbolic parameters serve best for interactive use in a notebook. In your regular code, avoid unnecessary creation of such functions. When using a symbolic parameter, you use a global variable even though you do not explicitly declare it. See [Global Variables](#) for information on global variables and recommendations on their use.

## Shortcut for Closing Statements

As a shortcut for a closing statement, use the `end` command. This command closes:

- Conditional structures `if ... then ... else` (a shortcut for `end_if`)
- Case Selection Structures `case ... of` (a shortcut for `end_case`)
- Loops (a shortcut for `end_for`, `end_repeat`, and `end_while`)
- Procedures (a shortcut for `end_proc`)

For example, the following two loops are equivalent:

```
for i in [0, 1, 0, 0] do
  if i = 1 then
    print(Unquoted, "True")
  else
    print(Unquoted, "False")
  end_if
end_for
```

False

True

False

False

```
for i in [0, 1, 0, 0] do
  if i = 1 then
    print(Unquoted, "True")
  else
    print(Unquoted, "False")
  end
end
```

False

True



False

False

## Debug MuPAD Code Using the Debugger

In this section...
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“Open the Debugger” on page 1-184
“Debug Step-by-Step” on page 1-186
“Set and Remove Breakpoints” on page 1-190
“Evaluate Variables and Expressions After a Particular Function Call” on page 1-197
“Watch Intermediate Values of Variables and Expressions” on page 1-199
“View Names of Currently Running Procedures” on page 1-200
“Correct Errors” on page 1-201

### Overview

Besides syntax errors such as misspelling a function name or omitting parenthesis, run-time errors can appear when executing your code. For example, you might modify the wrong variable or code a calculation incorrectly. Runtime errors are usually apparent when your code produces unexpected results. Debugging is the process of isolating and fixing these run-time problems.

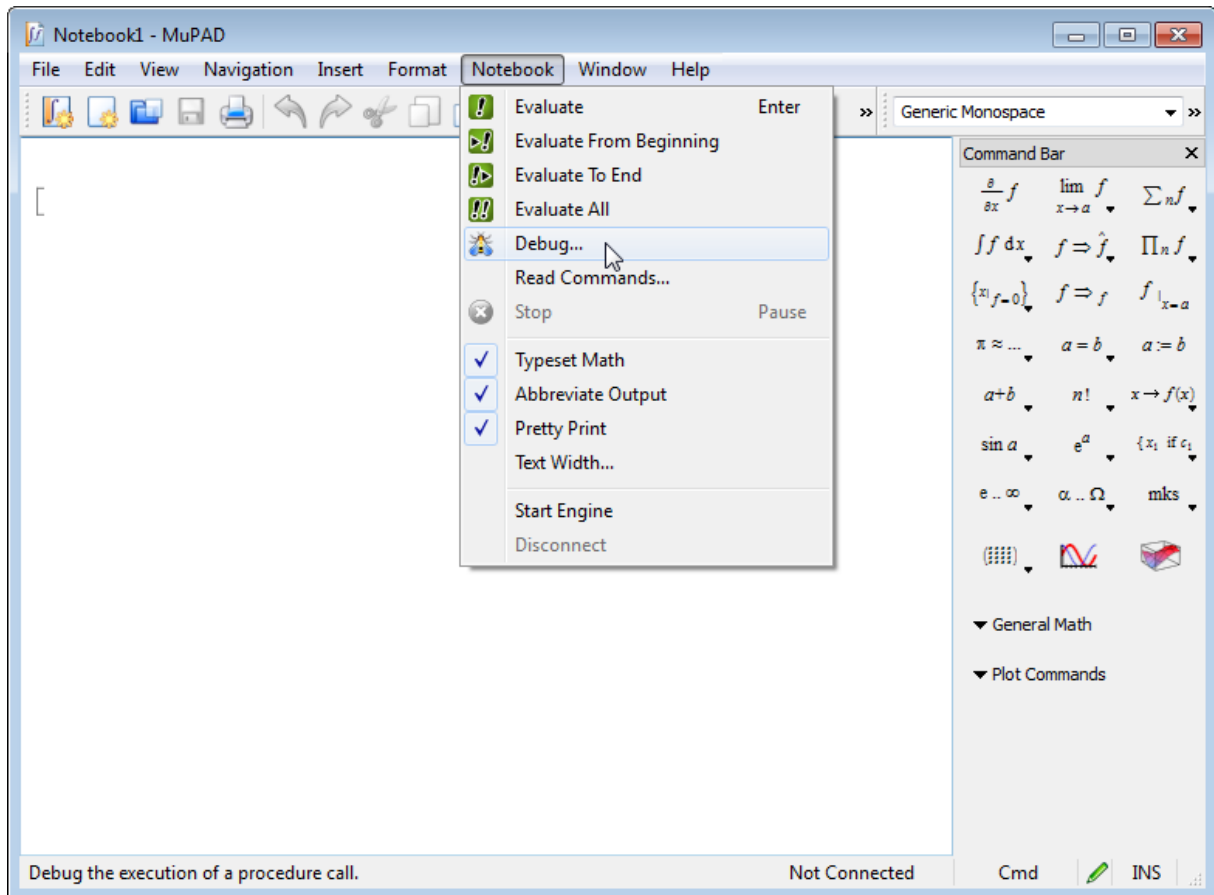
MuPAD provides a tool to help you with the debugging process. With the Debugger, you can:

- Run your procedure step-by-step.
- Set rigid and conditional breakpoints.
- Evaluate variables and expressions after a particular function call.
- Watch the changing intermediate values of the variables.
- View the name of the currently running procedure.

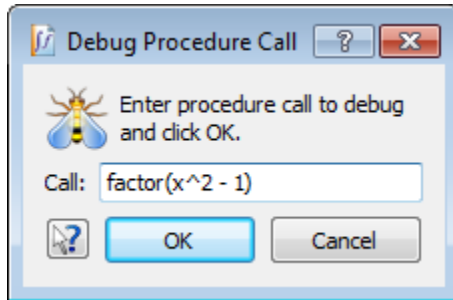
### Open the Debugger

To open the Debugger window from a notebook:

**1** Select **Notebook>Debug** from the main menu.



**2** In the Debug Procedure Call dialog box enter the procedure call you want to debug.



**3** Click **OK** to open the Debugger for this procedure call.

You also can open the Debugger directly from the command line using the `debug` command, for example:

```
debug(factor(x^2-1))
```

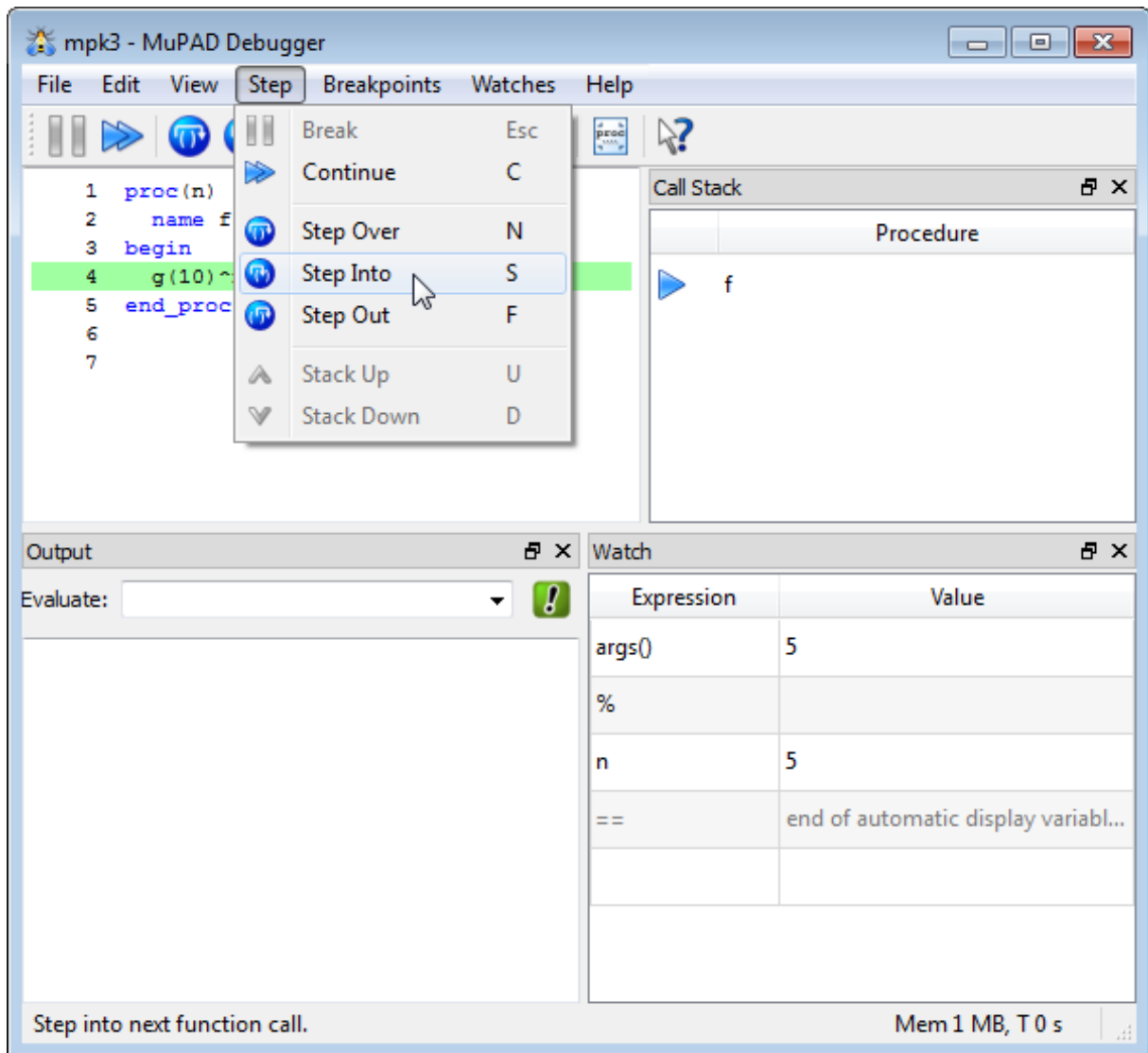
If you debug several lines of code, place the `debug` command in a separate input region. This allows you to avoid reevaluating the code every time you open the Debugger:

```
g := proc(x) begin x/(x+1) end_proc:  
f := proc(n) begin g(10)^n end_proc:
```

```
debug(f(5))
```

## Debug Step-by-Step

Using the MuPAD Debugger, you can run a procedure step by step. Running a procedure step by step helps you isolate the errors in your code. To start the step-by-step debugging process, select **Step** from the main menu.



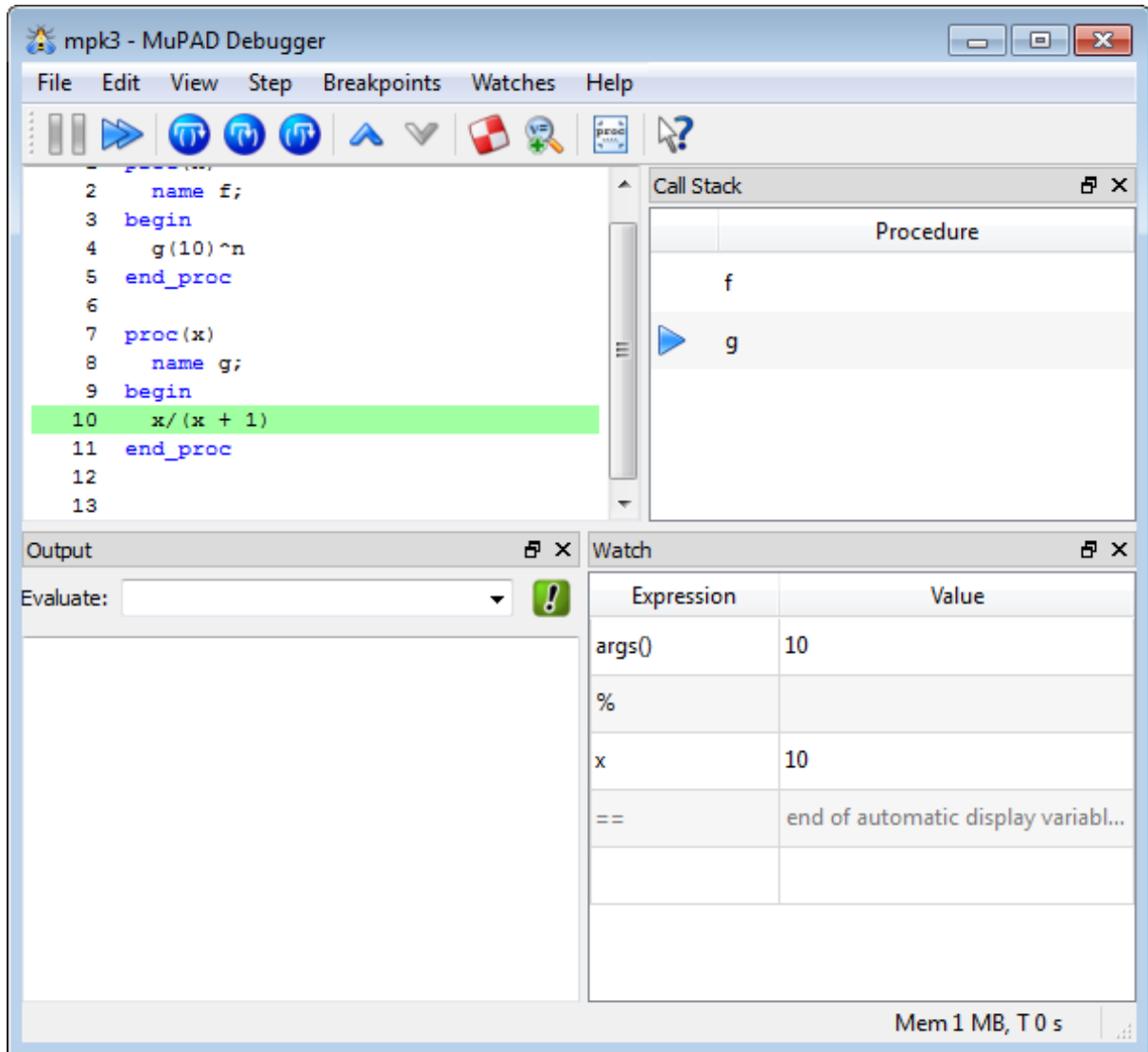
Also you can use the toolbar buttons.



Executing your code step by step you can:

- Use **Step Over** to execute the current line. If the code line contains a call to another function, the Debugger passes to the next code line without stepping into that function.
- Use **Step Into** to execute the current code line and, if the code line contains a call to another function, the Debugger steps into that function.
- After stepping in, use **Step Out** to run the rest of the current function, leave the called function, and pause.

For example, use **Step Into** to open and step through the inner procedure `g`.



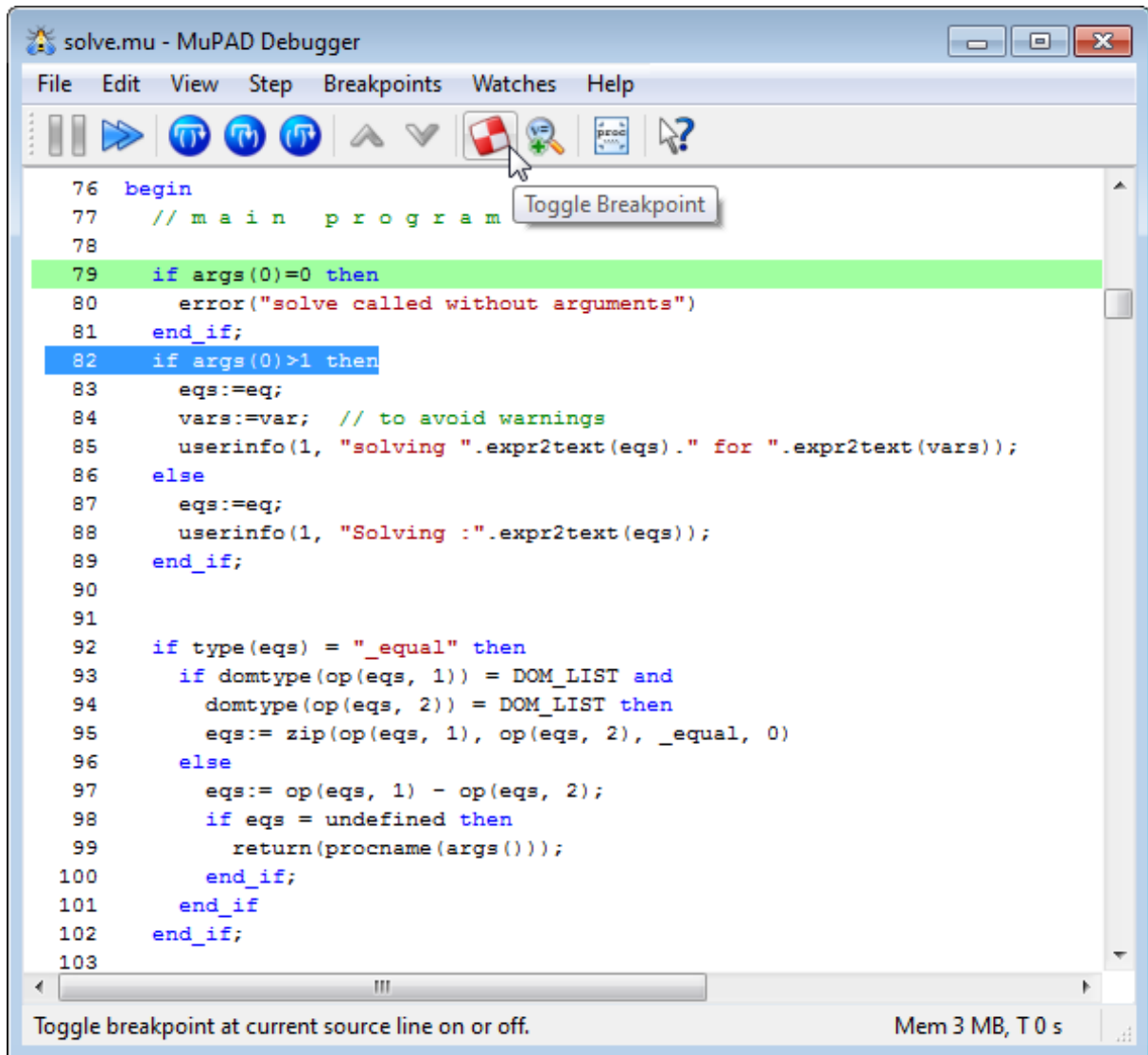
## **Set and Remove Breakpoints**

### **Set Standard Breakpoints**

Set breakpoints to pause execution of your code so you can examine values where you think the problem is. To set a breakpoint inside a procedure:

- 1** Select a line where you want to set a breakpoint.
- 2** Select **Breakpoints>Toggle Breakpoint** from the main menu or right-click to use the context menu. Also you can click the Toggle Breakpoint button on the toolbar.

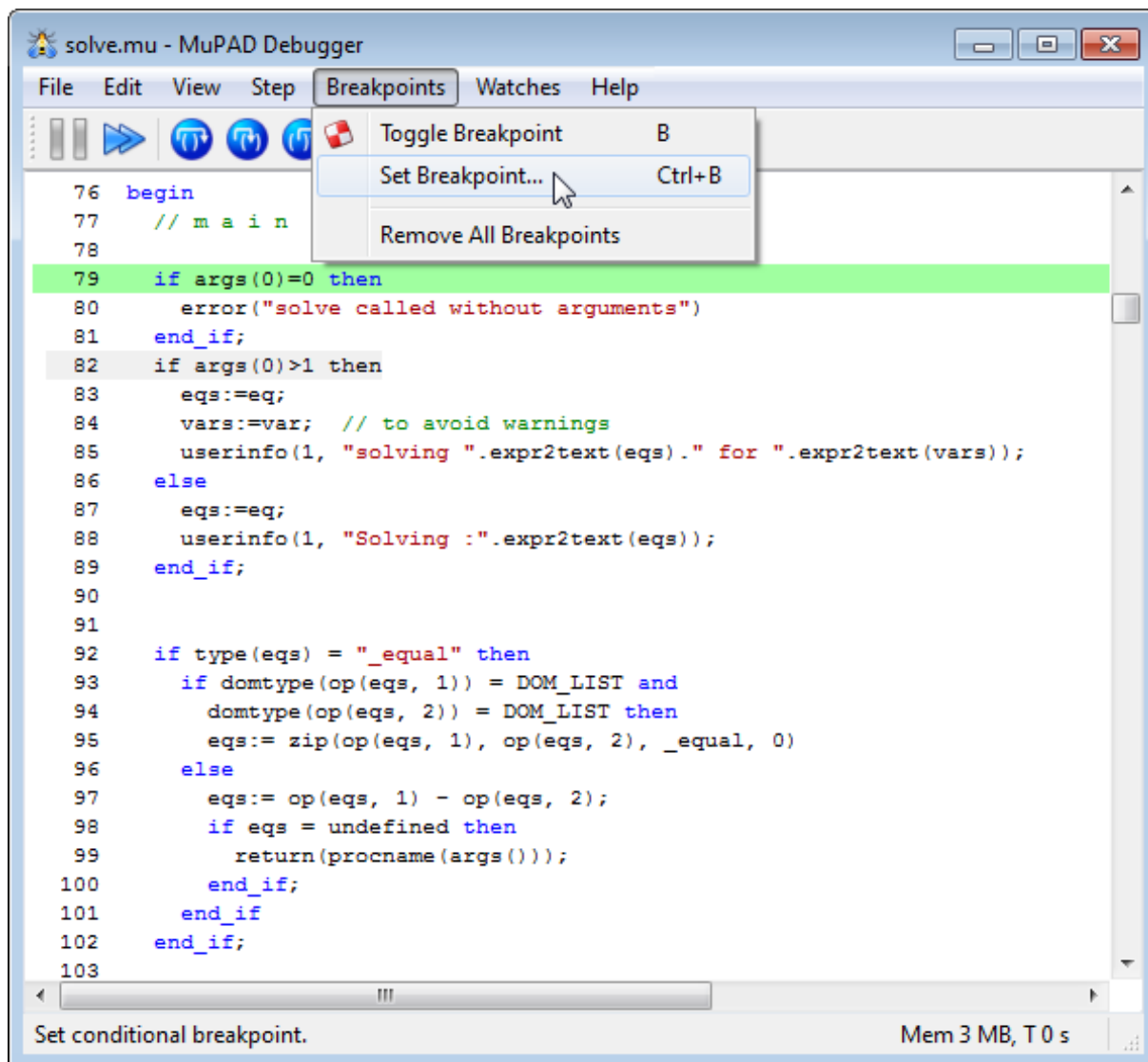




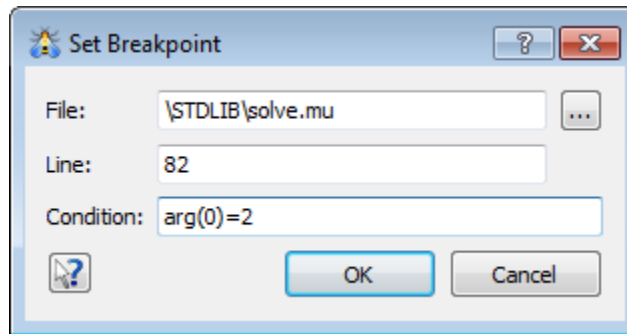
## Set Conditional Breakpoints

To set a conditional breakpoint:

- 1 Select a line where you want to set a breakpoint
- 2 Select **Breakpoints>Set Breakpoint** from the main menu.

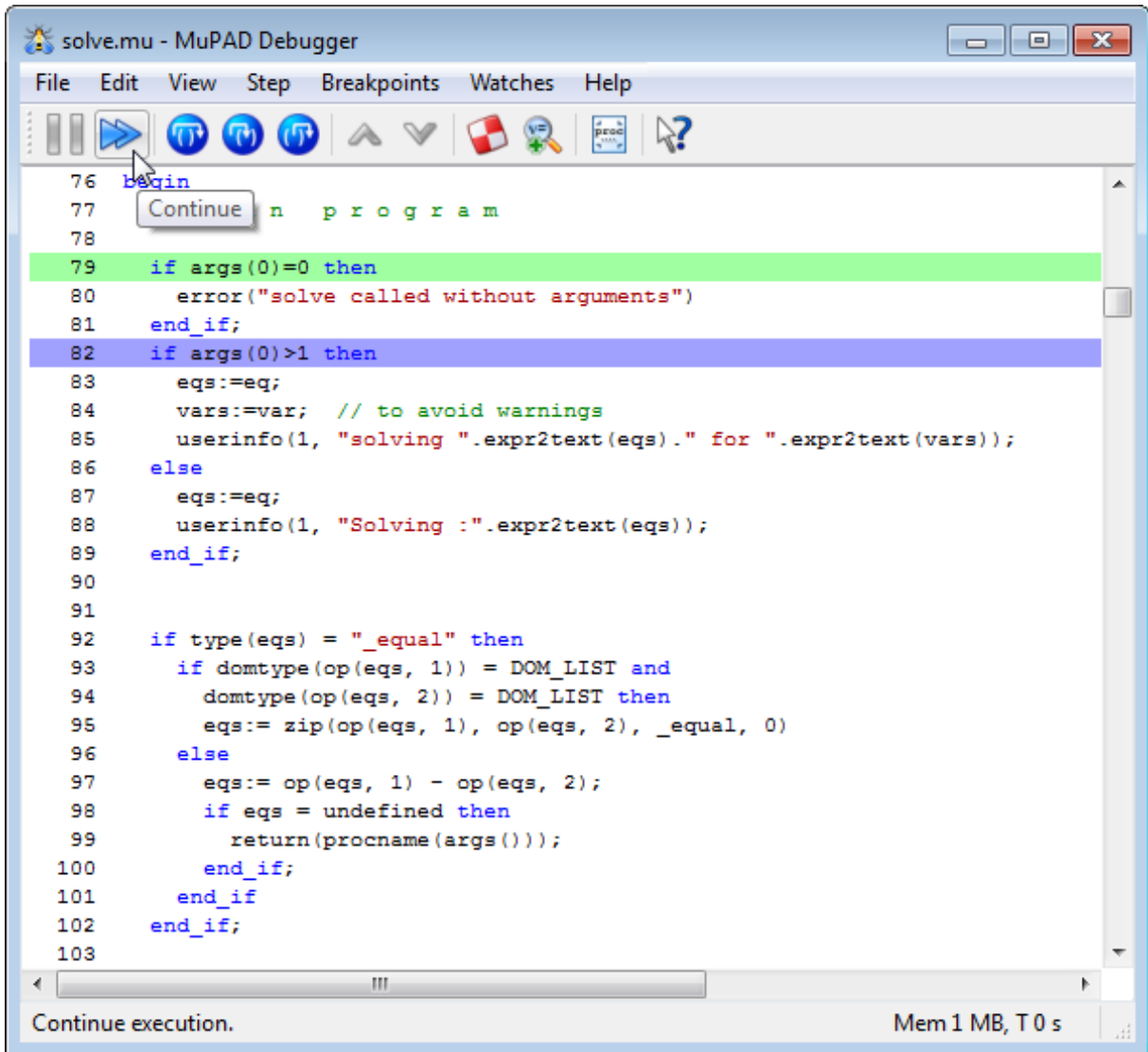


- 3 In the Set Breakpoint dialog box, type the condition under which you want the Debugger to stop on this line. In the dialog box, you also can change the placement of a breakpoint defining the file and line where you want to set the breakpoint.



## Use Breakpoints

After you set the breakpoint, you can continue the debugging process. The Debugger pauses at the breakpoints and executes the line with the breakpoint after you click the Continue button.



After setting breakpoints, you also can leave the current debugging session and start a new one. In the new session, the Debugger stops at all the breakpoints you previously set.

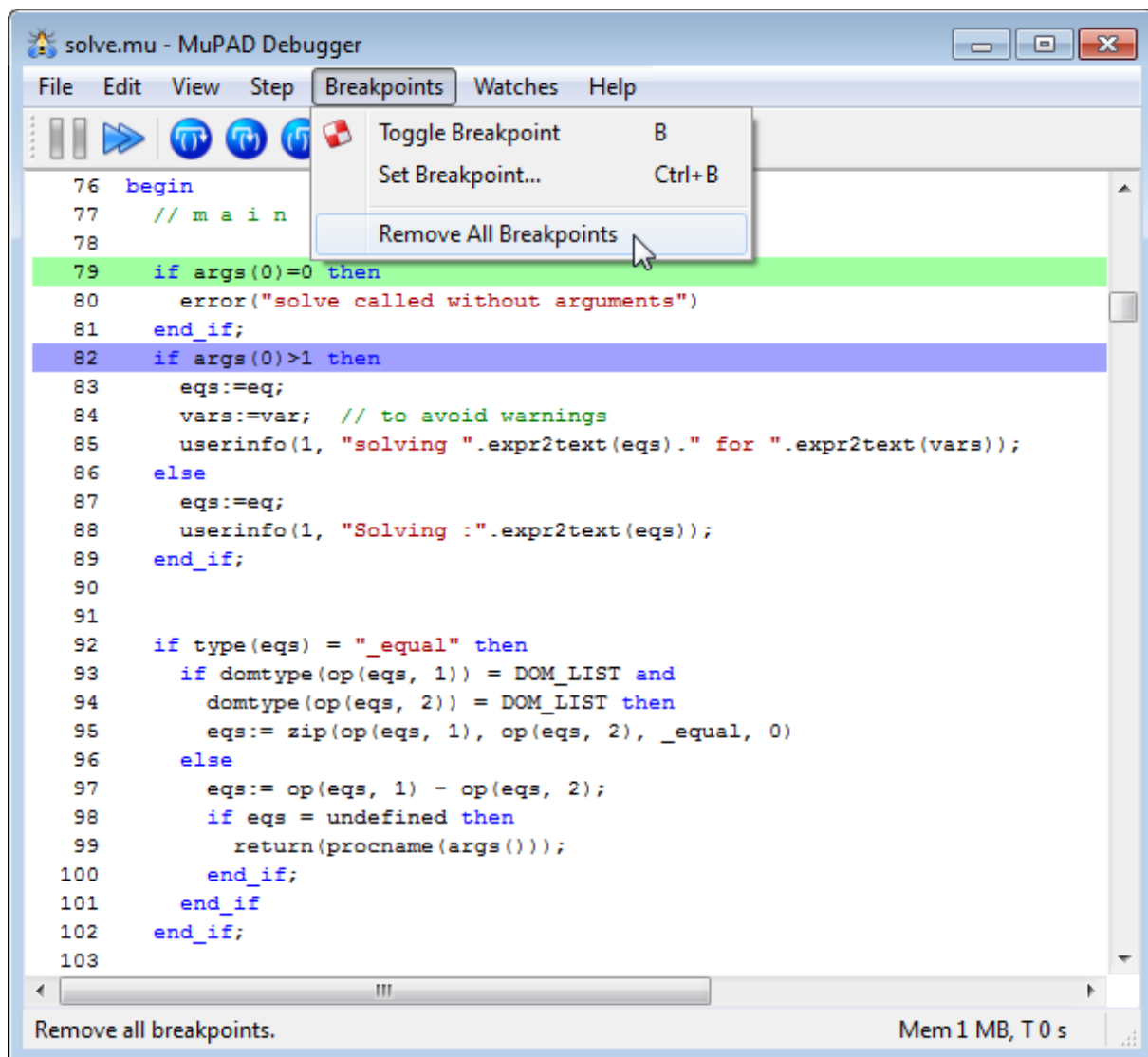
You can see the list of all breakpoints in the debugging process using the **Breakpoints** window. To open this window, select **View>Breakpoints**.

## **Remove Breakpoints**

To remove a breakpoint:

- 1** Select the breakpoint you want to remove.
- 2** Select **Breakpoints>Toggle Breakpoint** from the main menu or right-click to use the context menu. Also, you can click the Toggle Breakpoint button on the toolbar. The second click releases the button and removes the breakpoint.

If you want to remove all breakpoints, select **Breakpoints>Remove All Breakpoints** from the main menu.



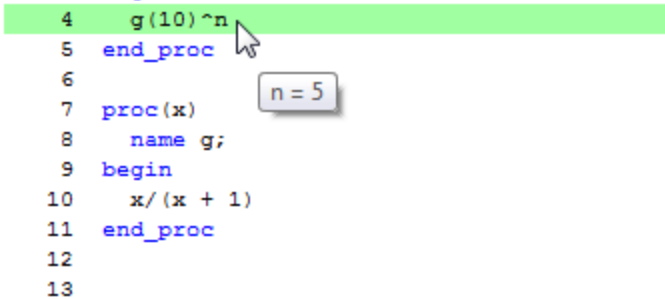
## Evaluate Variables and Expressions After a Particular Function Call

During the debugging process you can check the values of variables by hovering the cursor over a particular variable. The Debugger displays the current value of the variable.

```

1  proc(n)
2   name f;
3  begin
4   g(10)^n
5  end_proc
6
7  proc(x)
8   name g;
9  begin
10   x/(x + 1)
11  end_proc
12
13

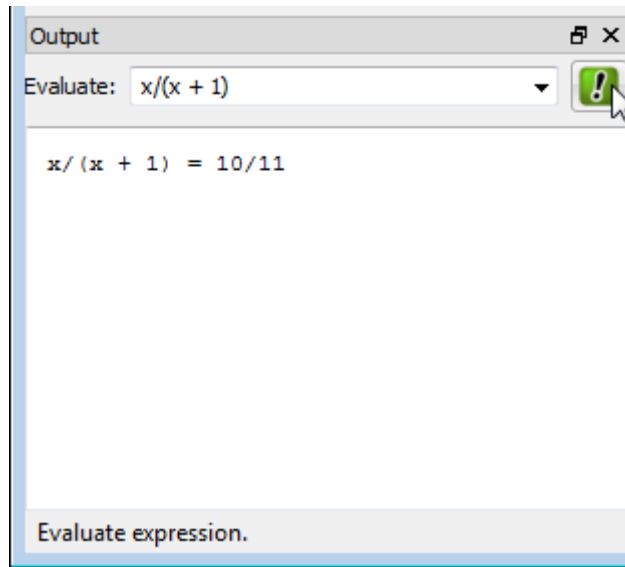
```



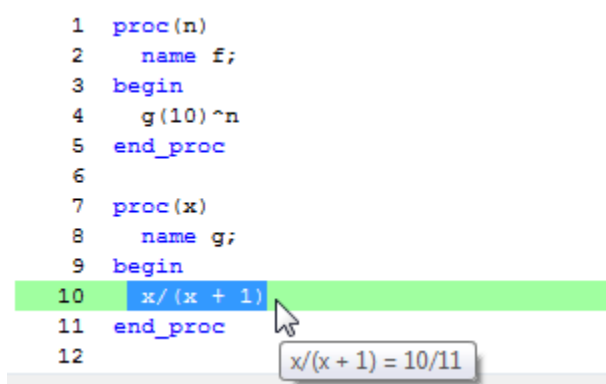
The image shows a code editor with two procedure definitions. The first procedure, `proc(n)`, has a line `g(10)^n` highlighted in green. A mouse cursor is hovering over the variable `n` in this expression. A tooltip box appears next to the cursor, displaying the value `n = 5`. The second procedure, `proc(x)`, defines a function `g` as `x/(x + 1)`.

To evaluate the value of an expression:

- 1** If you do not see the **Output** pane, select **View>Output** from the main menu.
- 2** In the **Output** pane, type the expression you want to evaluate or select one of the prior expressions from the drop-down menu.
- 3** Click the Evaluate button or press the **Enter** key.



Alternatively, you can select an expression and hover the cursor over it. If the expression is syntactically correct and can be computed fast, MuPAD displays the value of the expression in a tooltip.

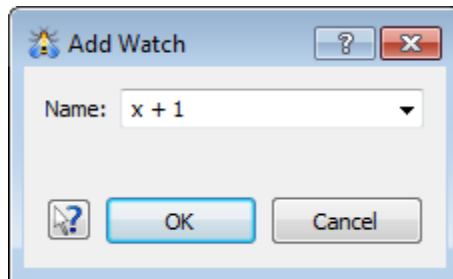




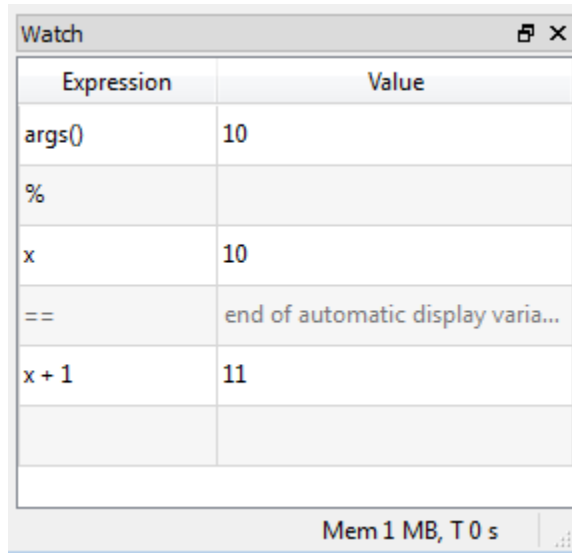
## Watch Intermediate Values of Variables and Expressions

You can observe the values of variables during the debugging process in the **Watch** pane. If you do not see the **Watch** pane, select **View>Watch**.

By default, the Debugger displays the values of the parameters used in the function call (args), the results of the most recent command (%), and the values of variables declared in the procedure. To watch the values of other variables and expressions during the debugging process, select **Watches>Add Watch** and enter the object or the expression you want to observe in the Add Watch dialog box.



You also can enter the expressions directly into the **Watch** table.



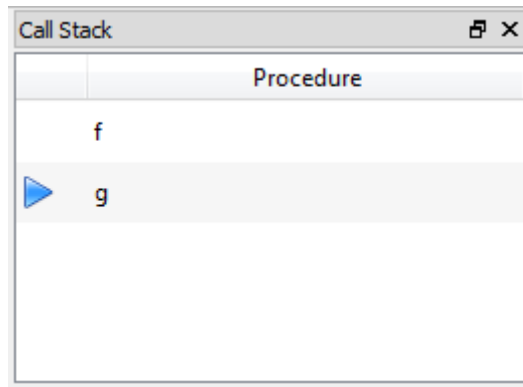
Expression	Value
args()	10
%	
x	10
==	end of automatic display varia...
x + 1	11

Mem 1 MB, T 0 s

## View Names of Currently Running Procedures

In the **Call Stack** pane, you can see the list of the procedures participating in the debugging process. If you do not see the **Call Stack** pane, select **View>Call Stack**.

The Debugger lists all the procedure calls in the **Call Stack** pane. The Debugger marks the name of the current procedure with a blue triangle and highlights the currently executed code line.



To switch between procedures, click the name of a procedure you want to switch to. Also, you can select **Step>Stack Up** or **Step>Stack Down** from the main menu or use the toolbar. As an alternative, you can press **u** and **d**. The **Call Stack** pane helps you navigate within nested calls of various procedures.

## Correct Errors

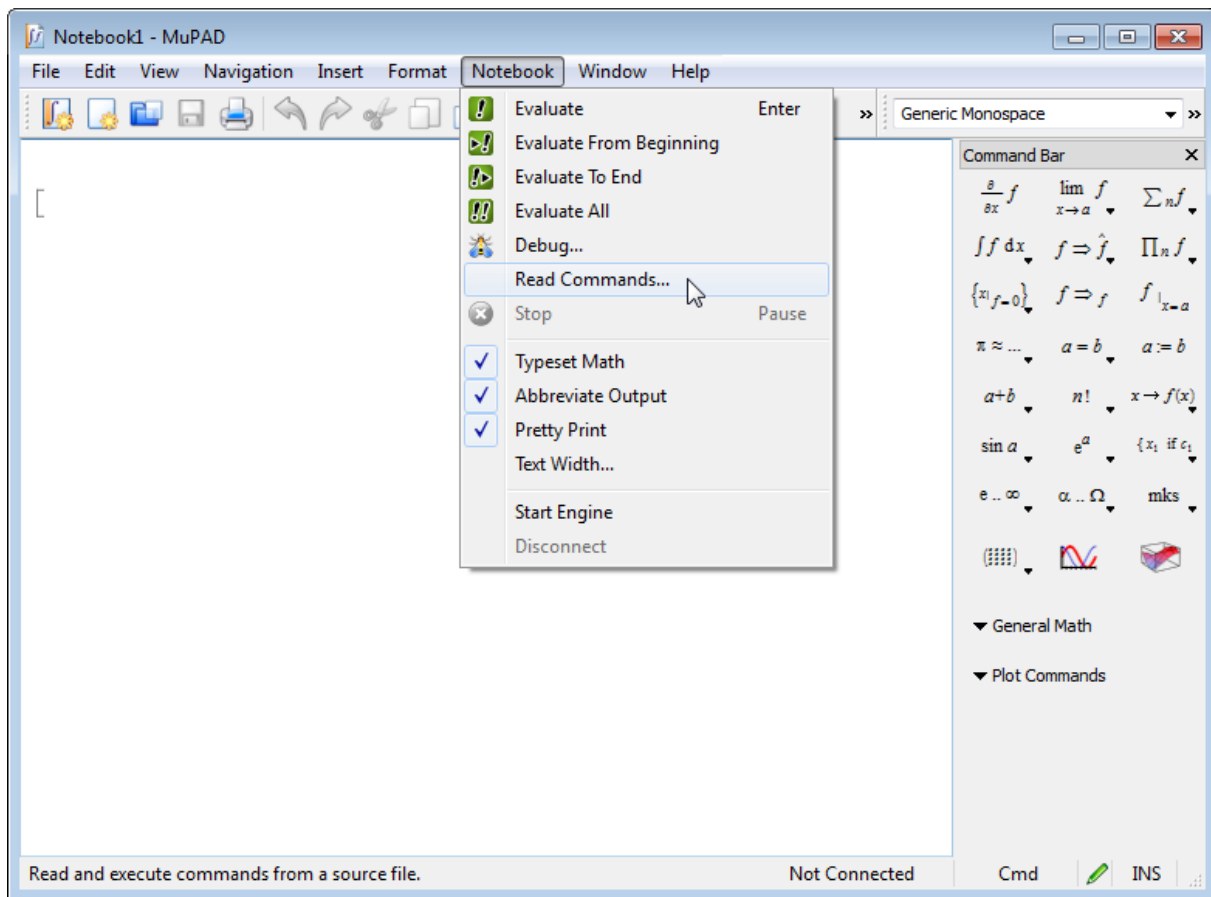
The Debugger displays procedures and helps you find errors in the code, but you cannot correct the errors in the Debugger window. To edit your code use the MATLAB Editor, a MuPAD notebook, or any text editor.

To open a new MATLAB Editor window, select **File>New Editor with Source** from the main menu or select **Open in Editor** from the context menu. The MATLAB Editor window contains the source that you see in the Debugger and lets you modify and save it.

Changes made in the Editor window do not automatically appear in the Debugger window. The Debugger presents the code that is already in the kernel. To run the Debugger on the corrected file:

- 1 Close the Debugger window if it is open.
- 2 In the Editor window select **File>Save** from the main menu to save changes.
- 3 Open a notebook.

4 Select **Notebook>Read Commands** from the main menu.



5 Select the file you want to run.

6 Start the Debugger from the notebook.

# Notebook Interface

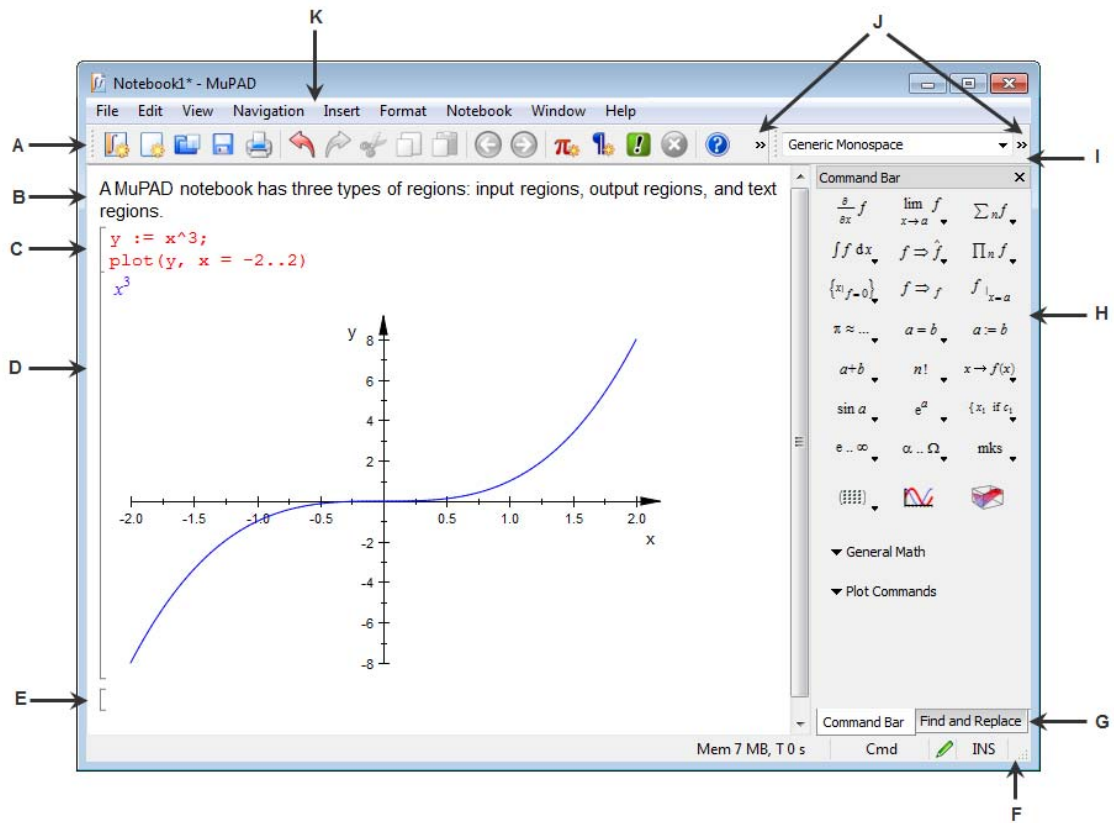
---

- “Notebook Overview” on page 2-3
- “Debugger Window Overview” on page 2-5
- “Arrange Toolbars and Panes” on page 2-8
- “Enter Data and View Results” on page 2-11
- “View Status Information” on page 2-13
- “Save Custom Arrangements” on page 2-14
- “Set Preferences for Notebooks” on page 2-15
- “Set Preferences for Dialogs, Toolbars, and Graphics” on page 2-20
- “Set Font Preferences” on page 2-24
- “Set Engine Preferences” on page 2-27
- “Get Version Information” on page 2-32
- “Use Different Output Modes” on page 2-33
- “Set Line Length in Plain Text Outputs” on page 2-40
- “Delete Outputs” on page 2-41
- “Greek Letters in Text Regions” on page 2-42
- “Special Characters in Outputs” on page 2-43
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- “Use Keyboard Shortcuts” on page 2-45
- “Use Mnemonics” on page 2-46
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- “Wrap Long Lines” on page 2-48

- “Hide Code Lines” on page 2-57
- “Change Font Size Quickly” on page 2-60
- “Scale Graphics” on page 2-63
- “Use Print Preview” on page 2-65
- “Change Page Settings for Printing” on page 2-69
- “Print Wide Notebooks” on page 2-70

## Notebook Overview

The first time you start MuPAD notebook, it appears with the default layout, as shown in the following illustration.



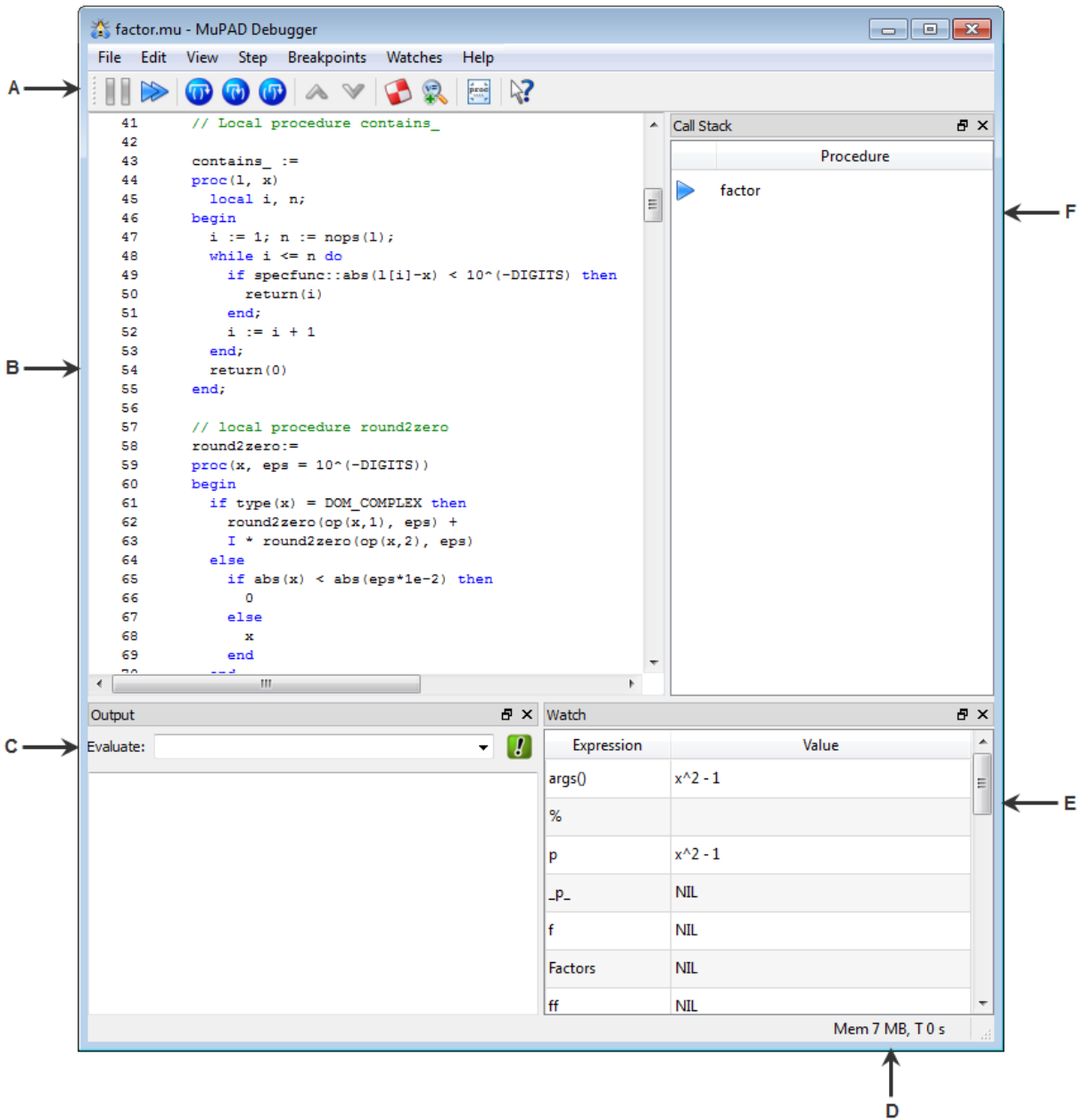
- **A** Perform most common tasks from the Standard toolbar.
- **B** Type your comments in the text regions.

- **C** Enter commands in the input regions.
- **D** View results (including graphics) in the output regions.
- **E** A new input region appears after evaluation of the bottom input region.
- **F** View current status information in the Status bar.
- **G** Find and Replace text in the input and text regions.
- **H** Quickly access standard functions from the Command Bar.
- **I** Format nongraphical objects in the input and output regions from the Format toolbar.
- **J** Use more items from the toolbars.
- **K** Menus change in the Graphics Format mode.



## **Debugger Window Overview**

The first time you start MuPAD Debugger window, it appears with the default layout, as shown in the following illustration.



- **A** Perform common tasks from the toolbar.
- **B** View the code that you debug.
- **C** Type an expression and evaluate it in the Output pane anytime during the debugging process.
- **D** Use the Status bar to view memory and time usage for the current or most recent debugging step.
- **E** Use the Watch pane to view values of variables during the debugging process.
- **F** Use the Call Stack pane to view the names of the procedures participating in the debugging process.

For information about the Debugger mode, see [Tracing Errors with the Debugger](#).

## Arrange Toolbars and Panes

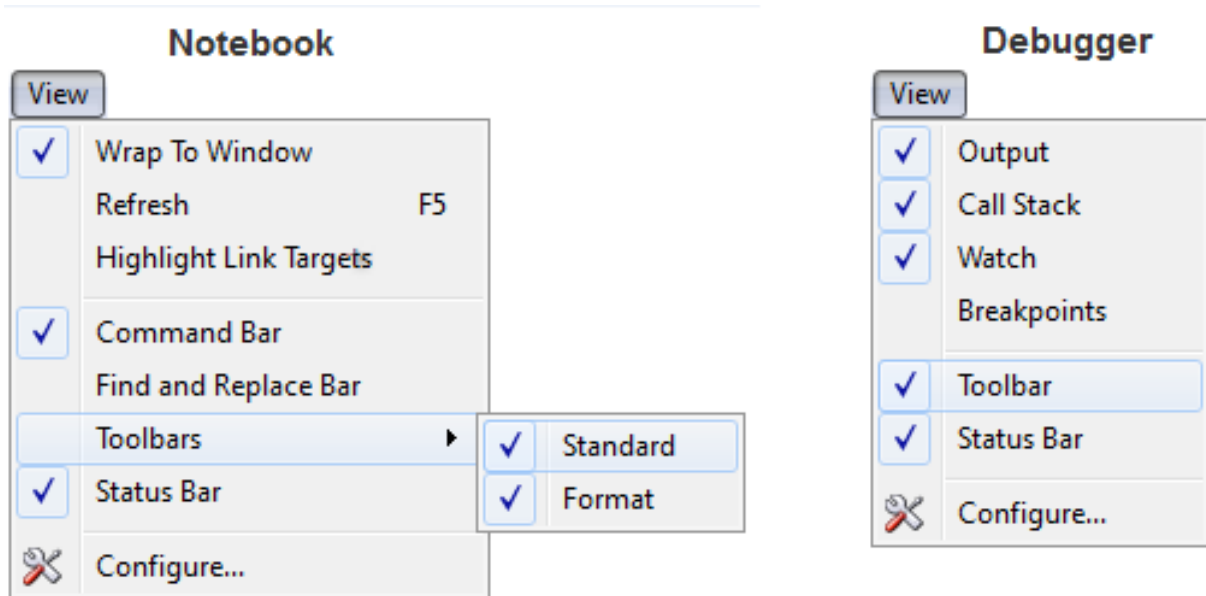
### In this section...

“Enabling and Disabling Toolbars and Panes” on page 2-8

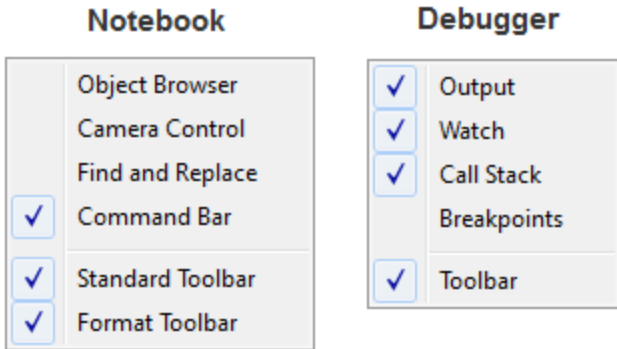
“Move Toolbars and Panes” on page 2-9

### Enabling and Disabling Toolbars and Panes

To enable or disable the Command Bar, Find and Replace Bar, toolbars, or Status Bar, select **View** from the main menu. The following illustration shows the **View** menu for a notebook and a Debugger window.

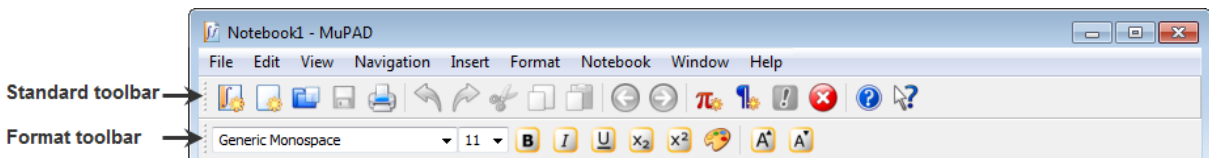
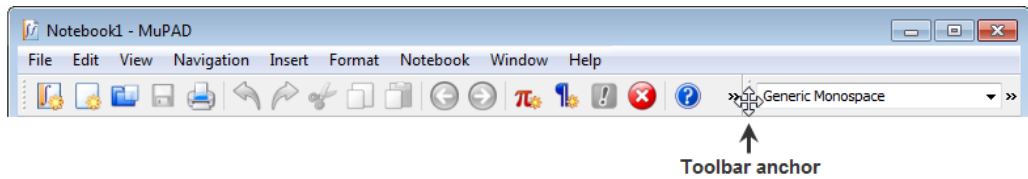


To use context menu, right-click any bar. The following illustration shows the context menu for the Notebook and Debugger windows.



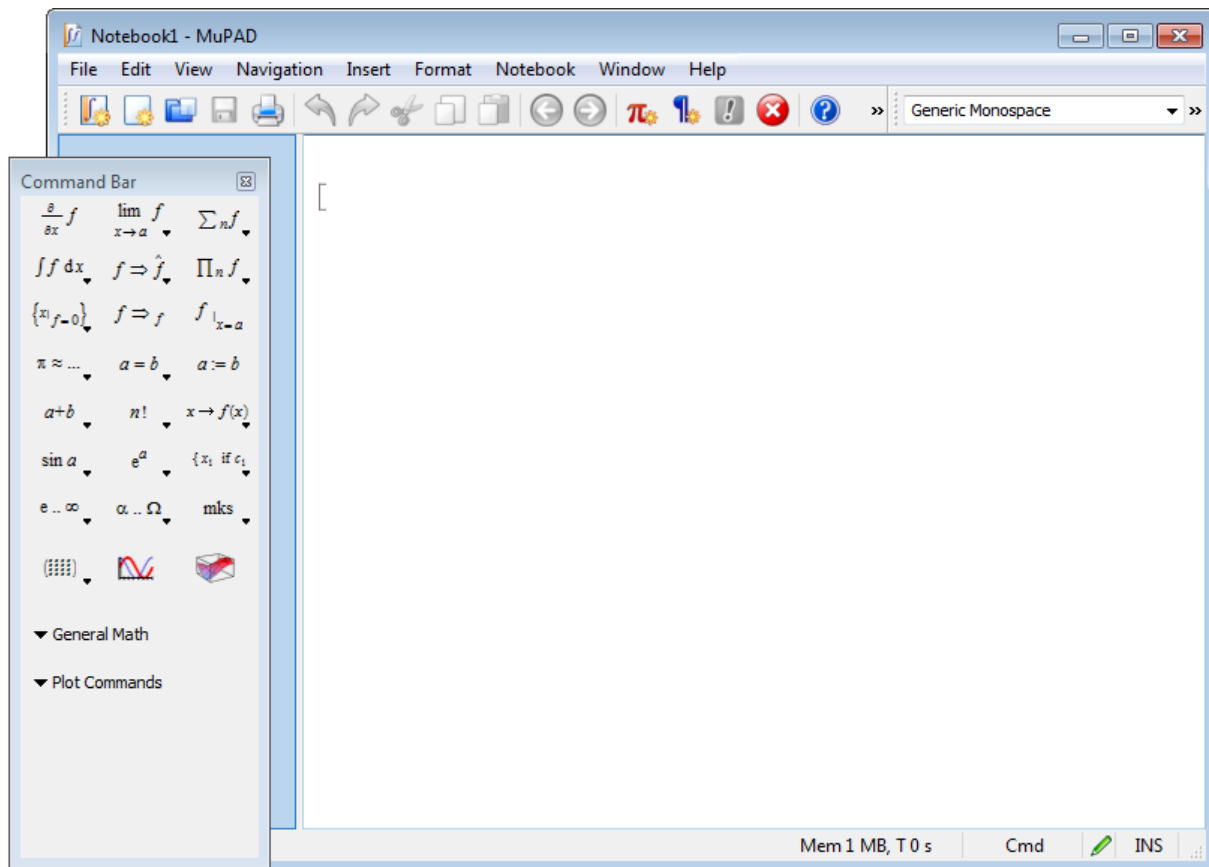
## Move Toolbars and Panes

To move a toolbar, grab the toolbar anchor and drag the toolbar to a different location. For example, line the toolbars one below the other.




To move the Command Bar or Find and Replace Bar, grab the pane title and drag the pane to a different location. For example, move the Command Bar to the left.

## 2 Notebook Interface



## Enter Data and View Results

MuPAD notebook has two types of regions for entering data:

- **Text regions** serve for entering your comments. They can contain text, tables, mathematical formulas, and graphics. Text regions do not have bracket markers. The default font color for text regions is black. To start a new text region, click outside the gray brackets and start typing. Alternatively, you can insert a new text region by selecting **Insert>Text Paragraph** or **Insert>Text Paragraph Above**. You cannot insert a text region inside a region of other type or between adjacent input and output regions.
- **Input regions** serve for entering your code. They can contain MuPAD mathematical expressions and commands in the MuPAD language. Input regions have gray bracket markers. The default font color for input regions is red. When you evaluate an expression in the bottom input region, MuPAD inserts a new input region below. To insert new input regions in other parts of a notebook, select **Insert>Calculation** or **Insert>Calculation Above** from the main menu. Alternatively, use the Insert Calculation button .

MuPAD notebook has a special type of region for viewing results:

- **Output regions** serve for viewing results. These regions automatically appear when you evaluate input regions. The results can include graphics and error messages. The default font color for output regions is blue. You cannot edit data in the output regions. To change the results, edit the associated input region and evaluate it by pressing **Enter**. Also, you can copy results from the output regions to the text and input regions. When you copy outputs to the input regions, MuPAD inserts ASCII equivalents of the results in the input regions.

If you want to change the default text color, font, or the appearance of brackets for the current notebook, see [Changing Default Format Settings](#).

If you want to change preferences for all notebooks, see [Setting Preferences for Notebooks](#).

---

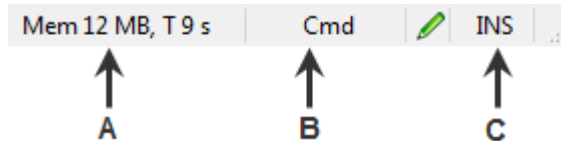
**Note** Most preferences affect only new notebooks. They do not affect existing notebooks.

---



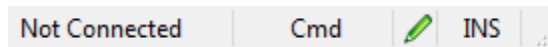
## View Status Information

The status bar shows the current status information.



- **A** Displays memory and time usage for the current or most recent computation.
- **B** Indicated the type of the currently active region.
- **C** Indicates insert or overwrite mode.

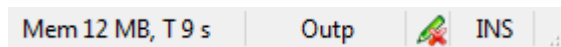
View the current engine state at the far left end of the status bar. If the engine is not connected to your notebook, the status bar displays **Not Connected**.



The status bar indicates the type of the region where you position the cursor. The indicator displays:

- **Cmd**, if the cursor is in an input region
- **Text**, if the cursor is in a text region
- **Outp**, if the cursor is in an output region

The status bar also indicates if the cursor is in a read-only part of a notebook, for example, in an output region.



For text and input regions, MuPAD notebook supports overwrite mode. Press the **Insert** key to enter text in overwrite mode. Press the **Insert** key again to return to entering text in insert mode. View the current state at the far right of the status bar.

### **Save Custom Arrangements**

You can change a notebook or a Debugger window arrangement to meet your needs, including resizing, moving, and closing toolbars and panes. When you end a session, MuPAD saves the arrangement. The next time you start MuPAD and open a notebook or a Debugger window, it appears the same way you left it.

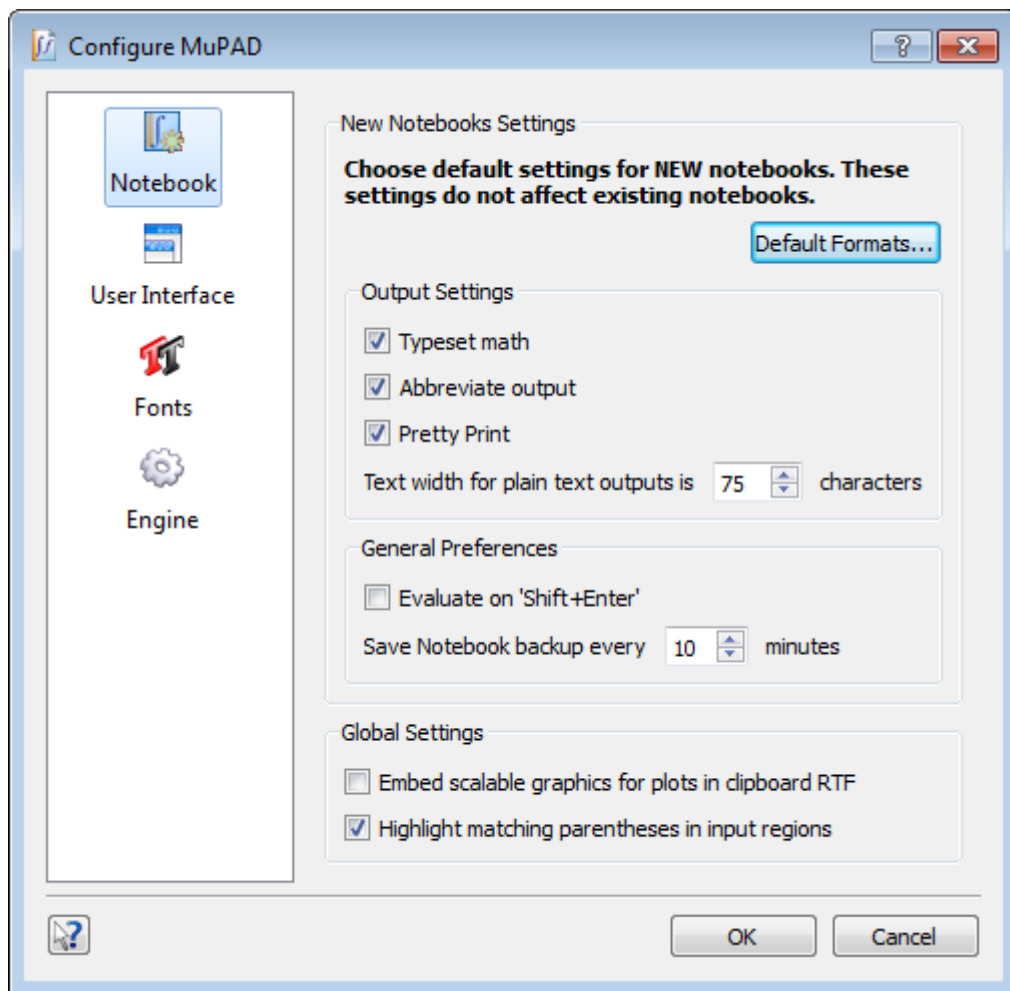
## Set Preferences for Notebooks

In this section...
“Preferences Available for Notebooks” on page 2-15
“Change Default Formatting” on page 2-17
“Scalable Format for Copying Graphics” on page 2-18

### Preferences Available for Notebooks

To change preferences for displaying the contents of notebooks:

- 1 Select **View > Configure**
- 2 In the left pane of the Configure MuPAD dialog box, click **Notebook**.



The right pane of the dialog box lets you change two types of settings: the settings that affect the new notebooks only and the global settings that affect all notebooks, including existing ones. For new notebooks, you can do the following:

- Change the default formatting settings of all new notebooks. For details, see [Changing Default Formatting for Notebook Elements](#).

- Change the default mode for displaying results. For details on output modes, see Using Different Output Modes.
- Change the default line length for displaying results in plain text format.
- Change key sequence for evaluation of input regions to **Shift+Enter**. The default key is **Enter**.
- Specify how often MuPAD automatically saves a backup document.

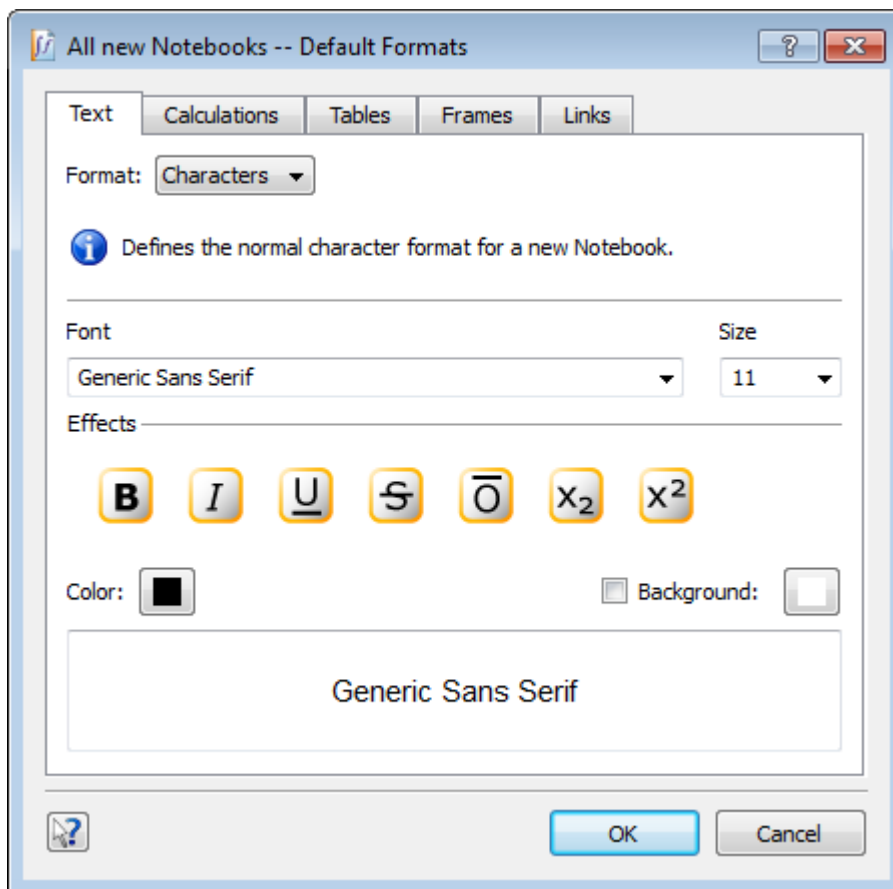
For all notebooks including existing ones, you can do the following:

- On Windows platforms, copy graphics using scalable Windows Metafile (WMF) format. For details, see “Scalable Format for Copying Graphics” on page 2-18.
- Highlight matched and mismatched delimiters (parentheses, brackets, and braces) in input regions. When you type a parenthesis, a bracket, or a brace, MuPAD highlights the matched delimiter in the pair. If a delimiter is missing or if a pair includes delimiters of different types (for example, a bracket and a brace), MuPAD uses a different color to highlight the delimiters. In addition to highlighting mismatched delimiters, MuPAD underlines them. To make the display of highlighting and underlining disappear, move the cursor away from a delimiter.

## Change Default Formatting

To specify default formatting for the elements of all new MuPAD notebooks:

- 1** Select **View>Configure**.
- 2** In the left pane of the Configure MuPAD dialog box, click **Notebook**.
- 3** In the right pane of the dialog box, click **Default Formats**.
- 4** In the **Default Formats** dialog box, use tabs to select the required element. You can specify default formatting for text regions, calculations, tables, frames, and links.



- 5 Specify default formatting for the element you selected. For example, set the default font size, style, and color for the text regions.

### Scalable Format for Copying Graphics

By default, MuPAD copies graphics to the clipboard using bitmap formats. On Windows platforms, you can choose to copy graphics using scalable Windows Metafile (WMF) format. To enable scalable WMF format for copying graphics, use the following steps:

- 1 Select **View>Configure**.

- 2** In the left pane of the Configure MuPAD dialog box, click **Notebook**.
- 3** In the right pane of the dialog box, select **Embed scalable graphics for plots in clipboard RTF**.

## Set Preferences for Dialogs, Toolbars, and Graphics

In this section...
“Preferences Available for Dialogs, Toolbars, and Graphics” on page 2-20
“Preferences for Toolbars” on page 2-22
“Preferences for Graphics” on page 2-22
“Preferences for Dialog Boxes” on page 2-22

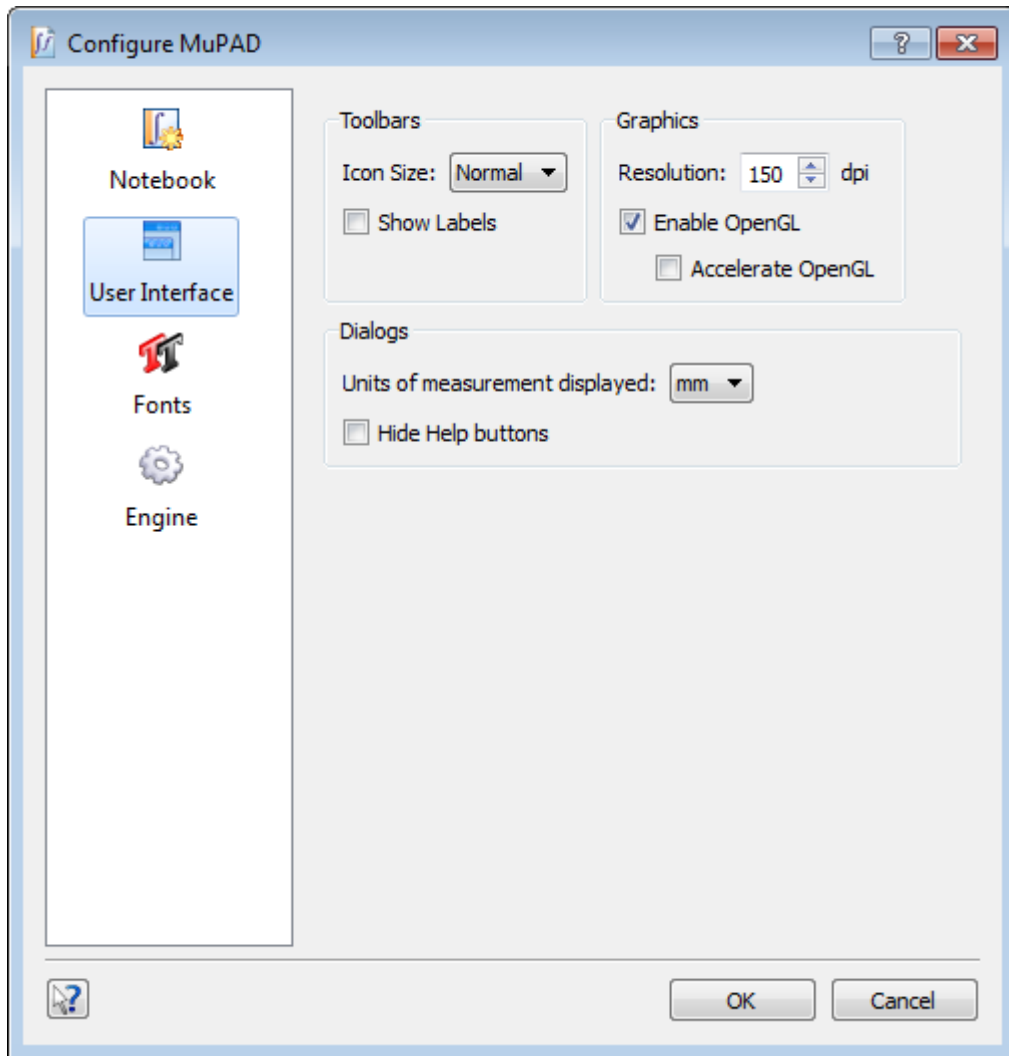
### Preferences Available for Dialogs, Toolbars, and Graphics

To specify the default settings for MuPAD toolbars, dialog boxes and graphics:

**1** Select **View > Configure**.

**2** In the left pane of the Configure MuPAD dialog box, click **User Interface**.



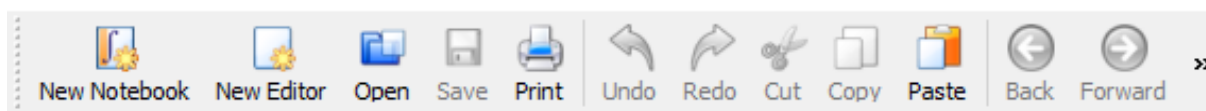


- 3 Use the right pane of the dialog box to specify the setting you want for the toolbars, graphics, units of measurements used in dialog boxes, and other options.

### Preferences for Toolbars

Specify the following preferences for the toolbars:

- **Icon Size.** Specify the size of icons in the toolbars.
- **Show Labels.** Select this option to display labels on the buttons of the toolbars.



### Preferences for Graphics

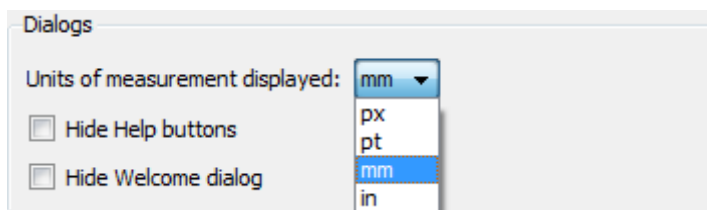
You can specify the following preferences for graphics:

- **Resolution.** Specify the dpi value for the resolution you want to use when displaying graphical results.
- **Enable OpenGL.** Select this option to use OpenGL®.
- **Accelerate OpenGL.** Select this option to use accelerated OpenGL. Mac OS platforms always use accelerated OpenGL.

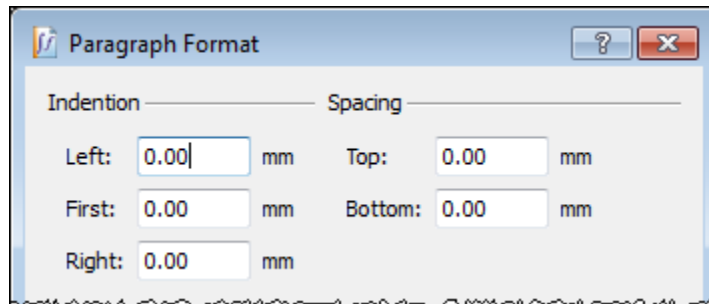
### Preferences for Dialog Boxes


You can specify the following preferences for dialogs:

- **Units of measurement displayed.** Specify the units of measurement you want to use in the MuPAD dialog boxes. For example, you can use millimeters.



Now all dialog boxes in MuPAD except those displaying the font sizes, use millimeters as the units of measurement. For example, the Paragraph Format dialog box shows the indentation and spacing sizes in millimeters.



- **Hide Help buttons.** Select this option if you do not want to see help buttons  in the dialog boxes.
- **Hide Welcome dialog.** Select this option if you do not want to see the welcome dialog box on startup.

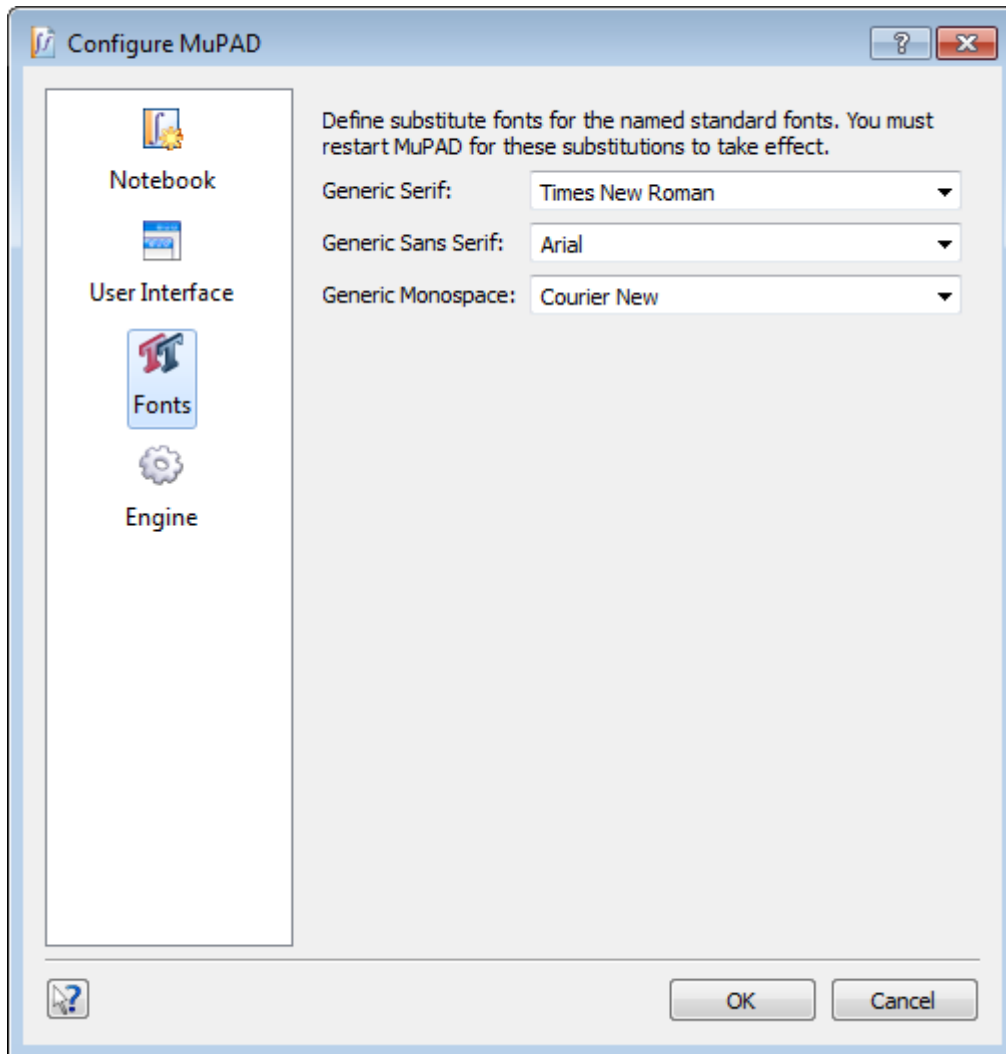
## Set Font Preferences

In this section...
“Select Generic Fonts” on page 2-24
“Default Generic Fonts for Microsoft Windows, Macintosh, and Linux” on page 2-26

### Select Generic Fonts

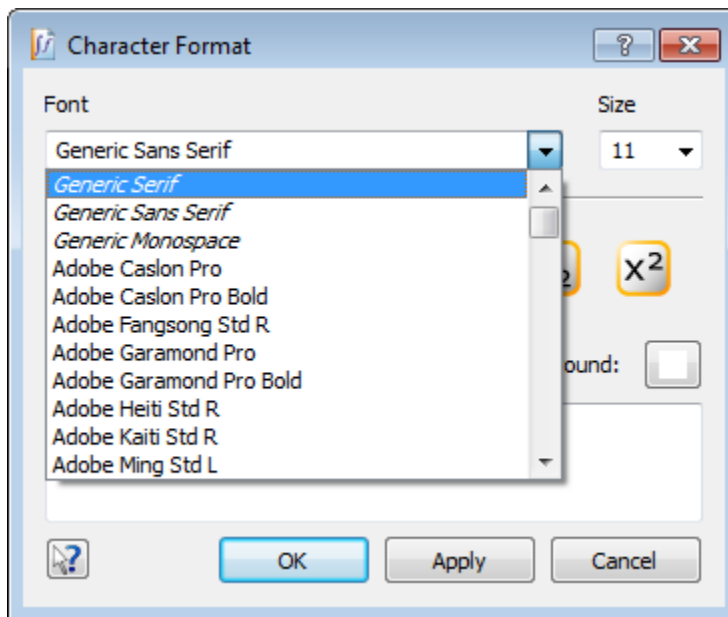
To specify the generic fonts for use in MuPAD notebooks:

- 1** Select **View > Configure**.
- 2** In the left pane of the Configure MuPAD dialog box, click **Fonts**.



- 3 Use the right pane of the dialog box to select the fonts you want to use as generic fonts.

Use these specified generic fonts to format text, mathematical expressions, or calculations in your notebooks.



## Default Generic Fonts for Microsoft Windows, Macintosh, and Linux

The default generic fonts in MuPAD depend on the platform you use.

Font	Windows	Macintosh
Generic Serif	Times New Roman	Times
Generic Sans Serif	Arial	Lucida Grande
Generic Monospace	Courier New	Monaco

On Linux® platforms, the default generic fonts depend on the system default fonts.

## Set Engine Preferences

### In this section...

“Change Global Settings” on page 2-27

“Restore Default Global Settings” on page 2-29

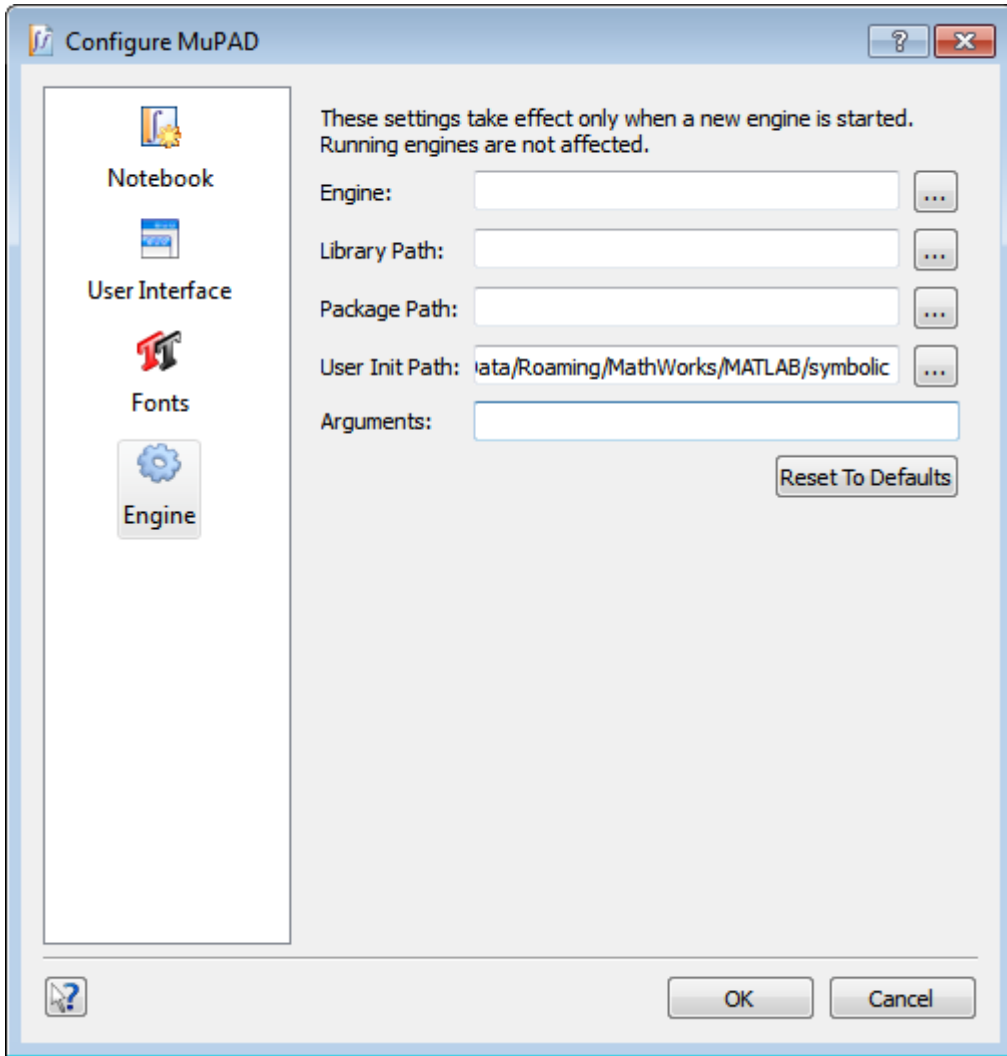
“Add Hidden Startup Commands to All Notebooks” on page 2-29

“Options Available for MuPAD Engine Startup” on page 2-29

### Change Global Settings

Global settings in MuPAD serve for specifying a particular engine you want to run, using additional libraries, startup commands and options. To change global settings:

- 1 Select **View > Configure**.
- 2 In the left pane of the Configure MuPAD dialog box, click **Engine**.



**3** Use the right pane to change the following settings:

- **Engine.** Path to the MuPAD engine that you want to use. Leaving this field empty indicates that you want to use the default engine.



- **Library Path.** Path to the standard set of the MuPAD libraries that you want to use. Leaving this field empty indicates that you want to use the default library.
- **Package Path.** Path to additional libraries (packages) you that want to use. Leaving this field empty indicates that you want to use the default packages.  
**User Init Path.** Path to the folder containing initialization file (userinit.mu). This file contains startup commands for MuPAD notebooks. You can add your own commands to the initialization file.
- **Arguments.** Options you want to use when starting MuPAD engine. See the list of Options Available When Starting Engine.

## Restore Default Global Settings

To restore the default global settings:

- 1 Select **View>Configure**.
- 2 In the left pane of the Configure MuPAD dialog box, click **Engine**.
- 3 In the right pane, click **Reset To Defaults** button. Clicking this button clears all fields, except for **User Init Path**.

## Add Hidden Startup Commands to All Notebooks

You can run MuPAD commands every time you start the engine without displaying these commands in a notebook. If you want to use hidden commands for a particular notebook, see Hiding Code Lines. To add startup commands to all MuPAD notebooks:

- 1 Select **View>Configure**.
- 2 In the left pane of the Configure MuPAD dialog box, click **Engine**.
- 3 In the **User Init Path** field, specify the path to the initialization file containing your commands.

## Options Available for MuPAD Engine Startup

You can use the following options on the startup of the MuPAD engine.

-g	Start the engine in debug mode. The engine creates debug nodes during the initial read of the MuPAD library. Without this option MuPAD creates the nodes during a debug session (after the first debug call). Then, it writes the information about passes through the nodes to a temporary file.
-v	Start the engine in the debug mode. MuPAD displays more detailed debug information in the output pane of the Debugger window.
-f	Suppress reading the initialization files. Use this option to test your code without deleting the files or the path specified in the <b>User Init Path</b> field.
-F	Suppress reading all the package files from the path specified in the <b>Package Path</b> field. Use this option to test your code without deleting the files or the path.
-L number	Set the limit of precalculated prime numbers. The engine calculates and stores all prime numbers that are less than the number at startup. The default startup number is 1000000. You can increase this number. The maximum value for this option is 436273009.
-U string	Specify options as strings. See <code>Pref::userOptions</code> for details.

- t Start the engine in the test coverage mode. See `prog::tcov` for details.
- T filename Start the engine in the test coverage mode and export the test coverage information to `filename`. See `prog::tcov` for details.

## Get Version Information

Type of Information You Want	To Get the Information
Version and Release Numbers	From the product, select <b>Help &gt; About MuPAD</b> . Alternatively, call <code>version()</code> or <code>Pref::kernel()</code> for a version number.
32-bit or 64-bit version	From the product, select <b>Help &gt; About MuPAD</b> . Alternatively, call <code>Pref::kernel(BitsInLong)</code> .
Build Number	For the build number of the kernel, call <code>Pref::kernel(BuildNr)</code> . For the build number of the MuPAD library, call <code>buildnumber</code> .

## Use Different Output Modes

### In this section...

“Abbreviations” on page 2-33

“Typeset Math Mode” on page 2-34

“Pretty Print Mode” on page 2-36

“Mathematical Notations Used in Typeset Mode” on page 2-38

### Abbreviations

MuPAD can display results of your calculations using different modes. By default, expressions in outputs use typeset mode and abbreviations:

```
solve(x^3 + x^2 + 1 = 0, x, MaxDegree = 3)
```

$$\left\{ -\frac{1}{9\sigma_1} - \sigma_1 - \frac{1}{3}, \frac{1}{18\sigma_1} + \frac{\sigma_1}{2} - \frac{1}{3} - \frac{\sqrt{3} \left( \frac{1}{9\sigma_1} - \sigma_1 \right) i}{2}, \frac{1}{18\sigma_1} + \frac{\sigma_1}{2} - \frac{1}{3} + \frac{\sqrt{3} \left( \frac{1}{9\sigma_1} - \sigma_1 \right) i}{2} \right\}$$

where

$$\sigma_1 = \left( \frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108} \right)^{1/3}$$

To disable abbreviations, clear **Notebook>Abbreviate Output**:

```
solve(x^3 + x^2 + 1 = 0, x, MaxDegree = 3)
```

$$\left\{ \begin{aligned} & -\frac{1}{9 \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}} - \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3} - \frac{1}{3}, \frac{1}{18 \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}} \\ & + \frac{\left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}}{2} - \frac{1}{3} - \frac{\sqrt{3} \left( \frac{1}{9 \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}} - \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3} \right) i}{2}, \\ & \frac{1}{18 \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}} + \frac{\left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}}{2} - \frac{1}{3} \\ & + \frac{\sqrt{3} \left( \frac{1}{9 \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3}} - \left( \frac{29}{54} - \frac{\sqrt{31} \sqrt{108}}{108} \right)^{1/3} \right) i}{2} \end{aligned} \right\}$$

To enable abbreviations, select **Notebook>Abbreviate Output**.

Alternatively, you can enable and disable abbreviations by using `Pref::abbreviateOutput`. For example, disable abbreviations:

`Pref::abbreviateOutput(FALSE):`

Enable abbreviations:

`Pref::abbreviateOutput(TRUE):`

## Typeset Math Mode

MuPAD can display results of your calculations using different formats. By default, expressions in outputs use typeset mode and abbreviations:

`solve(x^3 + x^2 + 1 = 0, x, MaxDegree = 3)`

$$\left\{ -\frac{1}{9\sigma_1} - \sigma_1 - \frac{1}{3}, \frac{1}{18\sigma_1} + \frac{\sigma_1}{2} - \frac{1}{3} - \frac{\sqrt{3}\left(\frac{1}{9\sigma_1} - \sigma_1\right)i}{2}, \frac{1}{18\sigma_1} + \frac{\sigma_1}{2} - \frac{1}{3} + \frac{\sqrt{3}\left(\frac{1}{9\sigma_1} - \sigma_1\right)i}{2} \right\}$$

where

$$\sigma_1 = \left( \frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108} \right)^{1/3}$$

To disable the typeset mode, clear **Notebook>Typeset Math**:

`solve(x^3 + x^2 + 1 = 0, x, MaxDegree = 3)`

$$\begin{aligned} & \left\{ \begin{array}{l} \frac{1}{9 \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}} - \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}} - \frac{1}{3} \\ \frac{1}{18 \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}} + \frac{\sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}}{2} - \frac{1}{3} - \frac{\sqrt{3}\left(\frac{1}{9 \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}} - \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}\right)i}{2} \\ \frac{1}{18 \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}} + \frac{\sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}}{2} - \frac{1}{3} + \frac{\sqrt{3}\left(\frac{1}{9 \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}} - \sqrt[3]{\frac{29}{54} - \frac{\sqrt{31}\sqrt{108}}{108}}\right)i}{2} \end{array} \right\} \end{aligned}$$





$$\begin{aligned}
 & \left\{ \sqrt[3]{\frac{\sqrt{54} \sqrt{108}}{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}} \right. \\
 & + \frac{\sqrt[3]{\frac{\sqrt{54} \sqrt{108}}{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}}}{2} \\
 & \left. \sqrt{3} \sqrt[3]{\frac{1}{9 \sqrt{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}}}} \right. \\
 & \left. \sqrt[3]{\frac{1}{\sqrt{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}}}} \right. \\
 & + \frac{\sqrt[3]{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}}}{2} \\
 & 18 \sqrt[3]{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}} \\
 & \left. \sqrt{3} \sqrt[3]{\frac{1}{9 \sqrt{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}}}} \right. \\
 & + \frac{\sqrt[3]{\frac{29 \sqrt{31} \sqrt{108} \sqrt[3]{1/3}}{54 \sqrt{108}}}}{2}
 \end{aligned}$$

To disable the pretty print mode, clear **Notebook>Pretty Print:**

```
solve(x^3 + x^2 + 1 = 0, x, MaxDegree = 3)
```

```
{- (1/9)/(29/54 - (1/108)*31^(1/2)*108^(1/2))^^(1/3) - (29/54 - (1/108)
(1/108)*31^(1/2)*108^(1/2))^^(1/3) + (1/2)*(29/54 - (1/108)*31^(1/2)*
1/108)*31^(1/2)*108^(1/2))^^(1/3) - (29/54 - (1/108)*31^(1/2)*108^(1/2)
108^(1/2))^^(1/3) + (1/2)*(29/54 - (1/108)*31^(1/2)*108^(1/2))^^(1/3) -
8^(1/2))^^(1/3) - (29/54 - (1/108)*31^(1/2)*108^(1/2))^^(1/3))*1/2*I}
```

When you copy some part of an output region to an input region, plain text outputs serve best. To obtain plain text outputs, disable both typeset and pretty print modes. When you copy an entire output region to an input region, typeset mode serves best.

To enable the pretty print mode, select **Notebook>Pretty Print**.

Alternatively, you can switch between the pretty print and plain text modes by using PRETTYPRINT. Note that first you must disable the typeset mode. For example, switch to the plain text mode:

```
PRETTYPRINT := FALSE:
```

Switch to the pretty print mode:

```
PRETTYPRINT := TRUE:
```

## Mathematical Notations Used in Typeset Mode

By default, MuPAD displays the output expressions using the Typeset Math mode. In this mode, MuPAD uses standard mathematical notations for special functions. For example:

```
Result:= int(sin(x^2), x)
```

$$\frac{\sqrt{2} \sqrt{\pi} S\left(\frac{\sqrt{2} x}{\sqrt{\pi}}\right)}{2}$$

If you are not familiar with a notation, you can see the corresponding MuPAD command using one of following methods:

- Disable typesetting mode by selecting **Notebook>Typeset Math**.  
Reevaluate the expression containing the unknown notation.
- Copy the output expression and paste it to an input region.
- Set PRETTYPRINT to FALSE, and then use the print command with the option Plain to display the results in plain text mode. For example:

```
PRETTYPRINT := FALSE:  
print(Plain, Result)
```

```
(1/2)*2^(1/2)*PI^(1/2)*fresnelS(2^(1/2)/PI^(1/2)*x)
```

For further computations, set PRETTYPRINT to TRUE.

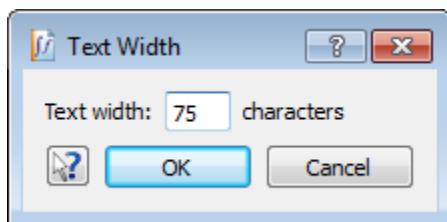
```
PRETTYPRINT := TRUE:
```

Knowing the MuPAD command, you can access the corresponding help page and get more information about the special function.

## Set Line Length in Plain Text Outputs

To see results in the plain text format, disable the typeset mode for outputs. By default, MuPAD limits lines in plain text outputs to 75 symbols. To change this setting for your current notebook:

- 1 Select **Notebook>Text Width**
- 2 In the **Text Width** dialog box, specify the line length limit.



Alternatively, assign the new value to the environment variable `TEXTWIDTH`:

```
TEXTWIDTH := 15:
```

To restore the setting to its default value, delete `TEXTWIDTH`:

```
delete TEXTWIDTH
```

MuPAD applies the new setting to all new outputs in a notebook. To apply the setting to existing output regions, re-evaluate the appropriate input regions. See “Evaluate Mathematical Expressions and Commands” on page 1-4.

To change the default line length limit for the current and all new notebooks, see Setting Preferences for Notebooks.

## Delete Outputs

To clear a particular output region in your notebook:

**1** Click the output region you want to delete or click the adjacent input region.

**2** Select **Edit>Delete Output**

To clear all the outputs in your current notebook, select **Edit>Delete All Outputs**.

## **Greek Letters in Text Regions**

You can convert characters in the text region to Greek letters one at a time:

- 1** Select the character you want to convert or place the cursor to the right of the character.
- 2** Select **Edit>Toggle Greek**.

To get the original font style use the same steps.

## Special Characters in Outputs

To produce special characters including greek letters in the output regions, use the `Symbol` command:

```
Symbol::Omega;  
Symbol::subScript(Symbol::omega, 0) + Symbol::omega*t;  
Symbol::subScript(M, Symbol::bigodot);  
Symbol::alpha;  
Symbol::LeftArrow;  
Symbol::ForAll
```

$\Omega$

$\omega_0 + \omega t$

$M$   
 $\odot$

$\alpha$

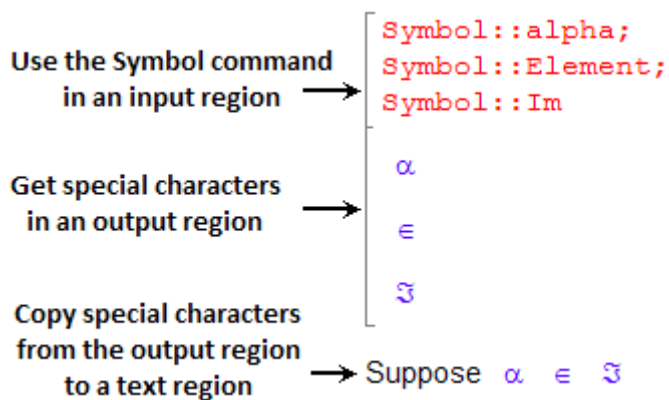
$\leftarrow$

$\forall$

## Non-Greek Characters in Text Regions

You can insert greek letters into text regions. To use other special characters in text regions:

- 1 Produce special characters in an output region.
- 2 Copy the characters from the output region and paste them into a text region.





## Use Keyboard Shortcuts

Using shortcut keys for your platform, you can access many of the desktop menu items. These shortcut keys are sometimes called accelerators or hot keys. For example, use the **Ctrl+X** shortcut to perform a cut on Microsoft® Windows platforms. Many of the menu items show the shortcuts. Additional standard shortcuts for your platform usually work, but only one is listed with each menu item.

Instructions in the documentation specify shortcuts using the key convention for Windows platforms, **Ctrl+**. With key bindings for Apple Macintosh platforms selected, use the **Command** key instead of the **Ctrl** key. On the Macintosh platform, to make full use of all keyboard shortcuts, you might need to enable full keyboard access. To access this option, select **Apple menu>System Preferences**, and click **Keyboard & Mouse**. Click **Keyboard Shortcuts**, and select **Turn full keyboard access on or off**.

In addition to common shortcuts, MuPAD provides special shortcuts to navigate within matched delimiters (parenthesis, brackets, braces).

On Microsoft Windows and Linux platforms:

- The **Ctrl+Alt+Shift+RIGHT** (the right arrow key) shortcut moves the cursor to the closing (right) delimiter.
- The **Ctrl+Alt+Shift+LEFT** (the left arrow key) shortcut moves the cursor to the opening (left) delimiter.

On Apple Macintosh platforms:

- The **Ctrl+Alt+RIGHT** (the right arrow key) shortcut moves the cursor to the closing (right) delimiter.
- The **Ctrl+Alt+LEFT** (the left arrow key) shortcut moves the cursor to the opening (left) delimiter.

For the pair of mismatched delimiters, you can correct the closing delimiter. To replace the closing delimiter with the one that matches the opening delimiter, place the cursor to the right of the closing delimiter and press the **Tab** key.

## Use Mnemonics

Using mnemonics, you can access menu items and buttons. Mnemonics are underlined on the menu item or button. For example, on the **File** menu, the **F** in **File** is underlined, which indicates that **Alt+F** opens the menu.

The Macintosh platform does not support mnemonics.

Some versions of Windows operating system do not automatically show the mnemonics on the menu. For example, you might need to hold down the **Alt** key while the tool is selected to see the mnemonics on the menus and buttons. Use the Windows Control Panel to set preferences for underlining keyboard shortcuts. See the Windows documentation for details.

## Overview

MuPAD provides functionality for creating electronic or printed documents, class notes, textbooks, and interactive presentations. Notebooks support the following options to help you design appealing and flexible documents:

- Separate formatting styles for commands in the input regions, text and mathematical formulas in the text regions
- Default formatting styles
- Different modes for displaying the results of your calculations
- Embedded graphics
- Links, tables, and frames
- Invisible startup commands
- Quick modifications of the font and graphics sizes during the presentation
- Export of your notebooks to PDF and HTML formats
- Export of your graphic results separately from notebooks

## Wrap Long Lines

In this section...
“Wrap Text” on page 2-48
“Wrap Expressions in Input Regions” on page 2-51
“Wrap Output Expressions” on page 2-53

### Wrap Text

To wrap text to a notebook window size, select **View > Wrap To Window**. If you use text wrapping and resize your notebook, MuPAD automatically adjusts text lines to a new window size. This option affects text regions only.

Notebook1\* - MuPAD

File Edit View Navigation Insert Format Notebook Window Help

Generic Sans Serif

The vertical dashed gray line shows the right margin of the printed page. When you print this notebook, MuPAD offers you to scale it down to fit the printer setting for a page size. If you use the actual page size, only the part of the notebook to the left of the vertical line appears on printed pages.

```
solve(300000*x^4 + 123456*x^3 + x^2 + 1000*x + 123456789 = 0, x, MaxDegree = 4)
```

$$\left\{ -\sigma_1 - \sigma_3 - \frac{643}{6250}, -\sigma_1 + \sigma_3 - \frac{643}{6250}, \sigma_1 - \sigma_2 - \frac{643}{6250}, \sigma_1 + \sigma_2 - \frac{643}{6250} \right\}$$

where

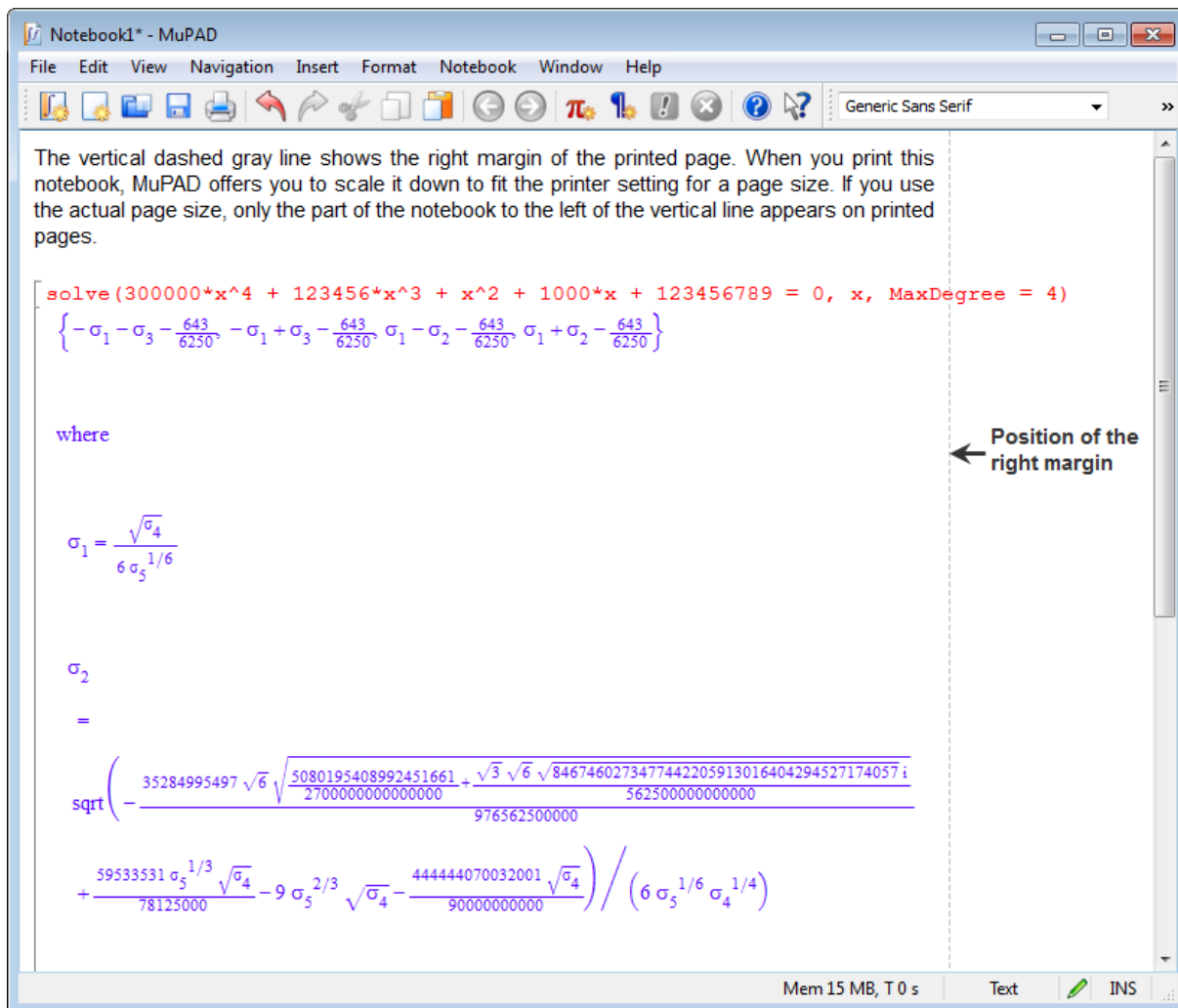
$$\sigma_1 = \frac{\sqrt{\sigma_4}}{6 \sigma_5^{1/6}}$$

$$\sigma_2 = \text{sqrt} \left( -\frac{35284995497 \sqrt{6} \sqrt{\frac{5080195408992451661 + \sqrt{3} \sqrt{6} \sqrt{8467460273477442205913016404294527174057 i}}{2700000000000000}}}{976562500000} + \frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4} - 9 \sigma_5^{2/3} \sqrt{\sigma_4} - \frac{444444070032001 \sqrt{\sigma_4}}{90000000000}}{(6 \sigma_5^{1/6} \sigma_4^{1/4})} \right)$$

$$\sigma_3 = \text{sqrt} \left( \frac{35284995497 \sqrt{6} \sqrt{\frac{5080195408992451661 + \sqrt{3} \sqrt{6} \sqrt{8467460273477442205913016404294527174057 i}}{2700000000000000}}}{976562500000} + \frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4}}{78125000} \right)$$

Mem 15 MB, T 0 s Text INS

When you wrap text in a notebook, and then unwrap it, the vertical line appears. This line shows you the position of the right margin of the current page format.

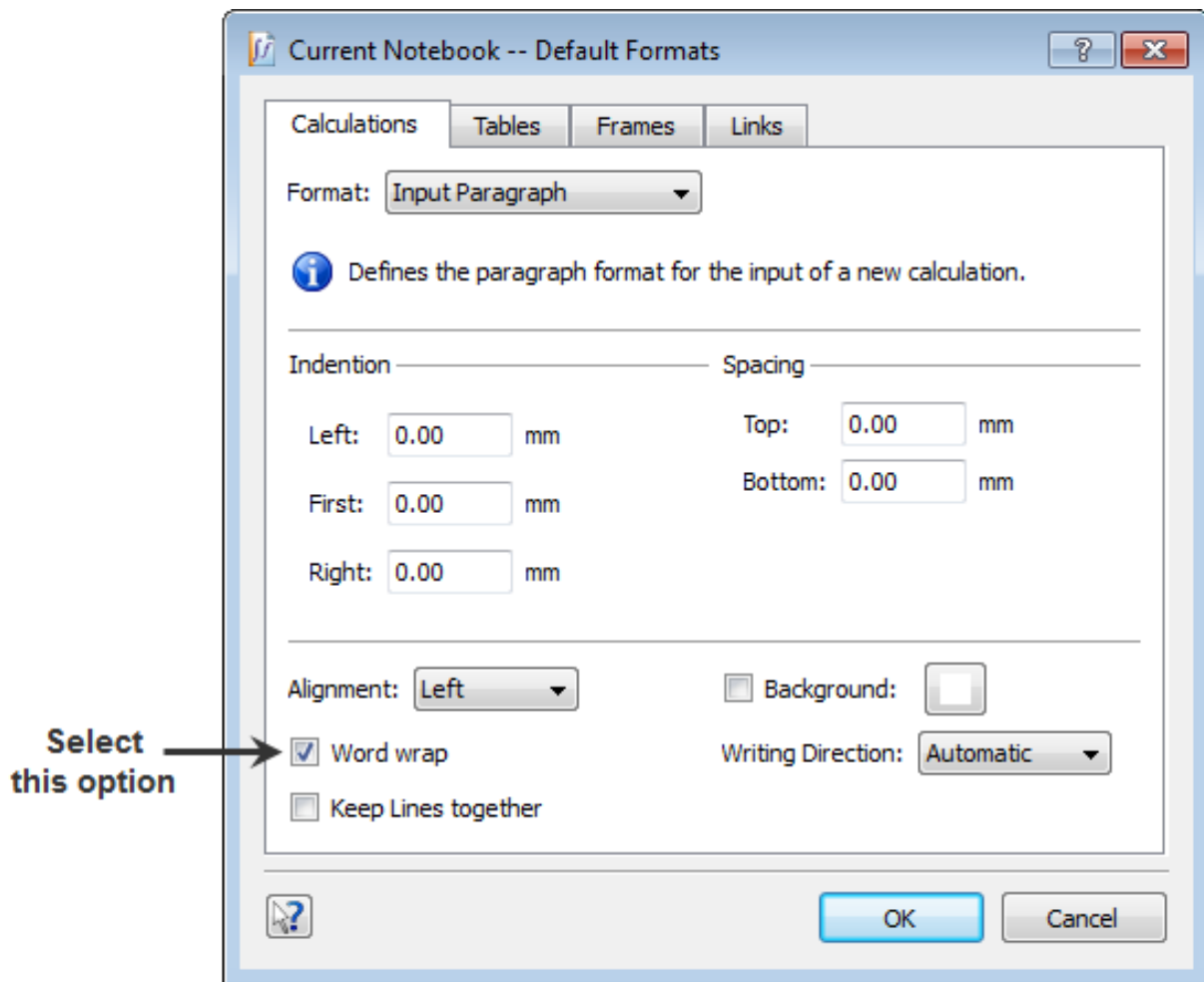


When you print the page, MuPAD lets you choose between scaling down the whole page or cropping the content to the right of the line. The line does not appear on printed pages. To remove this line, select **View > Wrap To Window**.

## Wrap Expressions in Input Regions

To wrap the contents of the input regions to a notebook window size:

- 1** Select **Format > Defaults**.
- 2** In the resulting dialog box, click the **Calculations** tab.
- 3** From the drop-down menu **Format**, select **Input Paragraph**.
- 4** Select the **Word wrap** check box.



Now MuPAD wraps all new expressions and commands in the input regions to a notebook window size. If you use wrapping for input regions and then resize your notebook, MuPAD automatically adjusts expressions and commands to a new window size.



The vertical dashed gray line shows the right margin of the printed page. When you print this notebook, MuPAD offers you to scale it down to fit the printer setting for a page size. If you use the actual page size, only the part of the notebook to the left of the vertical line appears on printed pages.

```
solve(300000*x^4 + 123456*x^3 + x^2 + 1000*x + 123456789 = 0, x,
MaxDegree = 4)
```

$$\left\{ -\sigma_3 - \sigma_2 - \frac{643}{6250}, -\sigma_3 + \sigma_2 - \frac{643}{6250}, \sigma_3 - \sigma_1 - \frac{643}{6250}, \sigma_3 + \sigma_1 - \frac{643}{6250} \right\}$$

where

$$\sigma_1 = \sqrt{\frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4} - 9 \sigma_5^{2/3} \sqrt{\sigma_4}}{78125000} - \frac{12764568061 \sqrt{6} \sqrt{\frac{205747883139494449 + \sqrt{3} \sqrt{6} \sqrt{1998184731169244615496586749324587 i}}{135000000000000000}}}{488281250000} - \frac{3599999629633 \sqrt{\sigma_4}}{90000000000} \Big/ (6 \sigma_5^{1/6} \sigma_4^{1/4})}$$

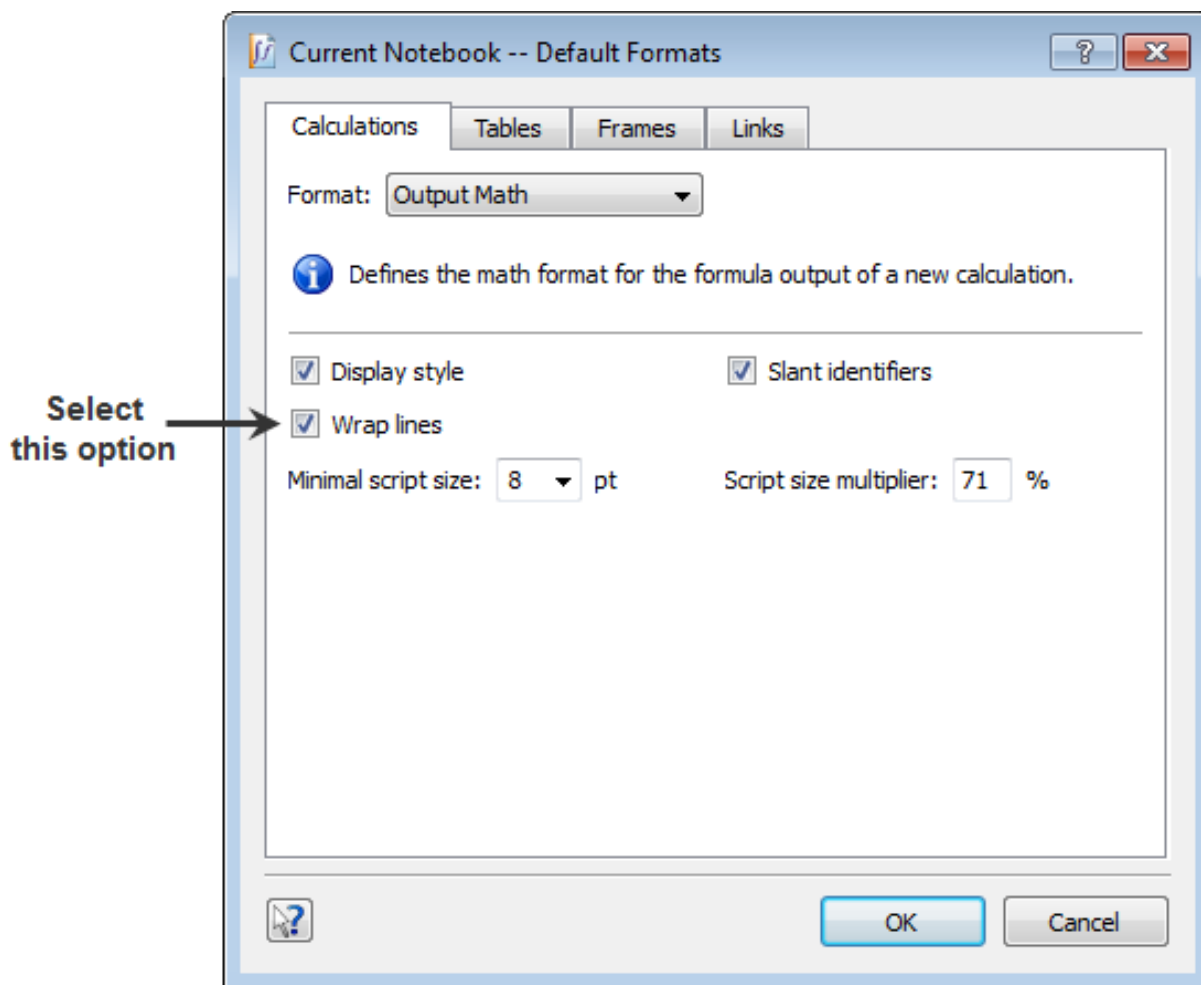
$$\sigma_2 = \sqrt{\frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4} - 9 \sigma_5^{2/3} \sqrt{\sigma_4}}{78125000} - \frac{12764568061 \sqrt{6} \sqrt{\frac{205747883139494449 + \sqrt{3} \sqrt{6} \sqrt{1998184731169244615496586749324587 i}}{135000000000000000}}}{488281250000} - \frac{3599999629633 \sqrt{\sigma_4}}{90000000000} \Big/ (6 \sigma_5^{1/6} \sigma_4^{1/4})}$$

Mem 15 MB, T 0 s    Cmd    INS

## Wrap Output Expressions

By default, MuPAD wraps results displayed in the output regions to a notebook window size. The system also wraps long lines in outputs when printing them. When wrapping breaks a number, an identifier, or a string, MuPAD inserts the line break symbol `\`. For example, the following result





If you disable wrapping, MuPAD does not insert line breaks in the new output expressions. If you want to remove line breaks in the existing output expressions, reevaluate these expressions.

ni-present-publish-notebook-wrap-text\* - MuPAD

File Edit View Navigation Insert Format Notebook Window Help

Generic Sans Serif

The vertical dashed gray line shows the right margin of the printed page. When you print this notebook, MuPAD offers you to scale it down to fit the printer setting for a page size. If you use the actual page size, only the part of the notebook to the left of the vertical line appears on printed pages.

```
solve(300000*x^4 + 123456*x^3 + x^2 + 1000*x + 123456789 = 0, x, MaxDegree = 4)
```

$$\left\{ -\sigma_3 - \sigma_2 - \frac{643}{6250}, -\sigma_3 + \sigma_2 - \frac{643}{6250}, \sigma_3 - \sigma_1 - \frac{643}{6250}, \sigma_3 + \sigma_1 - \frac{643}{6250} \right\}$$

where

$$\sigma_1 = \frac{\sqrt{-\frac{35284995497 \sqrt{6} \sqrt{\frac{5080195408992451661 + \sqrt{3} \sqrt{6} \sqrt{8467460273477442205913016404294527174057} i}}{2700000000000000}} + \frac{5625000000000000}{976562500000} + \frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4} - 9 \sigma_5^{2/3}}{78125000}}{6 \sigma_5^{1/6} \sigma_4^{1/4}}$$

$$\sigma_2 = \frac{\sqrt{\frac{35284995497 \sqrt{6} \sqrt{\frac{5080195408992451661 + \sqrt{3} \sqrt{6} \sqrt{8467460273477442205913016404294527174057} i}}{2700000000000000}} + \frac{5625000000000000}{976562500000} + \frac{59533531 \sigma_5^{1/3} \sqrt{\sigma_4} - 9 \sigma_5^{2/3}}{78125000}}{6 \sigma_5^{1/6} \sigma_4^{1/4}}$$

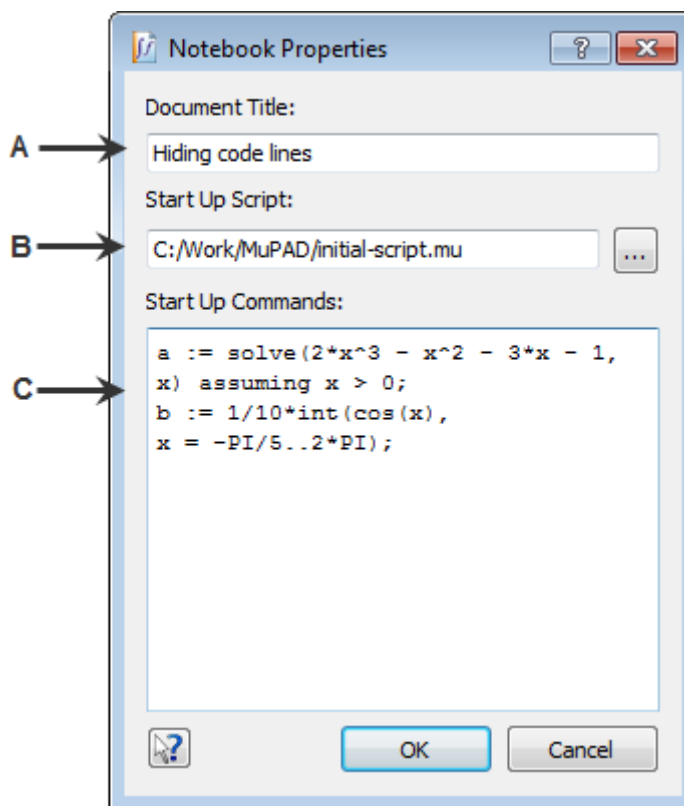
$$\sigma_3 = \frac{\sqrt{\sigma_4}}{6 \sigma_5^{1/6}}$$

Mem 7 MB, T 0 s    Text    INS

## Hide Code Lines

You can run MuPAD commands without displaying them in a notebook. MuPAD evaluates these commands every time you start a notebook engine. To specify the commands you want to execute invisibly:

- 1 Select **File>Properties**.
- 2 In the **Start Up Commands** field of the **Notebook Properties** dialog box, specify the commands you want to run without displaying in the notebook. Alternatively, attach an existing file containing the commands.



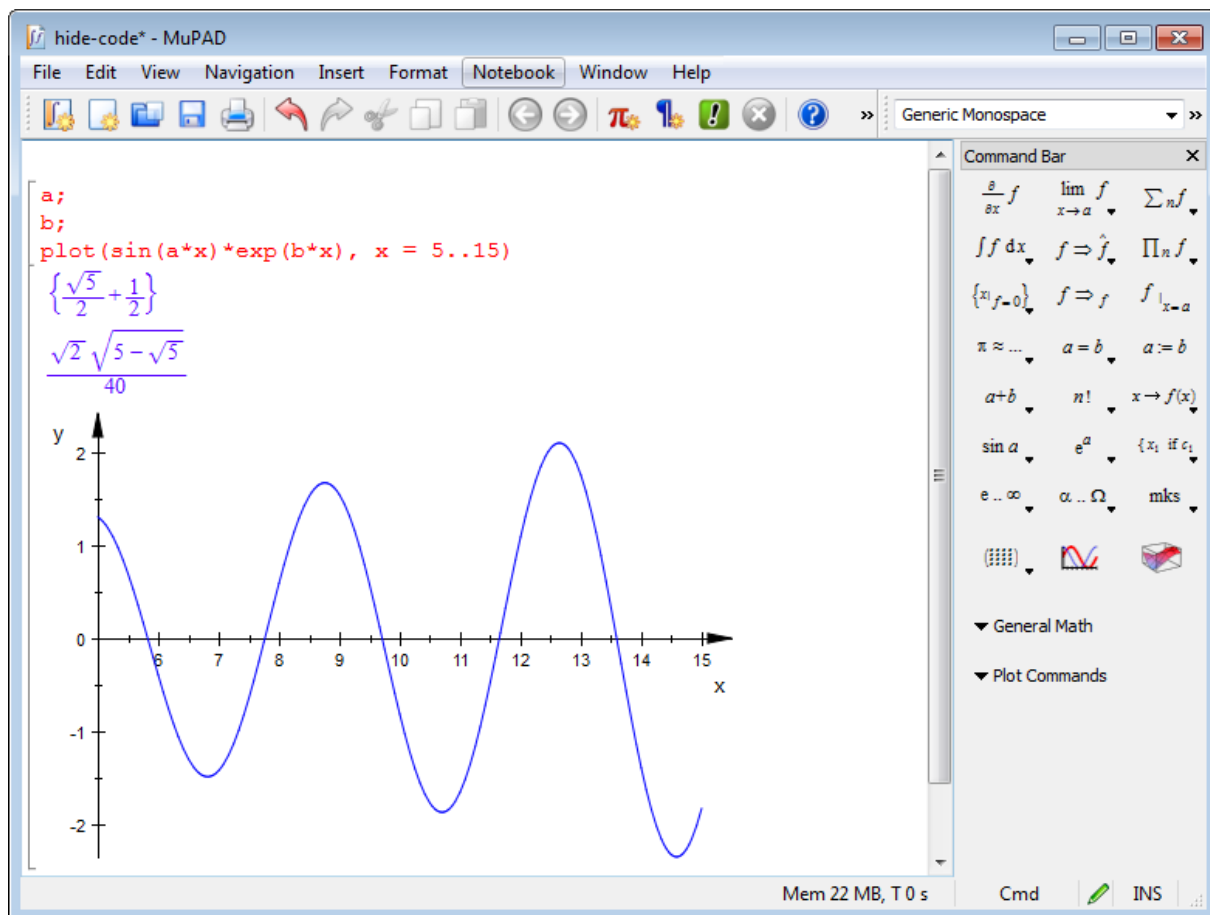
- A MuPAD uses this title only when exporting a notebook to HTML.

- **B** Attach an existing file with the MuPAD script that you want to execute invisibly at the notebook startup.
  - **C** Type the commands that you want to execute invisibly at the notebook startup.
- 3** Save the notebook. MuPAD saves the hidden code with the notebook. This code does not affect other notebooks. If you want to use hidden code for all notebooks, see Adding Hidden Startup Commands to All Notebooks.

To execute the commands you entered, restart the notebook engine. To restart an engine you can use one of the following methods:

- Select **Notebook>Disconnect**, and then **Notebook>Start Engine**.
- Close the notebook and reopen it.

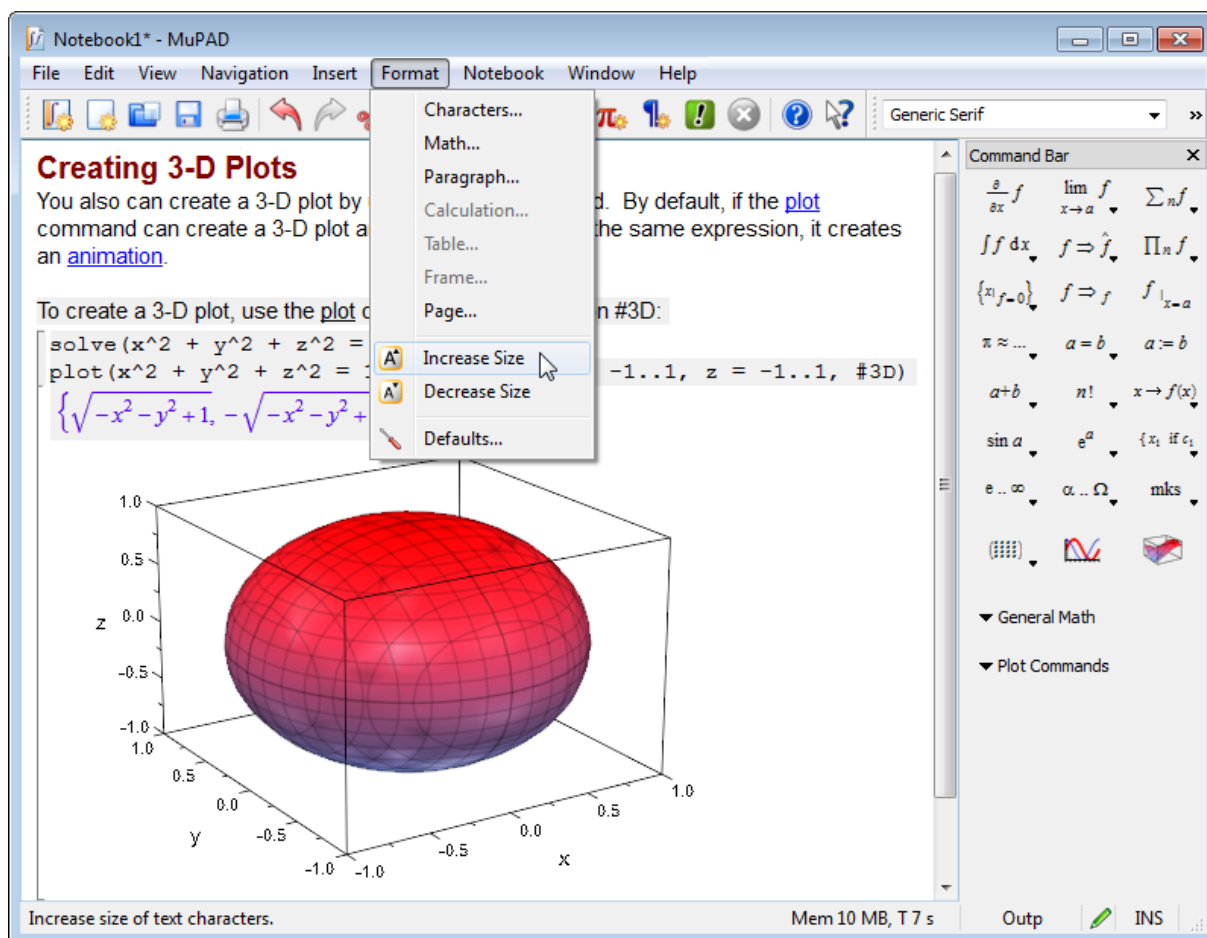
After you restart an engine, you can access all the objects defined in the **Notebook Properties** dialog box from your notebook.



## Change Font Size Quickly

To quickly resize all the characters in your notebook including the characters in the output regions:

- Select the part of a notebook you want to resize. If you want to resize fonts in a whole notebook, select **Edit>Select All**.
- Select **Format>Increase size** or **Decrease size**.





**Note** Graphics size does not change.

To change graphics size, see Scaling Graphics.

The screenshot shows the MuPAD Notebook1 interface. The main window displays a document titled "Notebook1\* - MuPAD" with a menu bar (File, Edit, View, Navigation, Insert, Format, Notebook, Window, Help) and a toolbar. The document content includes a section titled "Creating 3-D Plots" with the following text:

You also can create a 3-D plot by using the [plot](#) command. By default, if the [plot](#) command can create a 3-D plot and a 2-D animation for the same expression, it creates an [animation](#).

To create a 3-D plot, use the [plot](#) command with the option #3D:

```
solve(x^2 + y^2 + z^2 = 1, z);
plot(x^2 + y^2 + z^2 = 1, x = -1..1, y = -1..1, z = -
```

$$\left\{ \sqrt{-x^2 - y^2 + 1}, -\sqrt{-x^2 - y^2 + 1} \right\}$$

Below the code is a 3D plot of a red sphere with a grid, centered at the origin of a 3D coordinate system. The axes are labeled x, y, and z, with tick marks at 0.0, 0.5, and 1.0. The plot is contained within a 3D wireframe box.

On the right side of the interface is a "Command Bar" with a close button (X). It contains various mathematical symbols and functions, including:

- $\frac{\partial}{\partial x} f$ ,  $\lim_{x \rightarrow a} f$ ,  $\sum_n f$
- $\int f dx$ ,  $f \Rightarrow \hat{f}$ ,  $\prod_n f$
- $\{x|_{f=0}\}$ ,  $f \Rightarrow f$ ,  $f|_{x=a}$
- $\pi \approx \dots$ ,  $a = b$ ,  $a := b$
- $a + b$ ,  $n!$ ,  $x \rightarrow f(x)$
- $\sin a$ ,  $e^a$ ,  $\{x_i \text{ if } c_i\}$
- $e \dots \infty$ ,  $\alpha \dots \Omega$ ,  $\text{mks}$

Below the Command Bar are sections for "General Math" and "Plot Commands". At the bottom of the window, there is a status bar showing "Mem 10 MB, T 7 s" and buttons for "Outp", "INS", and a red arrow icon.

To undo font size changes, select **Edit>Undo** or use the toolbar button .

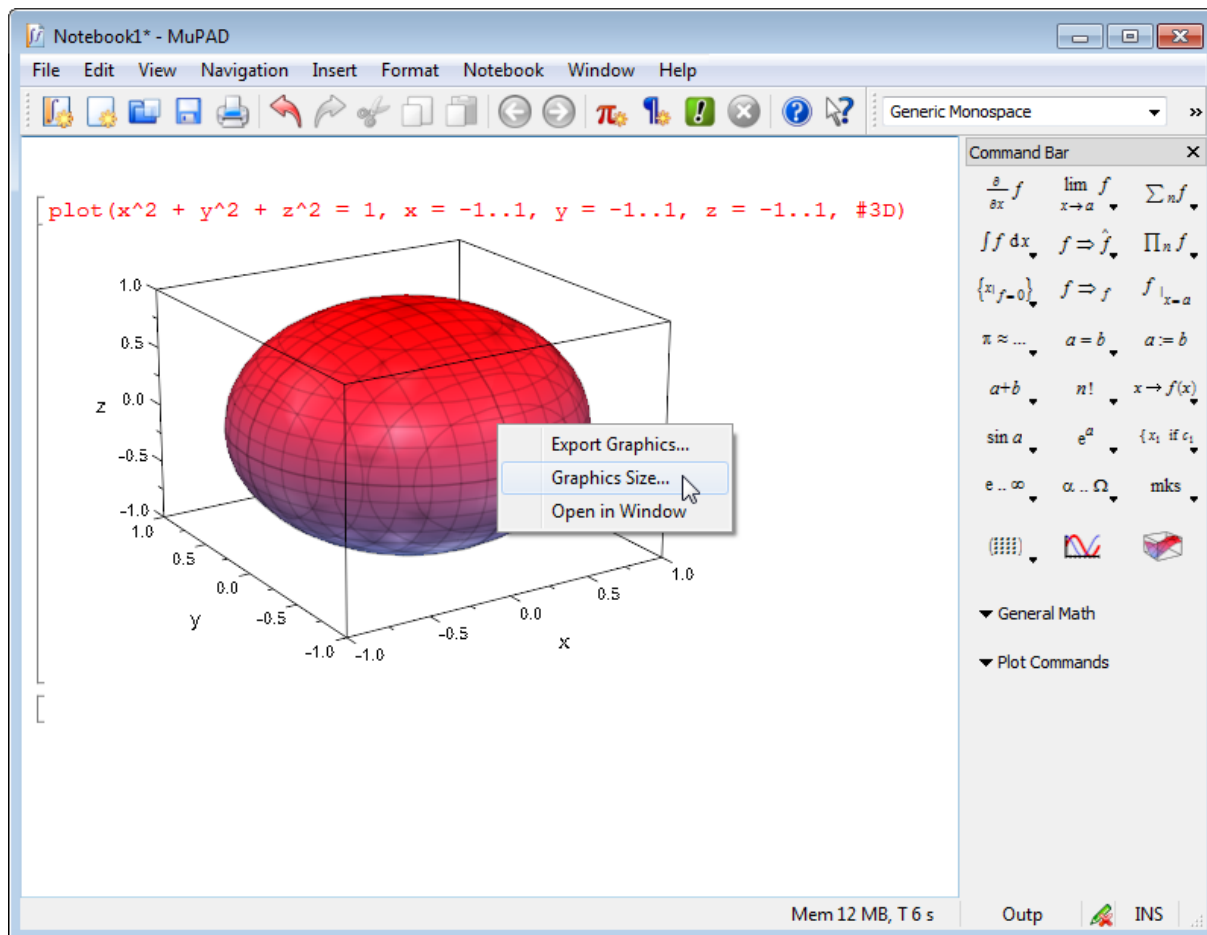
---

**Note** Using the opposite option (such as increasing the fonts you have decreased before) does not guarantee to restore the original font size.

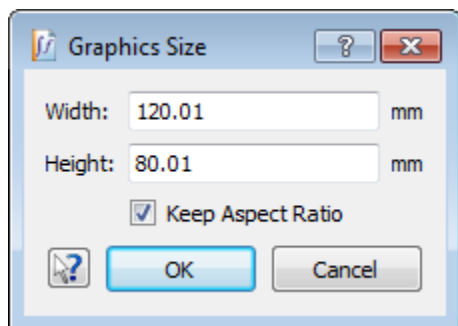
---

## Scale Graphics

To resize your graphics, right-click the graphics and select **Graphics Size**.



In the Graphics Size dialog box, set the height and width of the graphics. The option **Keep Aspect Ratio** lets you conserve the height to width ratio.



## Use Print Preview

### In this section...

“View Documents Before Printing” on page 2-65

“Print Documents from Print Preview” on page 2-65

“Save Documents to PDF Format” on page 2-66

“Get More Out of Print Preview” on page 2-67

### View Documents Before Printing

Before printing a notebook that displays long lines in the input or output regions or wide graphics that might not fit the page, use the Print Preview window to see how the document will look when printed. The dialog box lets you view and fix page layout problems, print your document, or save it to PDF format. Previewing documents before you print them helps to avoid printing unnecessary pages and thereby reduces paper waste.

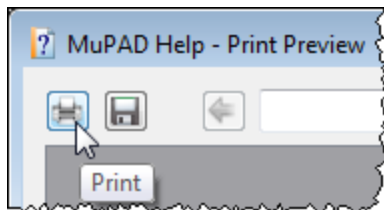
To see how the document looks when printed, select **File > Print Preview**. Alternatively, press **Alt+F+V** to open Print Preview.

Print Preview uses the properties of your current printer. If you print the document using any other printer or save it to PDF format, the result can differ from what you see in the Print Preview window.

To close Print Preview, click the **Close** button on the Print Preview toolbar.

### Print Documents from Print Preview

Suppose, you like the way your document appears in the Print Preview window and want to print the document. To print a document directly from Print Preview, click the Print button on the Print Preview toolbar.



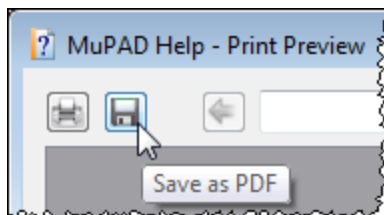
When you print directly from Print Preview, you can select and print specific pages. To print a particular page:

- 1 Open Print Preview.
- 2 Click the Print button in the Print Preview toolbar.
- 3 In the resulting dialog box, click the **Pages** option under **Page Range**.
- 4 Type the number of the page you want to print. To specify a range of pages, use a hyphen. For example, to print the second, third, and fourth pages, type 2-4.

If your document is wide and does not fit the page, use the Print Preview toolbar to adjust the document before printing. When you print a wide document from the Print Preview window, MuPAD does not prompt you to scale your document down. MuPAD prints documents exactly as you see them in the Print Preview window.

### Save Documents to PDF Format

If you like how your document appears in the Print Preview window, you can save it to PDF format without leaving Print Preview. To save a document as a PDF from the Print Preview window, click the Save as PDF button on the Print Preview toolbar.



PDF files created with the Save as PDF button are not editable. To create editable PDF files, click the Print button and try using a PDF printer available for your system.

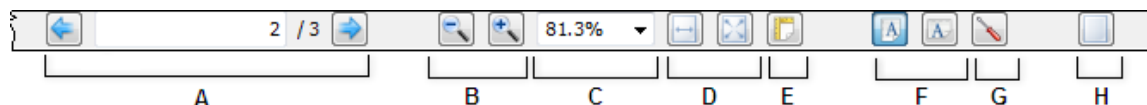
---



**Note** If a MuPAD document has links, these links are replaced by regular text in the resulting PDF file.

---

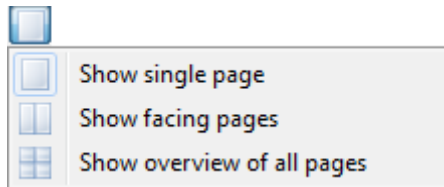
## Get More Out of Print Preview

Using the Print Preview toolbar, you can view one or more pages, zoom in and out, and switch between page orientations. If Print Preview shows that your document has layout problems visible when printed, you can fix these problems without leaving the Print Preview window. To see what a particular button lets you do, hover the pointer over the button and wait for a tooltip to appear.



- **A** Indicates the current page number. Use this field to jump to a particular page. Click the appropriate arrow to display the previous or the next page.
- **B** Lets you zoom in and out on a page.
- **C** Displays the current zoom factor. Use this field to zoom in or out by a fixed percentage. The drop-down menu lets you quickly select one of the commonly used zoom factors.
- **D** Fits the page to the Print Preview window. The button  adjusts the document so the width of the page matches the width of the Print Preview window. The button  adjusts the document so that an entire page fits in the Print Preview window. If you display two pages side-by-side, the Fit buttons adjust the document so that both pages fit in the window.
- **E** Scales the document so that all objects including graphics and calculation regions fit in the page width.
- **F** Selects page orientation. Lets you choose portrait or landscape layout.

- **G** Opens the Page Format dialog box for adjusting page settings. When you modify settings in the Format dialog box, MuPAD applies the new settings not only to a preview, but also to the notebook itself. If later you close the Print Preview window and save the notebook, MuPAD saves the new page settings with the notebook. See [Changing Page Settings for Printing](#) for details.
- **H** Lets you view multiple pages in the Print Preview window.



The option **Show facing pages** displays even pages on the left and odd pages on the right. If you select this option to display a document with multiple pages, the first page appears in the top-right corner.



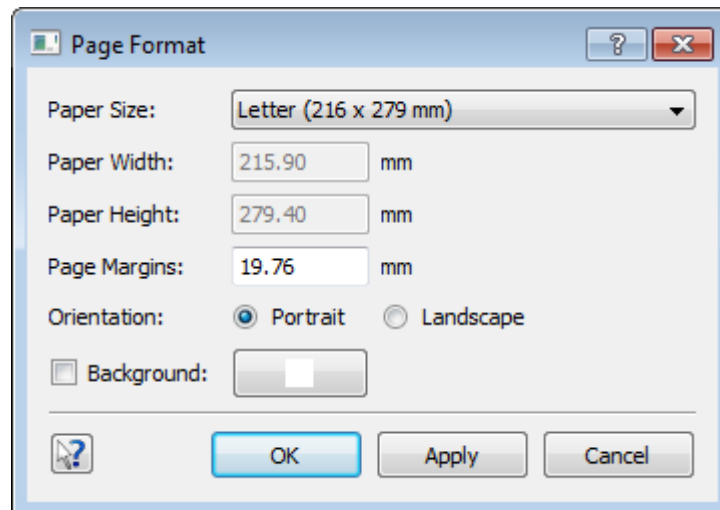
## Change Page Settings for Printing

If you do not specify page format, MuPAD uses the page settings of your default printer. If you do not install a printer, the default page size is A4. Note that A4 is narrower and longer than the Letter paper size.

<b>A4</b>	<b>Letter</b>
8.3 X 11.7 in	8.5 X 11 in
210 X 297 mm	216 X 279 mm

To change the page format for your notebook:

- 1 Select **Format>Page**
- 2 In the **Page Format** dialog box, specify paper size, margin size, orientation, and background color for your pages.



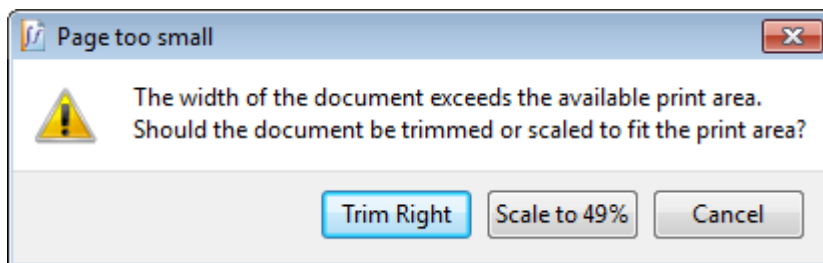
MuPAD saves your page settings with a notebook. These settings do not affect other existing or new notebooks.

## Print Wide Notebooks

Before printing a notebook that displays wide graphics or long lines in the input or output regions, use Print Preview to see how the content fits the page. Print Preview makes it easier to view and adjust wide documents. Alternatively, wrap and then unwrap text in a notebook by using the option **View > Wrap To Window**. When you wrap text in a notebook, and then unwrap it, the vertical line appears. This line shows you the position of the right margin of the current page format.

When you print a notebook that does not fit the page format, MuPAD prompts you to select one of the following options:

- Print only the part of the notebook to the left of the margin.
- Scale the whole notebook to the size that fits the page format.



Also, you can change the page format.

When you print a wide document from the Print Preview window, the above dialog box does not appear. Use the Print Preview toolbar to adjust the document before printing. MuPAD prints documents exactly as you see them in the Print Preview window.

# Mathematics

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## Evaluations in Symbolic Computations

Evaluation is one of the most common mathematical operations. Therefore, it is important to understand how and when MuPAD performs evaluations. For example, assign the value  $2 + 2$  to the variable  $y$ . Instead of assigning the expression  $2 + 2$ , MuPAD *evaluates* this expression, and assigns the result of the evaluation, the value 4, to the variable  $y$ :

```
y := 2 + 2: y
```

```
4
```

The variable  $y$  is an identifier, and the number 4 is the value of that identifier. Values of identifiers are not always numbers. For example, a value of an identifier can also contain identifiers. In the following assignment,  $y$  is an identifier, and the expression  $a + x$  is the value of that identifier:

```
y := a + x
```

```
a+x
```

The value of  $y$  is a sum of two identifiers,  $a$  and  $x$ . You can assign a value to any of these identifiers. For example, assign the value 10 to the identifier  $a$ . Now, MuPAD recognizes that  $a$  is equal to 10. Therefore, the system evaluates the value  $a + x$  of the identifier  $y$  to the expression  $x + 10$ :

```
a := 10: y
```

```
x+10
```

---

**Note** The value of an identifier is the value computed at the time of assignment.

---

The value of the identifier  $y$  is still  $x + a$ . If you assign any other value to  $a$ , MuPAD evaluates  $y$  using this new value:

```
a := 15: y
```

```
x+15
```

Now, assign the value 10 to the identifier `a`, and then assign the expression `x + a` to `y`. As in the previous example, MuPAD evaluates the identifier `y` and returns the expression `x + 10`:

```
a := 10:
```

```
y := a + x: y
```

```
x+10
```

Although the evaluation returns the same result as in the previous example, the value of `y` is different. Here the value of `y` is the expression `x + 10`. This value does not depend of the identifier `a`:

```
a := 15: y
```

```
x+10
```

For further computations, clear the identifiers `a`, `x`, and `y`:

```
delete a, x, y
```

The value of an identifier can be any MuPAD object. For example, the value of an identifier can be a list:

```
list := [x^k $ k = 1..10]
```

```
[x, x2, x3, x4, x5, x6, x7, x8, x9, x10]
```

If later you assign the value to `x`, the evaluation of the identifier `list` changes accordingly:

```
x := 1/2: list
```



$$\left[ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024} \right]$$

MuPAD applies the same evaluation mechanism to function names. For example, assign the function call  $f(\pi)$  to the identifier  $y$ :

```
y := f(PI)
```

$$f(\pi)$$

Now, assign the function `sin` to  $f$ . If you evaluate the identifier  $y$ , the system replaces the identifier  $f$  by its value `sin`. Then, the system evaluates the call `sin(PI)` and returns 0:

```
f := sin: y
```

$$0$$

If you change or delete the value of  $f$ , the evaluation of  $f(\pi)$  changes accordingly:

```
f := cos: y
```

$$-1$$

```
delete f: y
```

$$f(\pi)$$

## Level of Evaluation

### In this section...

“What Is an Evaluation Level?” on page 3-8

“Incomplete Evaluations” on page 3-9

“Control Evaluation Levels” on page 3-12

### What Is an Evaluation Level?

The value of an identifier can contain arbitrary MuPAD objects, including identifiers. If the value of an identifier contains another identifier, MuPAD tries to find the value of that second identifier. If the value of that second identifier also contains identifiers, MuPAD tries to find their values, and so on. Typically, evaluation continues until the system replaces all identifiers that have values by these values. The final result can contain identifiers that do not have assigned values. This recursive evaluation process is called a *complete evaluation*. Each evaluation step in this recursive process is an evaluation level. For example, evaluate the value of the identifier  $y$ :

```
y := a + x: a := 1: y
```

```
x+1
```

The resulting expression  $x + 1$  is the complete evaluation of  $y$ . The `level` function demonstrates each step of this recursive evaluation. The zero level of evaluation returns the identifier  $y$  itself:

```
level(y, 0)
```

```
y
```

The first level accesses the value of the identifier, and returns that value:

```
level(y, 1)
```

```
a+x
```

When you evaluate  $y$  up to the second level, the system recognizes that the expression  $x + a$  contains identifiers, which can also have assigned values. When searching for these values, the system finds that the identifier  $a$  has the value 1, and the identifier  $x$  does not have an assigned value:

```
level(y, 2)
```

```
x+1
```

In this example, MuPAD completely evaluates the identifier  $y$  by using just two evaluation steps. Evaluating  $y$  up to the third and higher levels returns the same expression:

```
level(y, 3)
```

```
x+1
```

```
delete a, x, y
```

## Incomplete Evaluations

MuPAD does not always evaluate identifiers completely. For some expressions, a complete evaluation requires a huge number of steps. To avoid very long or infinite evaluations, the system implements two environment variables, `LEVEL` and `MAXLEVEL`. These variables limit evaluation levels. If the current evaluation level exceeds the limitation set by one these variables, MuPAD stops the evaluation process before the system can replace all identifiers by their assigned values.

The environment variable `LEVEL` limits evaluation levels to a specified value. It does not try to detect and prevent an infinite evaluation loop. For interactive computations, the default value of the environment variable `LEVEL` is:

```
LEVEL
```

```
100
```

When the evaluation level reaches the value of `LEVEL`, MuPAD stops the evaluation and returns the result of the last computed evaluation step:

```
LEVEL := 10:
x := x + 1: x
```

```
x+10
```

```
delete LEVEL, x
```

MuPAD does not specify one uniform value of `LEVEL` for all computations. For most computations, the value is 100, but there are exceptions to this rule:

- If the evaluation occurs in a procedure, MuPAD limits the evaluation level to 1.
- If the evaluation occurs in a matrix, MuPAD limits the evaluation level to 1.
- MuPAD does not evaluate arrays, tables, and polynomials. (The evaluation level for these objects is 0.)
- MuPAD does not evaluate a returned value of the `last()` function call or its equivalent `%`. (The evaluation level is 0.)
- MuPAD does not evaluate returned values of some other system functions. For example, the system does not evaluate the results returned by the `subs` and `text2expr` functions. The help pages for such functions provide the information about the evaluation levels of the returned values.
- If the evaluation occurs in a function call `level(expression, n)`, MuPAD disregards the environment value `LEVEL`. Instead, the system uses the evaluation level `n`.

For example, although `LEVEL = 100` by default, the function call `level(a + x, 1)` evaluates the expression `a + x` to the first evaluations level:

```
a := b: b := 2: level(a + x, 1)
```

```
b+x
```

```
delete a, b, x
```

For more examples of incomplete evaluations and information about enforcing such evaluations, see [Enforcing Evaluation](#).

To detect and prevent infinite loops, MuPAD implements another environment variable, `MAXLEVEL`. The default value of `MAXLEVEL` for all computations is

```
MAXLEVEL
```

```
100
```

When evaluation level reaches the value of `MAXLEVEL`, MuPAD assumes that the evaluation is infinite and issues an error:

```
MAXLEVEL := 2:  
a := b: b := c: c := d: a
```

```
Error: Recursive definition. [See ?MAXLEVEL]
```

```
delete MAXLEVEL, a, b, c, d
```

If the value of `MAXLEVEL` is greater than the value of `LEVEL`, the global variable `MAXLEVEL` does not affect that evaluation. Otherwise, the value of `MAXLEVEL` limits the number of evaluation steps. For example, the default values of `LEVEL` and `MAXLEVEL` are equal (both values are 100). If an evaluation reaches the level 100, MuPAD uses the global variable `MAXLEVEL` and, therefore, issues an error:

```
x := x + 1: x
```

```
Error: Recursive definition. [See ?MAXLEVEL]
```

```
delete x
```

## Control Evaluation Levels

You can change the values of the environment variables `LEVEL` and `MAXLEVEL`. For example, the following equations define the identifiers  $x_k$  recursively:

```
(x[k] := (k + 1)*x[k + 1]) $ k = 1..9:
x[10] := 10:
```

Using the `level` function, evaluate the identifier  $x_1$  to the levels from 1 to 10. For this identifier, the level 10 returns the complete evaluation:

```
for l from 0 to 10 do
  print(level = l, hold(x[1]) = level(x[1], l))
end_for
```

level = 0,  $x_1 = x_1$

level = 1,  $x_1 = 2 x_2$

level = 2,  $x_1 = 6 x_3$

level = 3,  $x_1 = 24 x_4$

level = 4,  $x_1 = 120 x_5$

level = 5,  $x_1 = 720 x_6$

level = 6,  $x_1 = 5040 x_7$

level = 7,  $x_1 = 40320 x_8$

level = 8,  $x_1 = 362880 x_9$

```
level = 9, x1 = 3628800 x10
```

```
level = 10, x1 = 36288000
```

Since the default value of the environment value LEVEL = 100 is greater than 10, in interactive computations MuPAD returns the completely evaluated identifier  $x_1$ :

```
x[1]
```

```
36288000
```

Delete the identifiers  $x_k$ :

```
delete x
```

Set the value of the environment variable LEVEL to 2:

```
LEVEL := 2:
```

Now, MuPAD evaluates the identifier  $x_1$  only up to the second level:

```
(x[k] := (k + 1)*x[k + 1]) $ k = 1..9:
```

```
x[10] := 10:
```

```
x[1]
```

```
6 x3
```

The new value of LEVEL affects all interactive evaluations, except for evaluations in arrays, matrices, tables, and polynomials. For example, use the following recursive definition for the identifiers a, b, and c. Evaluation of the identifier a proceeds only to the second level:

```
a := b: b := c: c := 1: a
```

```
c
```

For further computations, delete the identifiers:

```
delete x, a, b, c:
```

The new value of LEVEL does not affect evaluations that happen in procedures. The evaluation level in procedures remains equal to 1. For example, create the procedure myProc that defines the values of the identifiers a, b, and c recursively:

```
myProc:= proc(d)
begin
  a := b: b := c: c := d: a
end_proc:
```

The procedure evaluates the identifier a up to the first evaluation level:

```
myProc(10)
```

*b*

```
delete a, b, c, d:
```

You can change the evaluation level inside a particular procedure. This change does not affect evaluations occurring in other procedures or inside interactive computations:

```
myProc:= proc(d)
begin
  LEVEL := 3:
  a := b: b := c: c := d: a
end_proc:
myProc(10)
```

10

For further computations, delete the identifiers and restore the value of LEVEL to its default:

```
delete a, b, c, d:
```





To evaluate  $x_1$  to the 111th evaluation level, you must change both LEVEL and MAXLEVEL variables. Also, you can use the level function instead of changing the value of LEVEL:

```
MAXLEVEL:= 112:  
(x[k] := (k + 1)*x[k + 1]) $ k = 1..110:  
x[111] := 1:  
level(x[1], 111)
```

```
1762952551090244663872161047107075788761409536026565516041574063347346955087248316\  
43655557459846231577319604766283797891314584749719987162332009625414533120000000\  
000000000000000000
```

Increase the value of MAXLEVEL only when you know that your code requires it. Do not increase this value for computations where you can avoid it. If your code has infinite loops, the increased level of MAXLEVEL can significantly decrease performance. Always restore the default value for further computations:

```
delete x, MAXLEVEL
```

## Enforce Evaluation

MuPAD automatically evaluates results returned by most of the system functions. However, a few functions can return unevaluated results. For example, the `text2expr` function does not evaluate the returned results:

```
text2expr("2 + 2")
```

2 + 2

The `last` function and its shortcut `%`, which return the previously computed object, also do not evaluate the results:

```
%
```

2 + 2

For such cases, MuPAD provides the `eval` function. This function enables you to enforce evaluation of an expression. For example, enforce evaluation of the previously returned expression:

```
eval(%);
```

4

Another example of the function that does not automatically evaluate returned results is the `subs` function. This function can simplify expressions that contain only purely arithmetical operations:

```
subs(x^2 + 1, x = 0)
```

1

However, the `subs` function does not evaluate expressions. For example, substitute the variable `x` with the value 0 in the following expression that contains the sine function:

```
subs(sin(x^2) + 1, x = 0)
```

```
sin(0) + 1
```

You can use the `eval` function to enforce evaluation of the results returned by `subs`. In this case, MuPAD evaluates the whole expression:

```
eval(%)
```

```
1
```

Alternatively, the `subs` function provides a more efficient method to evaluate its results. The `EvalChanges` option enforces evaluation of the modified parts of the expression, leaving the unchanged parts out of the evaluation process:

```
subs(sin(x^2) + 1, x = 0, EvalChanges)
```

```
1
```

Most efficiently, evaluate an expression at a particular value of a variable by using the `evalAt` function. See [Evaluation at a Point](#).

Also, MuPAD does not evaluate arrays, tables, and polynomials. For example, the system does not evaluate the identifiers `a` and `b` of the following array `A`:

```
A := array(1..2, [a, b]):  
b := 2*a: a := 1: A
```

```
(a b)
```

When you access the entries of the array `A` by using the `op` function, the system does not evaluate the entries of `A`. When you use the indexed access, the system evaluates the entries of arrays, matrices and tables:

```
op(A, 1), op(A, 2);  
A[1], A[2]
```

```
a, b
```

```
1, 2
```

To evaluate all entries of an array, a table, or a polynomial apply the `eval` function to that array, table, or polynomial. Use the `map` function to apply `eval` to an array or a table:

```
map(A, eval)
```

```
( 1 2 )
```

For polynomials, use the `mapcoeffs` function:

```
p := poly(c*x, [x]): c := 10:  
mapcoeffs(p, eval)
```

```
poly(10 x, [x])
```

```
delete a, b, c:
```

## Prevent Evaluation

When you perform interactive computations in MuPAD, the system tries to evaluate all expressions before returning them. For example, if the system can compute an integral, it returns the evaluated result. In most cases, the result is also simplified:

```
int(x^2*sin(x), x)
```

$$2x \sin(x) - \cos(x) (x^2 - 2)$$

The `hold` command enables you to prevent the evaluation of a MuPAD object. For example, `hold` lets you display the integral in its symbolic form:

```
hold(int)(x^2*sin(x), x) = int(x^2*sin(x), x)
```

$$\int x^2 \sin(x) \, dx = 2x \sin(x) - \cos(x) (x^2 - 2)$$

Also, you can prevent evaluation of an object by using the `level` function with the second argument 0. When you use `level` to prevent evaluation of identifiers, the results are equivalent to the results obtained with the `hold` function:

```
level(int(x^2*sin(x), x), 0)
```

$$\int x^2 \sin(x) \, dx$$

The `level` function only prevents evaluation of identifiers. If you create a function without a name, for example  $x \rightarrow \sin(x)$ , `level` does not prevent evaluation of that function:

```
level((x -> sin(x))(PI), 0)
```

```
0
```

In this case, use the `hold` function to prevent evaluation. For example, `hold` successfully prevents evaluation of the function  $x \rightarrow \sin(x)$  at the point  $x = \pi$ :

```
hold((x -> sin(x))(PI))
```

$$(x \rightarrow \sin(x))(\pi)$$

Both `hold` and `level` functions prevent the evaluation of an object only in the particular computation in which you explicitly use them. These functions do not prevent further evaluations. For example, if you assign an expression containing `hold` to a variable, and then call that variable, MuPAD evaluates the expression:

```
y := hold(int)(x^2*sin(x), x);
y
```

$$\int x^2 \sin(x) \, dx$$

$$2 x \sin(x) - \cos(x) (x^2 - 2)$$

## Actual and Displayed Results of Evaluations

When MuPAD evaluates an expression or executes a command, the output that the system displays can differ from the actual result. The simplest example of this behavior is that MuPAD does not display all computed results. You can suppress outputs by terminating commands with colons. For example, the evaluation of the following expression returns 4. However, MuPAD does not display any output because the expression is terminated with a colon:

```
2 + 2:
```

The function call `last(1)` returns the previously computed value. Alternatively, you can use the operator `%` to return that value:

```
%
```

```
4
```

MuPAD also suppresses intermediate results obtained within loops and procedures. For example, the evaluation of the following for loop returns five numbers. However, the output contains only the final result:

```
for x from 1 to 5 do
  hold(_power)(x, 2) = x^2
end_for
```

```
52 = 25
```

To display intermediate results obtained in loops and procedures, use the `print` function inside a loop or a procedure. For example, to display all five numbers obtained in the for loop, enter:

```
for x from 1 to 5 do
  print(hold(_power)(x, 2) = x^2)
end_for
```

```
12 = 1
```



$$2^2 = 4$$

$$3^2 = 9$$

$$4^2 = 16$$

$$5^2 = 25$$

Alternatively, use the `fprint` function. This function typically writes results to a file indicated by one of the arguments of `fprint`. When this argument is 0, the function displays the results on screen:

```
for x from 1 to 5 do
  fprint(Unquoted, 0, hold(_power)(x, 2) = x^2);
end_for
```

$$1^2 = 1$$

$$2^2 = 4$$

$$3^2 = 9$$

$$4^2 = 16$$

$$5^2 = 25$$

The `print` and `fprint` functions display outputs differently. The `print` function uses the typeset mode, which is how mathematical expressions are typically written on paper. The `fprint` function uses the ASCII format. For information about different output modes available in MuPAD, see Using Different Output Modes.

## Evaluate at a Point

To evaluate an expression for particular values of identifiers, use the `evalAt` function or its shortcut `|`. For example, evaluate the following expression at the point  $x = 0$ :

```
diff(x^2*exp(sin(x)), x $ 3) | x = 0
```

6

In MuPAD, all computations are symbolic by default. For example, evaluating the previous expression at  $x = 1$  returns the exact symbolic result:

```
diff(x^2*exp(sin(x)), x $ 3) | x = 1
```

$$5 \cos(1) e^{\sin(1)} - 6 e^{\sin(1)} \sin(1) + 6 \cos(1)^2 e^{\sin(1)} + \cos(1)^3 e^{\sin(1)} - 3 \cos(1) e^{\sin(1)} \sin(1)$$

To get a numeric approximation of the result, use the floating-point number to specify the point at which you want to evaluate an expression:

```
diff(x^2*exp(sin(x)), x $ 3) | x = 1.0
```

-4.180173868

Alternatively, you can evaluate an expression at a point by using the `subs` function with the `EvalChanges` option. For expressions that contain only free variables, `evalAt` and `subs` return identical results:

```
diff(sin(x)*cos(x^2), x $ 2) | x = PI,
subs(diff(sin(x)*cos(x^2), x $ 2), x = PI, EvalChanges)
```

$4 \pi \sin(\pi^2), 4 \pi \sin(\pi^2)$

`evalAt` and `subs` return different results for the expressions that contain dependent variables. The `subs` function does not distinguish between free and dependent variables. The function replaces both free and dependent variables with the new value, for example:

```
subs(x + int(f(x), x = 0..infinity), x = 1)
```

$$\int_0^{\infty} f(1) \, d1 + 1$$

The evalAt function replaces only free variables:

```
x + int(f(x), x = 0..infinity) | x = 1
```

$$\int_0^{\infty} f(x) \, dx + 1$$

## Choose a Solver

The general solvers (`solve` for symbolic solutions and `numeric::solve` for numeric approximations) handle a wide variety of equations, inequalities, and systems. When you use the general solver, MuPAD identifies the equation or the system as one of the types listed in the table that follows. Then the system calls the appropriate solver for that type. If you know the type of the equation or system you want to solve, directly calling the special solver is more efficient. When you call special solvers, MuPAD skips trying other solvers. Direct calls to the special solvers can help you to:

- Improve performance of your code
- Sometimes get a result where the general solver fails

The following table lists the types of equations and systems for which MuPAD offers special solvers. The `solve` and `numeric::solve` commands also handle these types of equations and systems (except systems presented in a matrix form). Define ordinary differential equations with the `ode` command before calling the general solver.

Equation Type	Symbolic Solvers	Numeric Solvers
General system of linear equations	<code>linsolve</code>	<code>numeric::linsolve</code>
General system of linear equations given in a matrix form $A \vec{x} = \vec{b}$	<code>linalg::matlinsolve</code>	<code>numeric::matlinsolve</code>
System of linear equations given in a matrix form $A \vec{x} = \vec{b}$ , where A is a Vandermonde matrix. For example:	<code>linalg::vandermondeSolve</code>	

Equation Type	Symbolic Solvers	Numeric Solvers
$\begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix}$ <p>See <code>linalg::vandermonde</code> for the definition and details.</p>		
<p>System of linear equations given in a matrix form <math>A \vec{x} = \vec{b}</math>, where A is a Toeplitz matrix. For example:</p> $\begin{pmatrix} a_0 & a_1 & a_2 & \dots & \dots & \dots & \dots & a_n \\ a_{-1} & a_0 & a_1 & a_2 & \dots & \dots & \dots & a_{n-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \dots & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{-n} & \dots & \dots & \dots & \dots & a_{-2} & a_{-1} & a_0 \end{pmatrix}$ <p>See <code>linalg::toeplitz</code> for the definition and details.</p>	<code>linalg::toeplitzSolve</code>	
<p>System of linear equations given in a matrix form <math>LU \vec{x} = \vec{b}</math>. The lower triangular matrix L and the upper triangular matrix U form an LU-decomposition.</p>	<code>linalg::matlinsolveLU</code>	

Equation Type	Symbolic Solvers	Numeric Solvers
Univariate polynomial equation. Call these functions to isolate the intervals containing real roots.	<code>polylib::realroots</code>	<code>numeric::polyroots</code> , <code>numeric::realroots</code>
Bivariate polynomial equation for which the general solver returns <code>RootOf</code> . Try calling <code>solve</code> with the option <code>MaxDegree</code> . If the option does not help to get an explicit solution, compute the series expansion of the solution. Expand the solution around the point where one of the variables is 0.	<code>series</code>	
System of polynomial equations		<code>numeric::polysysroots</code>
Arbitrary univariate equation		<code>numeric::realroot</code> , <code>numeric::realroots</code>
System of arbitrary equations		<code>numeric::fsolve</code>
Ordinary differential equation or a system of ODEs	<code>ode::solve</code>	<code>numeric::odesolve</code>
Ordinary differential equation or a system of ODEs. Call this function to get a procedure representing the numeric results instead of getting the numeric approximation itself.		<code>numeric::odesolve2</code>
Ordinary differential equations on homogeneous manifolds embedded in the space of $n \times m$ matrices.		<code>numeric::odesolveGeometri</code>
Linear congruence equation	<code>numlib::lincongruence</code>	
Quadratic congruence equation	<code>numlib::msqrts</code>	
Polynomial equation. Call this function to find modular roots.	<code>numlib::mroots</code>	

## Solve Algebraic Equations and Inequalities

### In this section...

“Specify Right Side of Equation” on page 3-29

“Specify Equation Variables” on page 3-29

“Solve Higher-Order Polynomial Equations” on page 3-31

“Find Multiple Roots” on page 3-33

“Isolate Real Roots of Polynomial Equations” on page 3-33

### Specify Right Side of Equation

The solver accepts both equations and expressions. If you call `solve` for an expression, the command assumes the right side to be 0:

```
solve(x^3 - 1, x);
solve(x^3 = 8, x);
solve(x^3 - 3*x^2 = 1 - 3*x, x)
```

$$\left\{1, -\frac{1}{2} - \frac{\sqrt{3}i}{2}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}\right\}$$

$$\{2, -1 - \sqrt{3}i, -1 + \sqrt{3}i\}$$

$$\{1\}$$

### Specify Equation Variables

Specifying the variable that you want to solve an equation for is optional. If you specify a variable, the `solve` command returns solutions as a set of numbers. Otherwise, the command returns a set of lists as a solution:

```
solve(x^2 - 3*x + 2 = 0, x);
solve(x^2 - 3*x + 2 = 0)
```

$$\{1, 2\}$$

$$\{[x = 1], [x = 2]\}$$

If your equation contains symbolic parameters, specify the variable for which you want to solve the equation:

$$\text{solve}(a*x^2 + b*x + c, x)$$

$$\left\{ \begin{array}{ll} \left\{ -\frac{b+\sigma_1}{2a}, -\frac{b-\sigma_1}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{array} \right.$$

where

$$\sigma_1 = \sqrt{b^2 - 4ac}$$

If you solve an equation with symbolic parameters and do not specify the variable, `solve` uses all parameters as variables and returns a set of all possible solutions. For example, solving the following equation the solver assumes that both  $x$  and  $y$  are free variables. When returning all possible solutions for this equation, the solver uses an arbitrary parameter  $z$ :

$$\text{solve}(x^3 + y^3)$$

$$\left\{ \left[ x = \frac{z}{2} - \frac{\sqrt{3}zi}{2}, y = z \right], \left[ x = \frac{z}{2} + \frac{\sqrt{3}zi}{2}, y = z \right], [x = -z, y = z] \right\}$$

To specify more than one variable, provide a list of variables as a second argument:

$$\text{solve}(a*x^2 + b*x + c, [a, b, c])$$



$$\begin{cases} \{[a = z, b = z1, c = 0]\} & \text{if } x = 0 \\ \left\{ \left[ a = -\frac{z1+xz}{x^2}, b = z, c = z1 \right] \right\} & \text{if } x \neq 0 \end{cases}$$

`solve` also can return an expression as `x` in `S`, where `x` is a list of variables for which you solve an equation, and `S` is a set of the solution vectors:

```
solve(a*x + 1/x)
```

$$\left( \frac{a}{x} \right) \in \left\{ \left( \begin{array}{c} -\frac{1}{z^2} \\ z \end{array} \right) \mid z \in \mathbb{C} \setminus \{0\} \right\}$$

## Solve Higher-Order Polynomial Equations

When you solve a higher-order polynomial equation, the solver sometimes uses `RootOf` to return the results:

```
solve(x^3 + 2*x + 1 = 0, x)
```

```
RootOf(z^3 + 2 z + 1, z)
```

To get an explicit solution for such equations, try calling the solver with the option `MaxDegree`. The option specifies the maximal degree of polynomials for which the solver tries to return explicit solutions. By default, `MaxDegree=2`. Increasing this value, you can get explicit solutions for higher-order polynomials. For example, specify `MaxDegree=3` and get explicit solutions instead of `RootOf` for the third-order polynomial:

```
solve(x^3 + 2*x + 1 = 0, x, MaxDegree = 3)
```

$$\left\{ \sigma_2 - \frac{2}{3\sigma_2}, \frac{1}{3\sigma_2} - \frac{\sigma_2}{2} - \sigma_1, \frac{1}{3\sigma_2} - \frac{\sigma_2}{2} + \sigma_1 \right\}$$

where

$$\sigma_1 = \frac{\sqrt{3} \left( \frac{2}{3\sigma_2} + \sigma_2 \right) i}{2}$$

$$\sigma_2 = \left( \frac{\sqrt{59} \sqrt{108}}{108} - \frac{1}{2} \right)^{1/3}$$

When you solve a fifth- or higher-order polynomial equation, the solver might be unable to return the solution explicitly, even with the option `MaxDegree`:

```
solve(x^5 + 2*x + 1 = 0, x);
solve(x^5 + 2*x + 1 = 0, x, MaxDegree = 5)
```

```
RootOf(z^5 + 2 z + 1, z)
```

```
RootOf(z^5 + 2 z + 1, z)
```

In general, there are no explicit expressions for the roots of polynomials of degrees higher than 4. Setting the option `MaxDegree` to 4 or a higher value makes no difference.

`RootOf` symbolically represents the set of the roots of a polynomial. You can use the expressions containing `RootOf` in your further computations. For example, find the sum over all roots of the polynomial:

```
sum(S^2 + S + 2, S in RootOf(X^5 + 2*X + 1, X));
```

```
10
```

To get the numeric approximation of the roots, use the `float` command:

```
float(RootOf(X^4 + X + 1, X))
```

```
{-0.7271360845 + 0.4300142883 i, -0.7271360845 - 0.4300142883 i,  
0.7271360845 + 0.9340992895 i, 0.7271360845 - 0.9340992895 i}
```

For more details on numeric approximations, see Solving Equations Numerically.

For univariate polynomial equations, MuPAD also can compute intervals containing the real roots. See Isolating Real Roots of Polynomial Equations.

## Find Multiple Roots

By default, the `solve` command does not return the multiplicity of the roots. When the solution of an equation contains multiple roots, MuPAD removes duplicates:

```
solve(x^2 - 6*x + 9 = 0, x)
```

```
{3}
```

The solver does not display multiple roots because it returns results as a set. A set in MuPAD cannot contain duplicate elements. To obtain polynomial roots with their multiplicities, use the option `Multiple`:

```
solve(x^2 - 6*x + 9 = 0, x, Multiple);  
solve((x - 1)^3*(x - 2)^7, x, Multiple)
```

```
{[3, 2]}
```

```
{[1, 3], [2, 7]}
```

## Isolate Real Roots of Polynomial Equations

For some polynomial equations, the solver cannot return the explicit symbolic solutions.

```
p:= x^5 - 31*x^4/32 + 32*x^3/33 - 33*x^2/34 - 34*x/35 + 35/36:
```

```
solve(p, x)
```

$$\text{RootOf}\left(z^5 - \frac{31 z^4}{32} + \frac{32 z^3}{33} - \frac{33 z^2}{34} - \frac{34 z}{35} + \frac{35}{36}, z\right)$$

If you prefer a solution in a form other than `RootOf` and want to avoid numeric methods, use `polylib::realroots` to find all intervals containing real solutions:

```
p:= x^5 - 31*x^4/32 + 32*x^3/33 - 33*x^2/34 - 34*x/35 + 35/36:  
polylib::realroots(p)
```

$$\left[[-2, 0], \left[\frac{3}{4}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right]\right]$$

## Solve Algebraic Systems

### In this section...

“Linear Systems of Equations” on page 3-35

“Linear Systems in a Matrix Form” on page 3-36

“Nonlinear Systems” on page 3-42

### Linear Systems of Equations

When solving a linear system of symbolic equations, the general solver returns a set of solutions:

```
solve([x + y = 1, 3*x - 2*y = 5], [x, y])
```

$$\left\{ \left[ x = \frac{7}{5}, y = -\frac{2}{5} \right] \right\}$$

The function `linsolve` returns a list of solutions:

```
linsolve([x + y = 1, 3*x - 2*y = 5], [x, y])
```

$$\left[ x = \frac{7}{5}, y = -\frac{2}{5} \right]$$

If there are more unknowns than independent equations in a system, `linsolve` solves the system for the first unknowns:

```
linsolve([x + y = a, 3*x - 2*y = b], [x, y, a, b])
```

$$\left[ x = \frac{2a}{5} + \frac{b}{5}, y = \frac{3a}{5} - \frac{b}{5} \right]$$

Providing the unknowns in different order affects the solution:

```
linsolve([x + y = a, 3*x - 2*y = b], [a, b, x, y])
```

$$[a = x + y, b = 3x - 2y]$$

## Linear Systems in a Matrix Form

To state the problem of solving the system of linear equations in a matrix form, use the following steps:

- 1 Create a matrix  $A$  containing the coefficients of the terms of linear equations. Each equation contributes to a row in  $A$ .
- 2 Create a column vector  $\vec{b}$  containing the right sides of the equations.
- 3 The matrix form of the linear system is  $A\vec{x} = \vec{b}$ . When solving a system in a matrix form, you provide a matrix  $A$  and a vector  $\vec{b}$ . The solver returns the solutions of the system as a vector  $\vec{x}$ .

The dimensions  $m \ n$  of the coefficient matrix define the following types of linear systems.

$m = n$	Square system	If the determinant of $A$ is not a zero, a unique solution of the system exists. Otherwise, the system has either infinitely many solutions or no solutions.
$m > n$	Overdetermined system	The system includes more equations than variables. The system can have one solution, infinitely many solutions, or no solutions.
$m < n$	Underdetermined system	The system includes more variables than equations. The system has either infinitely many solutions or no solutions.

To solve a linear system in a matrix form, use the `linalg::matlinsolve` command. For example, solve the following system of linear equations:

$$\begin{aligned} \text{eqn1} &:= 2*x + 3*y = 4: \\ \text{eqn2} &:= 3*x - 2*y = 1: \end{aligned}$$

To convert the system to a matrix form, use the `matrix` command to create a matrix of coefficients and a vector containing right sides of equations:

```
A := matrix([[2, 3],[3, -2]]);
b := matrix([4, 1])
```

$$\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

As a shortcut for converting a system of linear equations to a matrix form, use `linalg::expr2Matrix`:

```
Ab := linalg::expr2Matrix([eqn1, eqn2], [x,y])
```

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}$$

Now, use `linalg::matlinsolve` to solve the system:

```
linalg::matlinsolve(Ab)
```

$$\begin{pmatrix} \frac{11}{13} \\ \frac{10}{13} \end{pmatrix}$$

Alternatively, split the matrix `Ab` into a matrix of coefficients `A` and a vector `b` containing the right sides of equations. Use `linalg::matlinsolve` to solve the system:

```
A := Ab[1..2, 1..2]:
b := Ab[1..2, 3..3]:
linalg::matlinsolve(A, b)
```

$$\begin{pmatrix} \frac{11}{13} \\ \frac{10}{13} \end{pmatrix}$$

If your linear system is originally defined by a matrix equation, using the matrix form to solve the system is more intuitive. Also, the matrix form is convenient for solving equations with many variables because it avoids creating symbols for these variables. For example, the following matrices define a linear system:

```
A := linalg::hilbert(10);
b := matrix([i^(-2) $ i = 1..10])
```

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{pmatrix}$$



$$\begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{9} \\ \frac{1}{16} \\ \frac{1}{25} \\ \frac{1}{36} \\ \frac{1}{49} \\ \frac{1}{64} \\ \frac{1}{81} \\ \frac{1}{100} \end{pmatrix}$$

To solve this system, use `linalg::matlinsolve`:

```
linalg::matlinsolve(A, b)
```

$$\begin{pmatrix} \frac{1451}{252} \\ -99 \\ 1188 \\ -8008 \\ \frac{63063}{2} \\ -\frac{378378}{5} \\ 112112 \\ -\frac{700128}{7} \\ \frac{196911}{4} \\ -\frac{92378}{9} \end{pmatrix}$$

### Specialized Matrices

If your system of linear equations can be presented as a specialized matrix, you can solve the system by calling a special solver. Direct calls to the special solvers often improve the performance of your code. MuPAD offers special solvers for linear systems that can be represented by matrices of the following types:

- A matrix given by  $LU\vec{x} = \vec{b}$ , where L is a lower triangular matrix, and U is an upper triangular matrix (LU-decomposition of a matrix)
- Toeplitz matrix. For example, the following matrix is a Toeplitz matrix:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \\ a_{-1} & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_0 & a_1 & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{-1} & a_0 & a_1 \\ a_{-n} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{-2} & a_{-1} & a_0 \end{pmatrix}$$

See `linalg::toeplitz` for the definition and details.

- Vandermonde matrix. For example, the following matrix is a Vandermonde matrix::

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdot & \cdot & \cdot & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdot & \cdot & \cdot & a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n & a_n^2 & \cdot & \cdot & \cdot & a_n^{n-1} \end{pmatrix}$$

See `linalg::vandermonde` for the definition and details.

Suppose you want to solve the following system given by a Toeplitz matrix. Use `linalg::toeplitz` to define the system:

```
T := linalg::toeplitz(3, [0, 2, 5, 3, 0])
```

$$\begin{pmatrix} 5 & 3 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 5 \end{pmatrix}$$

The vector `y` defines the right sides of equations:

```
y := matrix([1, 2, 3])
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

To solve a system given by a Toeplitz matrix, use the `linalg::toeplitzSolve` special solver. This special solver is more efficient than `linalg::matlinsolve`. This solver accepts a vector or a list of diagonal elements `t` of a Toeplitz matrix instead of a Toeplitz matrix itself:

```
t := [0, 2, 5, 3, 0]:
x := linalg::toeplitzSolve(t, y)
```

$$\begin{pmatrix} \frac{16}{65} \\ -\frac{1}{13} \\ \frac{41}{65} \end{pmatrix}$$

For the list of special solvers available in MuPAD, see [Choosing Solver](#). For information on linear algebra functions, see [“Linear Algebra”](#).

## Nonlinear Systems

To solve a system of nonlinear equations symbolically, use the general solver. For example, solve the following system of trigonometric equations:

```
solve({4*cos(x) + 2*cos(y) = 3, 2*sin(x) + sin(y) = 1}, [x, y])
```

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} \in & \left\{ \begin{pmatrix} 2 \arctan\left(\sigma_1 + \frac{16}{49}\right) + 2\pi k \\ 2 \arctan\left(\frac{8}{13} - \sigma_2\right) + 2\pi l \end{pmatrix} \mid k \in \mathbb{Z}, l \in \mathbb{Z} \right\} \\ \cup & \left\{ \begin{pmatrix} 2 \arctan\left(\frac{16}{49} - \sigma_1\right) + 2\pi k \\ 2 \arctan\left(\sigma_2 + \frac{8}{13}\right) + 2\pi l \end{pmatrix} \mid k \in \mathbb{Z}, l \in \mathbb{Z} \right\} \end{aligned}$$

where

$$\sigma_1 = \frac{3\sqrt{23}}{49}$$

$$\sigma_2 = \frac{3\sqrt{23}}{13}$$

When you use the `VectorFormat` option, the solver returns the solutions as a set of vectors. If you want the solver to return one solution from the set, use the `PrincipalValue` option. When you use `PrincipalValue`, the solver still returns a set, although that set contains only one solution:

```
solve({4*cos(x) + 2*cos(y) = 3, 2*sin(x) + sin(y) = 1},
      [x, y], VectorFormat, PrincipalValue)
```

$$\left\{ \begin{pmatrix} 2 \arctan\left(\frac{3\sqrt{23}}{49} + \frac{16}{49}\right) \\ 2 \arctan\left(\frac{8}{13} - \frac{3\sqrt{23}}{13}\right) \end{pmatrix} \right\}$$

You can also approximate the exact symbolic solution numerically:

```
float(%)
```

$$\left\{ \begin{pmatrix} 1.110212402 \\ -0.9134005193 \end{pmatrix} \right\}$$

If `solve` cannot compute explicit solutions for a system, use numeric methods to approximate the solutions. For nonlinear systems, MuPAD offers the following special solvers:

- For a system of polynomial equations, use `numeric::polysysroots`.
- For an arbitrary system of equations, use `numeric::fsolve`.

### System of Polynomial Equations

Suppose you want to solve the following system of polynomial equations:

```
eqn1 := x^3 + x^2*y - 1:
eqn2 := x^2 - y^2 = 1/2:
```

The general symbolic solver produces the result in terms of the `RootOf` object:

```
solve({eqn1, eqn2}, [x, y])
```

$$\left( \begin{array}{c} x \\ y \end{array} \right) \in \left\{ \left( \begin{array}{c} \frac{z_1^2}{2} + z_1 + \frac{1}{4} \\ z_1 \end{array} \right) \mid z_1 \in \text{RootOf}\left(z^4 + 4z^3 + z^2 + 2z - \frac{7}{4}, z\right) \right\}$$

To approximate the solutions of the system of polynomial equations numerically, use the special solver `numeric::polysysroots`:

```
numeric::polysysroots({eqn1, eqn2}, [x, y])
```

```
{[x = -0.4142135624 - 0.6435942529 i, y = -0.2928932188 - 0.9101797211 i],
 [x = 0.8604395883, y = 0.4902614457], [x = 3.967987536, y = -3.904475008],
 [x = -0.4142135624 + 0.6435942529 i, y = -0.2928932188 + 0.9101797211 i]}
```

### System of Arbitrary Nonlinear Equations

Suppose you want to solve the following system of equations:

```
eqn1 := sin(a) - cos(b) + a*b:
eqn2 := (sin(a) + cos(b))^3:
```

The general solver cannot find a symbolic solution:

```
solve({eqn1, eqn2}, [a, b])
```

```
solve([sin(a) - cos(b) + a b, cos(b) + sin(a)], [a, b])
```

For numeric approximations of the solutions of the nonlinear system of equations, use the `numeric::fsolve`. The numeric solver returns only one solution:

```
numeric::fsolve({eqn1, eqn2}, [a, b])
```

```
[a = -221.473252, b = -0.009030067696]
```

---

**Note** When `numeric::fsolve` finds one solution, it stops looking for other solutions.

---

When you solve an arbitrary nonlinear system numerically, there is no general way to find all solutions. For more information, see *Solving Equations Numerically*.

## Solve Ordinary Differential Equations and Systems

### In this section...

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### General Solutions

An ordinary differential equation (ODE) contains derivatives of dependent variables with respect to the only independent variable. If  $y$  is a dependent variable and  $x$  is an independent variable, the solution of an ODE is an expression  $y(x)$ . The order of the derivative of a dependent variable defines the order of an ODE.

The solution of a single explicit first-order ODE can always be computed by integration, provided the solution exists. To define an ordinary differential equation, use the `ode` command:

```
o := ode(y'(x) = y(x)^2, y(x))
```

```
ode(y'(x) - y(x)^2, y(x))
```

---

**Note** `ode` does not accept multivariate expressions such as  $y(x, t)$ . It also does not accept `piecewise` expressions.

---

Now use the general `solve` to solve this equation:

```
solve(o)
```



$$\left\{0, -\frac{1}{C3+x}\right\}$$

Alternatively, you can call the ODE solver directly:

```
ode::solve(y'(x) = y(x)^2, y(x))
```

$$\left\{0, -\frac{1}{C7+x}\right\}$$

The general solutions of ODEs contain arbitrary constants of integration. The solver generates the constants of integration using the format of an uppercase letter C followed by an automatically generated number. For example, it generates C1, C2, and so on.

For higher-order ODEs, you can find explicit solutions only for special types of equations. For example, the following second-order equation has a solution in terms of elementary functions:

```
ode::solve(y''(x) = y(x), y(x))
```

$$\{C11 e^x + C10 e^{-x}\}$$

The solver introduces the solution of the following equation in terms of the Bessel functions. MuPAD uses standard mathematical notations for Bessel and other special functions:

```
ode::solve(y''(x) = y'(x) + y(x)*exp(x), y(x))
```

$$\left\{C13 e^{\frac{x}{2}} J_1\left(2 e^{\frac{x}{2}} i\right) + C14 e^{\frac{x}{2}} Y_1\left(2 e^{\frac{x}{2}} i\right)\right\}$$

If you have a second- or higher-order ODE, nonlinear ODE, or a system of ODEs, a symbolic solution does not always exist:

```
ode::solve(y''(x) = y'(x)^2 + y(x)*exp(x), y(x))
```

```
solve(y''(x) = y'(x)^2 + e^x y(x), y(x))
```

For ODEs that cannot be solved symbolically, try using numeric solvers.

## Initial and Boundary Value Problems

Many problems in engineering and physics involve solving differential equations with initial conditions or boundary conditions or both. To specify initial or boundary conditions, create a set containing an equation and conditions. For example, state the following initial value problem by defining an ODE with initial conditions:

```
IVP := ode({y'(x) = y(x), y(0) = 5, y'(0) = 1}, y(x))
```

```
ode({y'(0) = 1, y''(x) = y(x), y(0) = 5}, y(x))
```

When you solve an ODE with initial or boundary conditions, the solver adjusts the integration constants to fit these conditions:

```
solve(IVP)
```

```
{2 e^-x + 3 e^x}
```

The following equation has both initial and boundary conditions:

```
ode::solve({y''(x) = y(x), y(0) = 0, y(5) = 1, y'(0) = 0}, y(x))
```

```
{
  e^(-x/2) e^(5/2) cos((sqrt(3)x)/2) / sigma_1 - e^(5/2) e^x / sigma_1 +
  sqrt(3) e^(-x/2) e^(5/2) sin((sqrt(3)x)/2) / sigma_1
}
```

where

```
sigma_1 = cos((5*sqrt(3))/2) - e^(15/2) + sqrt(3) sin((5*sqrt(3))/2)
```

Each independent condition removes one integration constant:

```
ode::solve({y''(x) = y'(x)}, y(x))
```

$$\{C26 + C28 e^x + C27 e^{-x}\}$$

```
ode::solve({y''(x) = y'(x), y(0) = 0}, y(x))
```

$$\{C34 + C33 e^x - e^{-x} (C33 + C34)\}$$

```
ode::solve({y''(x) = y'(x), y(0) = 0, y(1) = 1}, y(x))
```

$$\left\{ C39 + \frac{e^x (C39 + e - C39 e)}{e^2 - 1} - \frac{e^{-x} (e - C39 e + C39 e^2)}{e^2 - 1} \right\}$$

```
ode::solve({y''(x)=y'(x), y(0)=0, y(1)=1, y'(0)=1/2}, y(x))
```

$$\left\{ \frac{e^{-x} (3 e - e^2)}{2 (e^2 - 2 e + 1)} - \frac{4 e - e^2 + 1}{2 (e^2 - 2 e + 1)} + \frac{e^x (e + 1)}{2 (e^2 - 2 e + 1)} \right\}$$

## Special Types of Ordinary Differential Equations

Suppose, the equation you want to solve belongs to the Clairaut type:

```
o:= ode(y(x) = x*y'(x) + y'(x)^3, y(x)):
solve(o)
```

$$\left\{ C44^3 + x C44, -\frac{2\sqrt{3}(-x)^{3/2}}{9}, \frac{2\sqrt{3}(-x)^{3/2}}{9} \right\}$$

The solver recognizes the type of the equation and applies the algorithm for solving Clairaut equations. To improve performance, call the solver with the option `Type = Clairaut`:

`solve(o, Type = Clairaut)`

$$\left\{ C45^3 + x C45, -\frac{2\sqrt{3}(-x)^{3/2}}{9}, \frac{2\sqrt{3}(-x)^{3/2}}{9} \right\}$$

The solver tries to recognize and tries to solve the following classes of ODEs.

Type	Equation	ODE Solver Option
Abel differential equation	$y'(x) = a_0(x) + a_1(x) y(x) + a_2(x) y(x)^2 + a_3(x) y(x)^3$	Abel
Bernoulli differential equation	$y'(x) + p(x) y = q(x) y^n$ where $n \neq 0$ and $n \neq 1$	Bernoulli
Chini differential equation	$y'(x) = a_0(x) + a_1(x) y(x) + a(x) y(x)^n$	Chini
Clairaut differential equation	$y(x) = x y'(x) + g(y'(x))$	Clairaut
Exact first-order ordinary differential equation	$y'(x) = f(x, y)$ that can be represented as $M(x, y)dx + N(x, y)dy = 0$ where $\frac{\partial}{\partial x} M(x, y) = \frac{\partial}{\partial y} N(x, y)$	ExactFirstOrder
Exact second-order ordinary differential equation	$y''(x) = f(x, y(x), y'(x))$ if a first integral turns this equation into a first-order ODE	ExactSecondOrder
Linear homogeneous ordinary differential equation	$Dy = 0$ , where D is a linear differential operator	Homogeneous
Lagrange differential equation	$y(x) = x f(y'(x)) + g(y'(x))$	Lagrange
Riccati differential equation	$y'(x) + p(x) y(x) = q(x) y(x)^2 + r(x)$	Riccati

If the solver cannot identify the equation with the type you indicated, it issues a warning and returns the special value FAIL:

```
ode::solve(y'(x) + y(x) = x, y(x), Type = Homogeneous)
```

```
Warning: Cannot detect the homogeneous ODE. [ode::homogeneous]
```

FAIL

## Systems of Ordinary Differential Equations

To solve a system of differential equations, specify the system as a set of equations:

```
s := {y'(x) = z(x), z'(x) = y(x) + 2*z(x)}:
```

Call the `ode::solve` function specifying the set of functions  $\{y(x), z(x)\}$  for which you want to solve the system:

```
ode::solve(s, {y(x), z(x)})
```

$$\left\{ \begin{aligned} z(x) &= e^{-x(\sqrt{2}-1)} \left( C47 - \sqrt{2} C47 + \sigma_1 + \sqrt{2} C46 e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)} \right), \\ y(x) &= e^{-x(\sqrt{2}-1)} (C47 + \sigma_1) \end{aligned} \right\}$$

where

$$\sigma_1 = C46 e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}$$

Now, suppose the system of differential equations appears in a matrix form. For example, define the system  $Y' = AY + B$ , where A, B, and Y represent the following matrices:

```
Y := matrix([x(t), y(t)]):
```

```
A:= matrix([[1, 2], [-1, 1]]):
B:= matrix([1, t]):
```

The `ode::solve` function does not accept matrices. To be able to use this solver, extract the components of the matrix and include them in a set. Use the `op` function to extract the equations from the matrix. Then, use the braces to create a set of the equations. You can omit the right sides of equations, in which case MuPAD assumes them to be 0:

```
s := {op(diff(Y, t) - A*Y - B)}
```

$$\left\{ \frac{\partial}{\partial t} y(t) - t + x(t) - y(t), \frac{\partial}{\partial t} x(t) - x(t) - 2 y(t) - 1 \right\}$$

Now, specify the set of functions  $\{x(t), y(t)\}$  for which you want to solve the system. Solve the system:

```
ode::solve(s, {x(t), y(t)})
```

$$\left\{ \left[ x(t) = e^t \cos(\sqrt{2} t) \sigma_2 - e^t \sin(\sqrt{2} t) \sigma_1, y(t) = -\frac{\sqrt{2} e^t \cos(\sqrt{2} t) \sigma_1}{2} - \frac{\sqrt{2} e^t \sin(\sqrt{2} t) \sigma_2}{2} \right] \right\}$$

where

$$\sigma_1 = C49 - \frac{e^{-t} (\sin(\sqrt{2} t) - 2\sqrt{2} \cos(\sqrt{2} t) + 6t \sin(\sqrt{2} t) - 3\sqrt{2} t \cos(\sqrt{2} t))}{9}$$

$$\sigma_2 = C48 + \frac{e^{-t} (\cos(\sqrt{2} t) + 2\sqrt{2} \sin(\sqrt{2} t) + 6t \cos(\sqrt{2} t) + 3\sqrt{2} t \sin(\sqrt{2} t))}{9}$$

If you are solving several similar systems of ordinary differential equations in a matrix form, create your own solver for these systems, and then use it as a shortcut. The solver for such systems must be a function that accepts matrices as input arguments, and then performs all required steps. For example, create a solver for a system of the first-order linear differential equations in a matrix form  $Y' = AY + B$ , where the components of functions depend on the variable  $t$ :

```
solveLinearSystem := (A, B, Y) ->
solve(ode({op(diff(Y, t) - A*Y - B)}, {op(Y)})):
```

The `solveLinearSystem` function accepts matrices as input parameters, creates a matrix of equations, extracts these equations to a set, and solves the system:

```
Y:= matrix([x(t), y(t)]):
A:= matrix([[1, 2], [-3, 1]]):
B:= matrix([2, t]):
solveLinearSystem(A, B, Y)
```

$$\left\{ \left[ x(t) = e^t \cos(\sqrt{6} t) \sigma_2 - e^t \sin(\sqrt{6} t) \sigma_1, y(t) = -\frac{\sqrt{6} e^t \cos(\sqrt{6} t) \sigma_1}{2} - \frac{\sqrt{6} e^t \sin(\sqrt{6} t) \sigma_2}{2} \right] \right\}$$

where

$$\sigma_1 = C51 + \frac{e^{-t} (30 \sin(\sqrt{6} t) + 37 \sqrt{6} \cos(\sqrt{6} t) - 42 t \sin(\sqrt{6} t) + 7 \sqrt{6} t \cos(\sqrt{6} t))}{147}$$

$$\sigma_2 = C50 + \frac{e^{-t} (37 \sqrt{6} \sin(\sqrt{6} t) - 30 \cos(\sqrt{6} t) + 42 t \cos(\sqrt{6} t) + 7 \sqrt{6} t \sin(\sqrt{6} t))}{147}$$

## Plot Solutions of Differential Equations

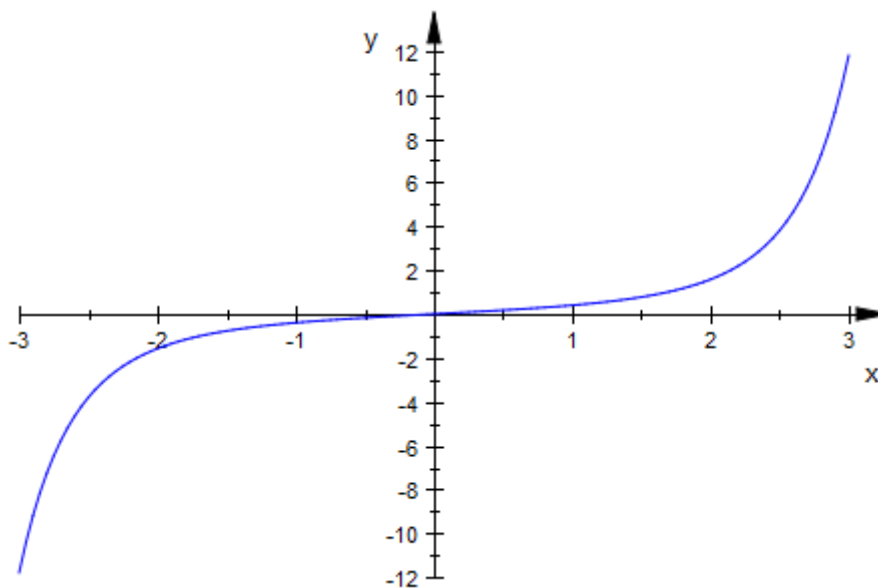
Suppose you want to solve the following equation. The solver returns the results as a set even if the set contains only one element:

```
f := ode::solve({y'(x) = x*y'(x), y(0) = 0, y'(0) = 1/3}, y(x))
```

$$\left\{ -\frac{\sqrt{2} \sqrt{\pi} \operatorname{erf}\left(\frac{\sqrt{2} x i}{2}\right) i}{6} \right\}$$

The plotting functions in MuPAD do not accept sets. To plot the solution, access the elements of a solution set using square brackets or the `op` command:

```
plotfunc2d(f[1], x = -3..3)
```



If you have more than one element of a solution set, you can access a particular element. For example, pick the second element of the solution set for the system of ODEs:

```
f := ode::solve(
    {y'(x) = z(x), z'(x) = y(x) + 2*z(x),
    y(0) = 0, z(0) = 1}, {y(x), z(x)})
```

$$\left\{ \left[ z(x) = e^{-x(\sqrt{2}-1)} \left( \frac{e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{2} - \frac{\sqrt{2}}{4} + \sigma_1 + \frac{1}{2} \right), \right. \right. \\ \left. \left. y(x) = -e^{-x(\sqrt{2}-1)} \left( \frac{\sqrt{2}}{4} - \sigma_1 \right) \right] \right\}$$

where

$$\sigma_1 = \frac{\sqrt{2} e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{4}$$



The solver returns results for a system as a set that contains a list. To open the set and access the list, use square brackets or the `op` command. To access a particular entry of this list, use square brackets:

```
f[1][2];
op(f)[2]
```

$$y(x) = -e^{-x(\sqrt{2}-1)} \left( \frac{\sqrt{2}}{4} - \frac{\sqrt{2} e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{4} \right)$$

$$y(x) = -e^{-x(\sqrt{2}-1)} \left( \frac{\sqrt{2}}{4} - \frac{\sqrt{2} e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{4} \right)$$

To access the right side of the equation, use square brackets or the `rhs` command:

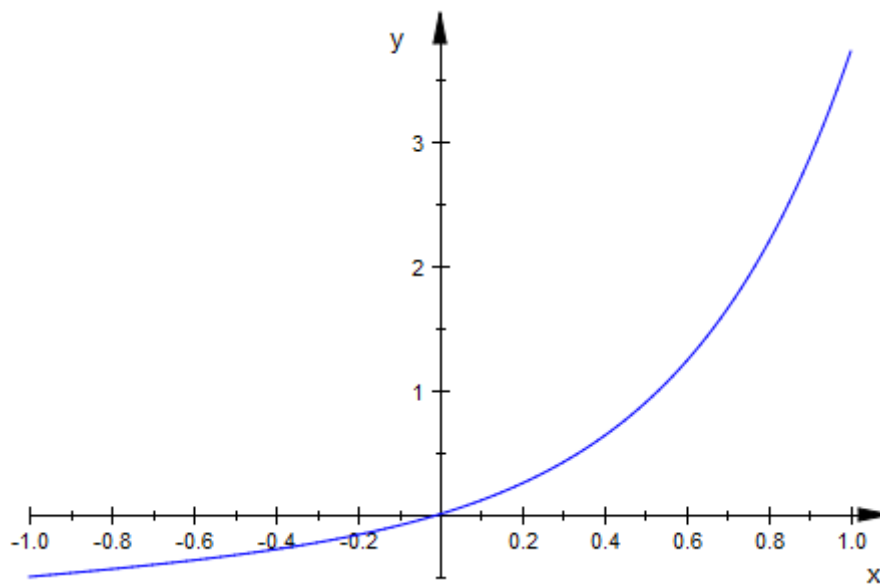
```
f[1][2][2];
rhs(f[1][2])
```

$$-e^{-x(\sqrt{2}-1)} \left( \frac{\sqrt{2}}{4} - \frac{\sqrt{2} e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{4} \right)$$

$$-e^{-x(\sqrt{2}-1)} \left( \frac{\sqrt{2}}{4} - \frac{\sqrt{2} e^{x(\sqrt{2}-1)} e^{x(\sqrt{2}+1)}}{4} \right)$$

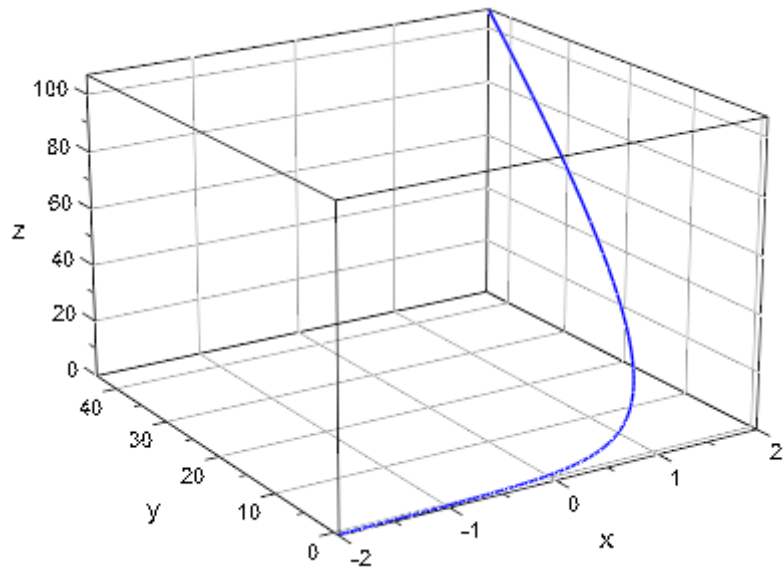
Plot this solution:

```
plotfunc2d(f[1][2][2], x = -1..1)
```



To plot a solution of the system of ODEs in 3-D, use the `plot::Curve3d` command:

```
solution := plot::Curve3d([x, f[1][2][2], f[1][1][2]],  
                           x = -2..2, GridVisible):  
plot(solution)
```



MuPAD provides the functions `plot::Ode2d` and `plot::Ode3d` for visualizing solutions of ODEs. Also, you can plot a vector field associated with an ODE. For all graphic capabilities available in MuPAD, see Graphics and Animations.

## Test Results

### In this section...

“Solutions Given in the Form of Equations” on page 3-58

“Solutions Given as Memberships” on page 3-60

“Solutions Obtained with IgnoreAnalyticConstraints” on page 3-62

### Solutions Given in the Form of Equations

Suppose you want to verify the solutions of this polynomial equation:

```
equation := x^3 + 4 = 0:
solution := solve(equation)
```

$$\left\{ \left[ x = -2^{2/3} \right], \left[ x = 2^{2/3} \left( \frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \right], \left[ x = -2^{2/3} \left( -\frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \right] \right\}$$

To verify the correctness of the returned solutions, substitute the solutions into the original equation. To substitute the results given in the form of equations, evaluate the original equations at the solution points. Use `evalAt` or the vertical bar `|` as a shortcut. For the first solution, the command returns the identity:

```
equation | solution[1]
```

$$0 = 0$$

To check that the left side of the equation is equal to the right side, use the `testeq` command:

```
testeq(equation | solution[1])
```

TRUE

For the second solution, `evalAt` returns an equation with an unsimplified left side. In many cases, MuPAD does not automatically simplify expressions, for example:

```
equation | solution[2];
```

$$4 \left( \frac{1}{2} + \frac{\sqrt{3}i}{2} \right)^3 + 4 = 0$$

`testeq` simplifies the expressions on both sides of the equation:

```
testeq(equation | solution[2])
```

TRUE

As an alternative to evaluating at a point, use the `subs` command to substitute the solution into the original equation:

```
equation := x^3 + 4 = 0;
solution := solve(equation);
testeq(subs(equation, solution[1]));
testeq(subs(equation, solution[2]));
testeq(subs(equation, solution[3]))
```

$$\left\{ \left[ x = -2^{2/3} \right], \left[ x = 2^{2/3} \left( \frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \right], \left[ x = -2^{2/3} \left( -\frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \right] \right\}$$

TRUE

TRUE

TRUE

To verify the solutions of a system of equations, test each equation separately:

```
equations := {x^2 + 2*y = 3, 4*x^2 + 5*y = 6}:
```

```
solutions := solve(equations, {x, y});
testeq((equations|solutions[1])[1]);
testeq((equations|solutions[1])[2]);
testeq((equations|solutions[2])[1]);
testeq((equations|solutions[2])[2])
```

```
{[x = -i, y = 2], [x = i, y = 2]}
```

```
TRUE
```

```
TRUE
```

```
TRUE
```

```
TRUE
```

## Solutions Given as Memberships

Suppose you want to verify the solutions of this trigonometric equation:

```
equation := sin(x)/x = 0:
solution := solve(equation, x)
```

```
{ $\pi k \mid k \in \mathbb{Z} \setminus \{0\}$ }
```

To verify the results, evaluate the original equation at the solution points. Evaluating at a point requires a solution to be in the form of an equation. If you have a solution in the form of membership, `evalAt` returns an error:

```
equation | op(solution)
```

```
Error: An equation is expected.
```

You cannot use the expression `x = solution` directly because `solution` is represented by a set. This set contains the solution for the variable `x`, the independent variable `k`, and the condition on the variable `k`:

```
op(solution)
```

```
 $\pi k, [k], [\mathbb{Z} \setminus \{0\}]$ 
```

Extract the solution for `x`, the variable `k`, and the conditions on the variable `k` from the set. MuPAD returns the variable `k` and its conditions as lists. Use the additional square brackets to extract `k` and the conditions from the lists:

```
op(solution)[1];
op(solution)[2][1];
op(solution)[3][1]
```

```
 $\pi k$ 
```

```
 $k$ 
```

```
 $\mathbb{Z} \setminus \{0\}$ 
```

Now evaluate the original equation at the solution points  $x = \pi k$  under the conditions for `k`:

```
testeq(equation | x = op(solution)[1])
      assuming op(solution)[2][1]
      in op(solution)[3][1]
```

```
TRUE
```

Alternatively, use the `subs` command to substitute the solution into the original equation:

```
testeq(subs(equation, x = op(solution)[1]))
      assuming op(solution)[2][1]
      in op(solution)[3][1]
```

TRUE

### Solutions Obtained with IgnoreAnalyticConstraints

If you verify solutions of an equation or a system solved with the IgnoreAnalyticConstraints option, testeq can return FALSE or UNKNOWN, for example:

```
equation := ln(x) + ln(x + 10) = ln(y):
solutions := solve(equation, x, IgnoreAnalyticConstraints);
testeq(subs(equation, x = solutions[1]));
testeq(subs(equation, x = solutions[2]))
```

$$\{\sqrt{y+25}-5, -\sqrt{y+25}-5\}$$

UNKNOWN

FALSE

When you solve an equation, inequality or a system using the IgnoreAnalyticConstraints option, the solver uses an additional set of simplified mathematical rules. These rules intentionally trade off mathematical strictness and correctness for simplicity of the results. Although this option often leads to the most practical and expected results, it also can lead to incorrect results. For the set of rules IgnoreAnalyticConstraints applies, see the help page of the solve command.

To verify such solutions, try using the same IgnoreAnalyticConstraints option for testeq. When you use this option, the testeq command does not guarantee that the solutions are correct everywhere on the complex plane. The command checks that the solutions are correct for the values of the parameters for which the rules applied by IgnoreAnalyticConstraints are valid:

```
testeq(subs(equation, x = solutions[1]),
        IgnoreAnalyticConstraints);
testeq(subs(equation, x = solutions[2]),
```



```
IgnoreAnalyticConstraints)
```

```
TRUE
```

```
FALSE
```

The `testeq` command did not verify both solutions. When trying to prove the equivalence of two expressions, `testeq` runs random tests before applying `IgnoreAnalyticConstraints`. If tests for random values of identifiers show that expressions are not equivalent, `testeq` disregards the `IgnoreAnalyticConstraints` option and returns `FALSE`. To suppress running random tests, set the number of these tests to zero:

```
testeq(subs(equation, x = solutions[2]),
        NumberOfRandomTests = 0,
        IgnoreAnalyticConstraints)
```

```
TRUE
```

Verifying numeric results returned by the solver using `IgnoreAnalyticConstraints` does not require using the same option. Substitute numeric results into the original equations and call `testeq` to prove equivalence of the expressions on both sides of the equations:

```
equation := x^(11/2) = 1:
solution := solve(equation, IgnoreAnalyticConstraints);
testeq(equation | solution[1])
```

```
{[x = 1]}
```

```
TRUE
```

## If Results Look Too Complicated

### In this section...

“Use Options to Narrow Results” on page 3-64

“Use Assumptions to Narrow Results” on page 3-66

“Simplify Solutions” on page 3-67

### Use Options to Narrow Results

By default, the MuPAD solvers return all possible solutions regardless of their length. Also, by default the solvers assume the solutions are complex numbers. To limit the number of the solutions to some specific ones, the solvers provide a number of options. For information about the options accepted by a particular solver, see the page for that solver. For example, for the list of options provided by the general solver, see the `solve` help page.

The following equation has five solutions:

```
solve(x^5 - 1, x)
```

$$\left\{ 1, -\frac{\sqrt{5}}{4} - \frac{1}{4} - \sigma_1, -\frac{\sqrt{5}}{4} - \frac{1}{4} + \sigma_1, \frac{\sqrt{5}}{4} - \frac{1}{4} - \sigma_2, \frac{\sqrt{5}}{4} - \frac{1}{4} + \sigma_2 \right\}$$

where

$$\sigma_1 = \frac{\sqrt{2} \sqrt{5 - \sqrt{5}} i}{4}$$

$$\sigma_2 = \frac{\sqrt{2} \sqrt{\sqrt{5} + 5} i}{4}$$

If you need a solution in real numbers, use the `Real` option. The only real solution of this equation is 1:

```
solve(x^5 - 1, x, Real)
```

{1}

For the following standard quadratic equation, the solver returns the solutions for all possible values of symbolic parameters a, b, and c:

`solve(a*x^2 + b*x + c, x)`

$$\left\{ \begin{array}{ll} \left\{ -\frac{b+\sigma_1}{2a}, -\frac{b-\sigma_1}{2a} \right\} & \text{if } a \neq 0 \\ \left\{ -\frac{c}{b} \right\} & \text{if } a = 0 \wedge b \neq 0 \\ \mathbb{C} & \text{if } a = 0 \wedge b = 0 \wedge c = 0 \\ \emptyset & \text{if } a = 0 \wedge b = 0 \wedge c \neq 0 \end{array} \right.$$

where

$$\sigma_1 = \sqrt{b^2 - 4ac}$$

To disregard special cases, use the `IgnoreSpecialCases` option:

`solve(a*x^2 + b*x + c, x, IgnoreSpecialCases)`

$$\left\{ -\frac{b + \sqrt{b^2 - 4ac}}{2a}, -\frac{b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

For the following equation, the solver returns a complete, but rather long and complicated solution:

`solve(x^(5/2) + 1/x^(5/2) = 1, x)`

$$\left\{ \frac{1}{\sigma_1}, \frac{1}{\sigma_2}, -\frac{\sigma_3}{\sigma_1}, -\frac{\sigma_4}{\sigma_1}, -\frac{\sigma_3}{\sigma_2}, -\frac{\sigma_4}{\sigma_2} \right\}$$

where

$$\sigma_1 = \left( \frac{1}{2} - \frac{\sqrt{3}i}{2} \right)^{2/5}$$

$$\sigma_2 = \left( \frac{1}{2} + \frac{\sqrt{3}i}{2} \right)^{2/5}$$

$$\sigma_3 = \frac{\sqrt{5}}{4} + \frac{1}{4} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}i}{4}$$

$$\sigma_4 = \frac{\sqrt{5}}{4} + \frac{1}{4} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}i}{4}$$

If you want a simpler and more practical solution, try the `IgnoreAnalyticConstraints` option. With this option, the solver uses a set of simplified mathematical rules that are not generally correct. The returned solutions tend to be most useful for many problems in engineering and physics. Note that with this option the solver does not guarantee the correctness and completeness of the result:

```
solve(x^(5/2) + 1/x^(5/2) = 1, x, IgnoreAnalyticConstraints)
```

$$\left\{ \frac{1}{\left( \frac{1}{2} - \frac{\sqrt{3}i}{2} \right)^{2/5}}, \frac{1}{\left( \frac{1}{2} + \frac{\sqrt{3}i}{2} \right)^{2/5}} \right\}$$

See the list of the options accepted by the general solver `solve`.

### Use Assumptions to Narrow Results

If you want to limit the number of solutions, but the list of options available for `solve` does not have an appropriate option, try using assumptions. Suppose, for the following polynomial expression you need only positive solutions. Use

the `assuming` command to temporarily assume that  $x$  is a positive number. Under this assumption, the solver returns four positive solutions:

```
solve(x^7 + 2*x^6 - 59*x^5 - 106*x^4 + 478*x^3 + 284*x^2
      - 1400*x + 800, x) assuming x > 0
```

$$\left\{1, \frac{\sqrt{17}}{2} - \frac{1}{2}, 5\sqrt{2}, \sqrt{5} - 1\right\}$$

Without the assumption, the solver returns all seven solutions:

```
solve(x^7 + 2*x^6 - 59*x^5 - 106*x^4 + 478*x^3 + 284*x^2
      - 1400*x + 800, x)
```

$$\left\{1, -\sqrt{5} - 1, -\frac{\sqrt{17}}{2} - \frac{1}{2}, \frac{\sqrt{17}}{2} - \frac{1}{2}, -5\sqrt{2}, 5\sqrt{2}, \sqrt{5} - 1\right\}$$

To make several assumptions, combine them with `and`:

```
solve([a*x + b*y = c, h*x - g*y = f], [x, y])
      assuming f = c and a = h and a <> 0
```

```
{ { --      f + g z      -- }
  { { | x = -----, y = z | } if b + g = 0
    { { --      h      -- }
  {
  { { --      f      -- }
    { { | x = -, y = 0 | } if b + g <> 0
      { { --      h      -- }
```

For more information, see “Properties and Assumptions”.

## Simplify Solutions

While solving equations, MuPAD automatically simplifies many objects such as some function calls and arithmetical expressions with numbers. Automatic

simplifications reduce the complexity of expressions used in intermediate steps, which improves performance of the solvers.

MuPAD solvers do not call the simplification functions for final results. When you call the `solve` command, you can get a long and complicated solution:

```
S:= solve(ln(1/x) + ln(5) = 1/x + ln(3), x)
```

$$\left\{ -\frac{1}{W_0\left(-e^{\ln(3)-\ln(5)}\right)} \right\}$$

To simplify such results, use `simplify` or `Simplify`. The `simplify` function is faster:

```
simplify(S)
```

$$\left\{ e^{\ln\left(\frac{5}{3}\right)+W_0\left(-\frac{3}{5}\right)} \right\}$$

`Simplify` is slower, but more powerful:

```
Simplify(S)
```

$$\left\{ -\frac{1}{W_0\left(-\frac{3}{5}\right)} \right\}$$

For more information, see “Simplification”.

## If Results Differ from Expected

### In this section...

“Verify Equivalence of Expected and Obtained Solutions” on page 3-69

“Verify Equivalence of Solutions Containing Arbitrary Constants” on page 3-70

“Completeness of Expected and Obtained Solutions” on page 3-73

## Verify Equivalence of Expected and Obtained Solutions

Symbolic solutions can be returned in different, but mathematically equivalent forms. MuPAD continuously improves its functionality, including solvers and simplifiers. These improvements can cause different releases of MuPAD to return different forms of the same symbolic expressions. For example, when you solve the equation

```
eq := ode({y'(t)=-a^2*y(t), y(0)=1, y'(PI/a)=0}, y(t)):
```

MuPAD 5.1 (R2008b) returns:

```
solution := solve(eq):
eval(solution) assuming a <> 0
```

$$\{\cos(at)\}$$

For the same equation, MuPAD 5.2 (R2009a) returns:

```
solution := solve(eq):
eval(solution) assuming a <> 0
```

$$\left\{ \frac{1}{2} \frac{e^{ati}}{2} + \frac{e^{ati}}{2} \right\}$$

---

**Note** `testeq` cannot compare sets. To test mathematical equality of the solutions returned as sets, compare each pair of the solutions individually.

---

If a returned solution differs from what you expect, test mathematical equality of the solutions:

```
testeq(cos(a*t), (1/exp(a*t*I))/2 + exp(a*t*I)/2)
```

```
TRUE
```

## Verify Equivalence of Solutions Containing Arbitrary Constants

### Equal Arbitrary Constants

Verifying solutions with arbitrary constants can be a lot more complicated than verifying simple solutions without constants. In such solutions, consider arbitrary constants as symbolic parameters. In simple cases, you can assume that the parameters are equal. For example, solve the following ordinary differential equation. The form of the returned solution depends on the type of an equation. The solver can identify an equation with different equation types:

```
reset()
```

```
o:= ode(y'(x) = (- 1/x + 2*I)*y(x) + 1/x*y(x)^2, y(x)):
o1 := solve(o)
```

$$\left\{ 0, \frac{e^{-\ln(x)+2xi}}{C2+2Ei(1,-2xi)i+\frac{e^{2xi}}{x}} \right\}$$

If you explicitly specify the type as Bernoulli, the solver returns another form of the result:

```
o2 := solve(o, Type = Bernoulli)
```



$$\left\{ 0, \frac{x e^{-\ln(x)+2xi}}{e^{2xi} + C3 x + 2 x Ei(1, -2xi) i} \right\}$$

Check the equality of these two solutions by calling the `testeq` command for each pair of the solutions. Remember that `testeq` cannot compare sets.

```
testeq(o1[1], o2[1]), testeq(o1[2], o2[2])
```

TRUE, FALSE

The second solution returns `FALSE` because `testeq` does not know that `C2` and `C3` are arbitrary constants. When you explicitly assume the equality of the constants, `testeq` confirms that the solutions are mathematically equal:

```
testeq(o1[1], o2[1]) assuming C2 = C3,
testeq(o1[2], o2[2]) assuming C2 = C3
```

TRUE, TRUE

### Arbitrary Constants Representing Different Expressions

Verifying mathematical equality of the results by assuming that the arbitrary constants are equal does not always work. The solver can choose arbitrary constants to represent different expressions. For example, one form of a solution can include `C1` and another form of the same solution can include  $e^{C2}$ . In this case, you need to assume that  $C1 = e^{C2}$ .

If the results include arbitrary constants, the number of elements in a solution set can depend on the form in which MuPAD returns the results. For example, if you specify the type of the following ordinary differential equation as `Chini`, the solver returns three separate solutions:

```
o:= ode(y'(x) = x*y(x)^2 + x*y(x) + x, y(x)):
L:= solve(o, Type = Chini)
```

$$\left\{ -\frac{1}{2} - \frac{\sqrt{3}i}{2}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \frac{\sqrt{3} \tan\left(\frac{\sqrt{3}\left(\frac{x^2}{2} + C4\right)}{2}\right)}{2} - \frac{1}{2} \right\}$$

Specifying the type of the same ordinary differential equation as Riccati gives you two separate solutions:

```
M := solve(o, Type = Riccati)
```

$$\left\{ -\frac{1}{2} - \frac{\sqrt{3}i}{2}, \frac{e^{-\frac{\sqrt{3}x^2i}{2}}}{C5 - \frac{\sqrt{3}e^{-\frac{\sqrt{3}x^2i}{2}}}{3}i} - \frac{1}{2} - \frac{\sqrt{3}i}{2} \right\}$$

When you specify the equation type as Riccati, the solver returns a more general result. This result combines the second and third elements of the set returned for the Chini type. The additional solution for the Chini equation appears at a particular value of the integration constant for the Riccati equation. Find the value of the constant at which the more general solution for the Riccati equation turns to the second solution for the Chini equation:

```
solve(L[2] = M[2], C5)
```

```
{0}
```

Use `evalAt` to verify that if the integration constant is 0, the solution for Riccati equation gives the additional solution that you see for Chini type:

```
evalAt(M[2], C5 = 0)
```

$$-\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

You can find the dependency between the constants in the solutions returned for Riccati and Chini types. As a first step, rewrite the results using similar

terms. For example, rewrite the expression with exponents in terms of tangents:

```
m2 := rewrite(M[2], tan)
```

$$-\frac{\tan\left(\frac{\sqrt{3}x^2}{4}\right) + i}{\left(\tan\left(\frac{\sqrt{3}x^2}{4}\right) - i\right) \left(C5 + \frac{\sqrt{3} \left(\tan\left(\frac{\sqrt{3}x^2}{4}\right) + i\right) i}{3 \left(\tan\left(\frac{\sqrt{3}x^2}{4}\right) - i\right)}\right)} - \frac{1}{2} - \frac{\sqrt{3}i}{2}$$

Define the constant C5 in terms of C4:

```
C5 := simplify(solve(L[3] = m2, C5, IgnoreSpecialCases))
```

$$\left\{ -\sqrt{3} \left( -\frac{\sin(\sqrt{3} C4)}{3} + \frac{\cos(\sqrt{3} C4) i}{3} \right) \right\}$$

Now if you want to verify that the two forms of the solution are equivalent, substitute the constant in Riccati solution with this expression.

For more information on testing mathematical equivalence of the solutions, see Testing Results.

## Completeness of Expected and Obtained Solutions

### Special Cases

The returned results can differ from what you expected due to incompleteness of the expected or the returned solution set. Complete sets of solutions account for all the special cases for all symbolic parameters and variables included in an equation. Complete solutions can be very large and complicated. Often you need only one practical solution from a long or infinite solution set. Some solvers are designed to return incomplete, but practical solutions. For example, consider the equation  $ax + b = y$ . When solving this equation in the MATLAB Command Window, the toolbox ignores special cases:

```
>> solve('a*x + b = y')
```

```
ans =
```

```
-(b - y)/a
```

MuPAD returns the complete set of solutions accounting for all possible values of the symbolic parameters  $a$ ,  $b$ , and  $y$ :

```
solve(a*x + b = y, x)
```

$$\begin{cases} \left\{ -\frac{b-y}{a} \right\} & \text{if } a \neq 0 \\ \mathbb{C} & \text{if } b=y \wedge a=0 \\ \emptyset & \text{if } b \neq y \wedge a=0 \end{cases}$$

Solving the equation in MuPAD with the `IgnoreSpecialCases` option, you get the same short result as in the MATLAB Command Window:

```
solve(a*x + b = y, x, IgnoreSpecialCases)
```

$$\left\{ -\frac{b-y}{a} \right\}$$

### Infinite Solution Sets

Some equations have an infinite number of solutions. When solving the same equation in the MATLAB Command Window, you get only one solution from the infinite set:

```
>> syms x;
```

```
>> solve(sin(x))
```

```
ans =
```

```
0
```

By default, the MuPAD solver returns all the solutions:

```
S := solve(sin(x), x)
```

```
{ $\pi k \mid k \in \mathbb{Z}$ }
```

To get one element of the solution set, use the `solveLib::getElement` command:

```
solveLib::getElement(S)
```

```
0
```

If you want the solver to return just one solution, use the `PrincipalValue` option:

```
S := solve(sin(x), x, PrincipalValue)
```

```
{0}
```

`PrincipalValue` can help you shorten the results omitting all solutions, except one. The option does not allow you to select a particular solution.

## Solve Equations Numerically

### In this section...

“Get Numeric Results” on page 3-76

“Solve Polynomial Equations and Systems” on page 3-78

“Solve Arbitrary Algebraic Equations and Systems” on page 3-79

“Isolate Numeric Roots” on page 3-85

“Solve Differential Equations and Systems” on page 3-85

### Get Numeric Results

There are two methods to get numeric approximations of the solutions:

- Solve equations symbolically and approximate the obtained symbolic results numerically. Using this method, you get numeric approximations of all the solutions found by the symbolic solver. If the symbolic solver fails to find any solutions, MuPAD calls the numeric solver directly. For nonpolynomial equations, the numeric solver returns only one solution. Using the symbolic solver and post-processing its results method requires more time than a purely numeric solver and can significantly decrease performance.
- Solve equations using numeric methods from the beginning. This method is faster, but for nonpolynomial equations the numeric solver returns only the first solution it finds. You can use the `AllRealRoots` option to make the solver look for other real roots. Even with this option, the solution set can be incomplete. Using `AllRealRoots` can significantly decrease performance.

### Approximate Symbolic Solutions Numerically

When you solve an equation symbolically, the solver does not always return results in an explicit form. For example, the solver can represent solutions using `RootOf`:

```
solve(x^3 + x + 1, x)
```

```
RootOf(z^3 + z + 1, z)
```

If you have a symbolic solution that you want to approximate numerically, use the `float` command:

```
float(%)
```

```
{-0.6823278038, 0.3411639019 - 1.1615414 i, 0.3411639019 + 1.1615414 i}
```

Use `float` to approximate the solutions of the symbolic system:

```
float(solve([x^3 + x^2 + 2*x = y, y^2 = x^2], [x, y]))
```

```
{[x = -0.5 + 1.658312395 i, y = 0.5 - 1.658312395 i], [x = -0.5 + 0.8660254038 i, y = -0.5 + 0.8660254038 i], [x = -0.5 - 0.8660254038 i, y = -0.5 - 0.8660254038 i], [x = -0.5 - 1.658312395 i, y = 0.5 + 1.658312395 i], [x = 0.0, y = 0.0]}
```

Approximating symbolic solutions numerically, you get the complete set of solutions. For example, solve the following equation symbolically:

```
S := solve(sin(x^2) = 1/2, x)
```

```
{-σ2 | k ∈ ℤ} ∪ {σ2 | k ∈ ℤ} ∪ {-σ1 | k ∈ ℤ} ∪ {σ1 | k ∈ ℤ}
```

where

$$\sigma_1 = \frac{\sqrt{6} \sqrt{\pi} \sqrt{12k+5}}{6}$$

$$\sigma_2 = \frac{\sqrt{6} \sqrt{\pi} \sqrt{12k+1}}{6}$$

Suppose, you want to get numeric results instead of expressions containing `PI`. The `float` command returns the infinite solution set:

```
float(S)
```

$$\begin{aligned} & \{0.7236012546 \sqrt{12.0 k + 1.0} \mid k \in \mathbb{Z}\} \cup \{0.7236012546 \sqrt{12.0 k + 5.0} \mid k \in \mathbb{Z}\} \\ & \cup \{-0.7236012546 \sqrt{12.0 k + 1.0} \mid k \in \mathbb{Z}\} \cup \{-0.7236012546 \sqrt{12.0 k + 5.0} \mid k \in \mathbb{Z}\} \end{aligned}$$

### Solve Equations Numerically

To avoid getting symbolic solutions and to proceed with numeric methods, use the `numeric::solve` command. If an equation is not polynomial, the numeric solver returns only one solution:

```
numeric::solve(sin(x^2) = 1/2, x)
```

```
{-226.9444724}
```

### Solve Polynomial Equations and Systems

For polynomial equations, the numeric solver returns all solutions:

```
numeric::solve(4*x^4 + 3*x^3 + 2*x^2 + x + 5 = 0, x)
```

```
{-0.8801137713 - 0.7633158339 i, 0.5051137713 + 0.8159896507 i,  
0.5051137713 - 0.8159896507 i, -0.8801137713 + 0.7633158339 i}
```

When called with the option `AllRealRoots`, the solver omits all complex roots. For example, when you symbolically solve the polynomial equation and approximate the solutions, you get all solutions:

```
numeric::solve(4*x^4 + 3*x^3 + 2*x^2 + x - 1 = 0, x)
```

```
{-0.8763850947, 0.3971412064, -0.1353780559 + 0.8366380364 i,  
-0.1353780559 - 0.8366380364 i}
```

To limit the solution set to the real solutions only, use the option `AllRealRoots`:

```
numeric::solve(4*x^4 + 3*x^3 + 2*x^2 + x - 1 = 0, x,  
AllRealRoots)
```



```
{-0.8763850965, 0.3971412051}
```

Using `numeric::solve`, you also can solve a system of polynomial equations:

```
numeric::solve([x^3 + 2*x = y, y^2 = x], [x, y])
```

```
{[x = -0.2812406534 - 1.234872424 i, y = 0.7018735689 - 0.8796971979 i],
 [x = 0.2365742943, y = 0.4863890359], [x = 0.1629535062 + 1.615154465 i,
 y = -0.9450680868 - 0.8545175144 i], [x = 0.1629535062 - 1.615154465 i,
 y = -0.9450680868 + 0.8545175144 i], [x = 0, y = 0],
 [x = -0.2812406534 + 1.234872424 i, y = 0.7018735689 + 0.8796971979 i]}
```

To solve linear systems numerically, use the `numeric::linsolve` command. For example, solve the following system symbolically and numerically:

```
linsolve([x = y - 1, x + y = 5/2], [x, y]);
numeric::linsolve([x = y - 1, x + y = 5/2], [x, y])
```

```
[x = 3/4, y = 7/4]
```

```
[x = 0.75, y = 1.75]
```

## Solve Arbitrary Algebraic Equations and Systems

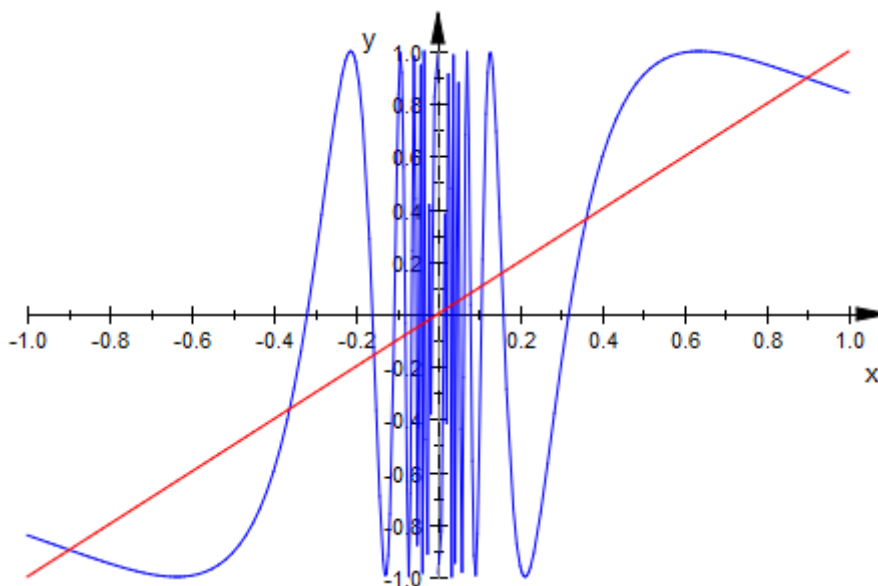
For nonpolynomial equations, there is no general method of finding all the solutions numerically. When you solve a nonpolynomial equation or a system numerically, and the solutions exist, the solver returns only one solution:

```
numeric::solve(sin(1/x) = x, x)
```

```
{-0.00884125012}
```

This equation obviously has more than one solution:

```
plot(sin(1/x), x, x = -1..1)
```



To get more real solutions of a single equation containing one variable, call the numeric solver with the option `AllRealRoots`. The `AllRealRoots` option does not guarantee that the solver finds all existing real roots. For example, the option helps to find additional solutions for the equation:

```
numeric::solve(sin(1/x) = x, x, AllRealRoots)
```

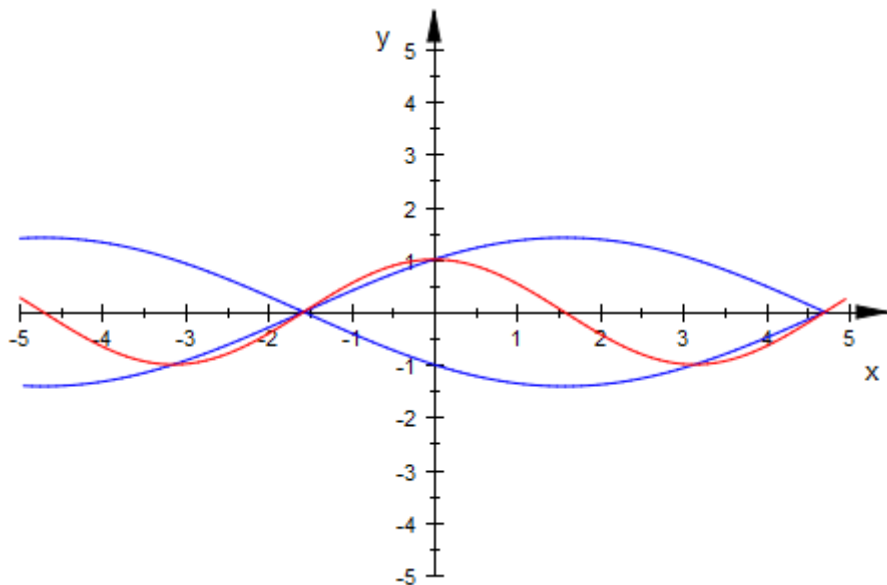
Warning: Problem in isolating search intervals. Some roots might be 1

```
{-0.8975394613, -0.3606716807, -0.1553007909, -0.1073278873, -0.07907931072,
-0.06392228125, -0.05290310069, -0.0455672924, -0.03972592668, -0.03179879686,
-0.0289615375, -0.01988649969, -0.01591146474, -0.007402961185, 0.007402961185,
0.01591146474, 0.01988649969, 0.0289615375, 0.03179879686, 0.03972592668, 0.0455672924,
0.05290310069, 0.06392228125, 0.07907931072, 0.1073278873, 0.1553007909, 0.3606716807,
0.8975394613}
```

For a system of nonpolynomial equation, the solver also returns only one solution. Plotting the equations, you see that the system has more than one solution:

```
numeric::solve([sin(x) = y^2 - 1, cos(x) = y], [x, y]);
plot(sin(x) = y^2 - 1, cos(x) = y)
```

```
{[x = -1.570796327, y = 0.0000000002599318689]}
```



The AllRealRoots option does not work for systems:

```
numeric::solve([sin(x) = y^2 - 1, cos(x) = y], [x, y],
AllRealRoots)
```

```
Error: Only one equation is allowed with the 'AllRealRoots' option. [
```

To find numerical approximations of other solutions, specify intervals that contain the solutions. You can use the command `numeric::solve` that internally calls `numeric::fsolve`. However, to speed up your calculations, call `numeric::fsolve` directly. Note that `numeric::solve` returns a set of solutions, and `numeric::fsolve` returns a list:

```
numeric::solve([sin(x) = y^2 - 1, cos(x) = y],
               [x = 2.5..3.5, y = -1.5..-0.5]);
numeric::fsolve([sin(x) = y^2 - 1, cos(x) = y],
                [x = 4..5, y = -0.2..0.2])
```

```
{[x = 3.141592653, y = -1.0]}
```

```
[x = 4.71238898, y = -0.0000000003741742246]
```

The `MultiSolutions` option also serves to find more than one numeric approximation. Without this option, the numeric solver looks for a solution inside the specified interval and disregards any solutions it finds outside of the interval. When the solver finds the first solution inside the interval, it stops and does not look for other solutions. If you use `MultiSolutions`, the solver returns the solutions found outside of a specified interval.

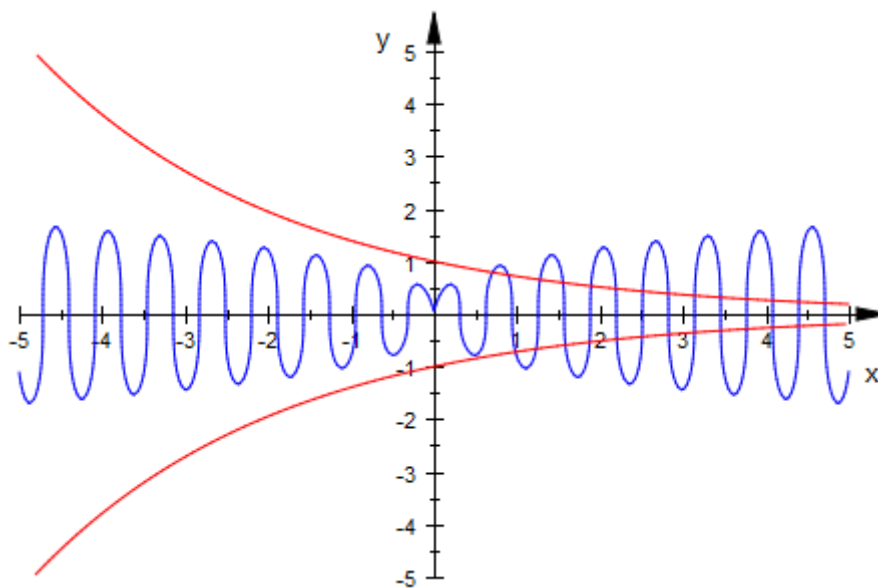
---

**Note** If you use the option `MultiSolutions` and do not specify any interval, the numeric solver returns only the first solution it finds.

---

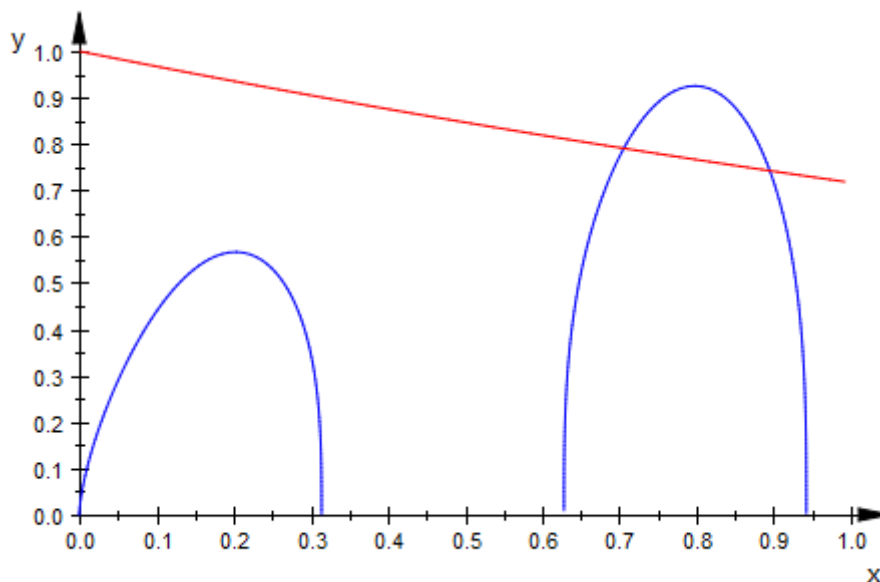
With the `MultiSolutions` option, the solver also stops after it finds the first solution inside the specified interval. For example, find several numeric approximations for the following system:

```
eqs := [x*sin(10*x) = y^3, y^2 = exp(-2*x/3)]:
plot(x*sin(10*x) = y^3, y^2 = exp(-2*x/3))
```



Specify the interval where you want to search for the solutions. For example, consider the interval  $x = 0..1$  that contains two solutions:

```
plot(x*sin(10*x) = y^3, y^2 = exp(-2*x/3), x = 0..1, y = 0..1)
```



Call the `numeric::solve` or `numeric::fsolve` command with the `MultiSolutions` option. Both solvers return one solution that belongs to the specified interval and one solution outside of the interval:

```
numeric::fsolve(eqs, [x = 0..1, y = 0..1], MultiSolutions);
numeric::solve(eqs, [x = 0..1, y = 0..1], MultiSolutions)
```

```
[x = 0.705791548, y = 0.7903622856], [x = 1.278589279, y = 0.6529880654]
```

```
{[x = 1.278589279, y = 0.6529880654], [x = 0.705791548, y = 0.7903622856]}
```

Specifying the interval that does not contain any solutions can help you find more approximations. In this case, the solver cannot find the solution inside the interval and continues searching. Before the solver quits, it can find many solutions outside the specified interval:

```
numeric::fsolve(eqs, [x = -10..0, y = 0..1], MultiSolutions)
```

```
[x = 1.876790003, y = -0.5349421505], [x = 0.8950751882, y = 0.7420353495],
[x = 2.194032234, y = 0.4812617019], [x = 1.58378888, y = -0.5898248642],
[x = 0.705791548, y = 0.7903622856], [x = 1.278589279, y = 0.6529880654],
[x = 2.204123351, y = -0.4796455988], [x = 0.9816416007, y = -0.7209295436]
```

## Isolate Numeric Roots

If you know an interval containing the solutions of a polynomial equation, you can significantly speed up numeric approximations. To find one solution inside a specified interval, use the `numeric::realroot` command. The command returns real solutions and omits the complex ones:

```
numeric::realroot(1/4*x^4 + x^3 + x + 1 = 0, x = -5..0)

-4.174547488
```

You also can specify an interval and search for all subintervals that can contain real roots. The `numeric::realroots` command returns a complete list of such subintervals:

```
numeric::realroots(1/4*x^4 + x^3 + x + 1 = 0, x = -5..0)

[[-4.1875, -4.15625], [-0.7109375, -0.70703125]]
```

If the equation you solve is polynomial, each subinterval contains exactly one root. For nonpolynomial equations, `numeric::realroots` can return subintervals that do not contain any roots. `numeric::realroots` guarantees that the search interval does not contain any real roots outside the returned subintervals.

## Solve Differential Equations and Systems

There is no general way to solve an arbitrary second- or higher-order ordinary differential equation (ODE). Often, such an equation does not have a symbolic solution. In some cases, the solution exists, but it can be presented only by special functions. For example, the solution of the following second-order ODE includes the error function `erf`:

```
o:=ode({y''(t) = t*y'(t), y(0) = 0, y'(0) = 1/3}, y(t)):
ode::solve(o)
```

$$\left\{ -\frac{\sqrt{2} \sqrt{\pi} \operatorname{erf}\left(\frac{\sqrt{2} t i}{2}\right) i}{6} \right\}$$

Suppose, you do not need an exact symbolic solution, but you want to approximate the solution for several values of the parameter  $t$ . For numeric approximations of the solutions of ODEs, MuPAD provides two functions:

- `numeric::odesolve` returns a numeric approximation of the solution at a particular point.
- `numeric::odesolve2` returns a function representing a numeric approximation of the solution.

Both functions accept either a first-order ODE or a system of first-order ODEs. To solve a higher-order equation, convert it to a system of the first-order equations. For example, represent the second-order ODE you solved symbolically as a system of two first-order equations:

$y' = z, z' = t z, y(0) = 0, z(0) = \frac{1}{3}$ . The solution vector for this system is  $Y = [y, z]$ .

### Approximate at Particular Points

The function `numeric::odesolve` requires the system of first-order ODEs (dynamic system) to be represented by a procedure:

```
f := proc(t, Y) begin [Y[2], t*Y[2]] end_proc
proc f(t, Y) ... end
```

The second parameter of `numeric::odesolve` is the range over which you want to solve an ODE. The third parameter is a list of initial conditions

$(y(0) = 0, z(0) = \frac{1}{3})$ . Approximate the solutions  $y(t)$  for  $t = 1, t = 3$ , and  $t = \frac{1}{127}$ :

```
numeric::odesolve(f, 0..1, [0, 1/3]);
numeric::odesolve(f, 0..3, [0, 1/3]);
```



```
numeric::odesolve(f, 0..1/127, [0, 1/3])
```

```
[0.3983192206, 0.5495737569]
```

```
[11.79725153, 30.00571043]
```

```
[0.002624699038, 0.3333436668]
```

MuPAD also offers an alternate way to generate parameters for the ODE numeric solvers:

**1** Define your initial value problem as a list or a set:

```
IVP := {y''(t) = t*y'(t), y(0) = 0, y'(0) = 1/3}:
```

**2** Define a set of fields over which you want to get a solution:

```
fields := [y(t), y'(t)]:
```

**3** Convert the initial value problem and the fields into a procedure acceptable by `numeric::odesolve`. The function `numeric::ode2vectorfield` generates the required procedure:

```
[ODE, t0, Y0] := [numeric::ode2vectorfield(IVP, fields)]
```

```
[proc ODE(t, Y) ... end, 0, [0, 1/3]]
```

Now call `numeric::odesolve` to approximate the solution at particular values of `t`:

```
numeric::odesolve(ODE, t0..1, Y0)
```

```
[0.3983192206, 0.5495737569]
```

### Represent Numeric Approximations as Functions

The function `numeric::odesolve2` also requires the system of first-order ODEs (dynamic system) to be represented by a procedure. To generate parameters for `numeric::odesolve2`, use the following steps:

- 1 Define your initial value problem as a list or a set:

```
IVP := {y''(t) = t*y'(t), y(0) = 0, y'(0) = 1/3}:
```

- 2 Define a set of fields over which you want to get a solution:

```
fields := [y(t), y'(t)]:
```

- 3 Convert the initial value problem and the fields into a procedure acceptable by `numeric::odesolve2`. The function `numeric::ode2vectorfield` generates the required procedure:

```
ODE := numeric::ode2vectorfield(IVP, fields)
```

```
proc(t, Y) ... end, 0, [0, 1/3]
```

Now call `numeric::odesolve2` to approximate the solution:

```
numApprox := numeric::odesolve2(ODE)
```

```
proc numApprox(t) ... end
```

Using the function generated by `numeric::odesolve2`, find the numeric approximation at any point. For example, find the numeric solutions for the values  $t = 1$ ,  $t = 3$ , and  $t = \frac{1}{127}$ . You get the same results as with `numeric::odesolve`, but the syntax is shorter:

```
numApprox(1);
numApprox(3);
numApprox(1/127)
```

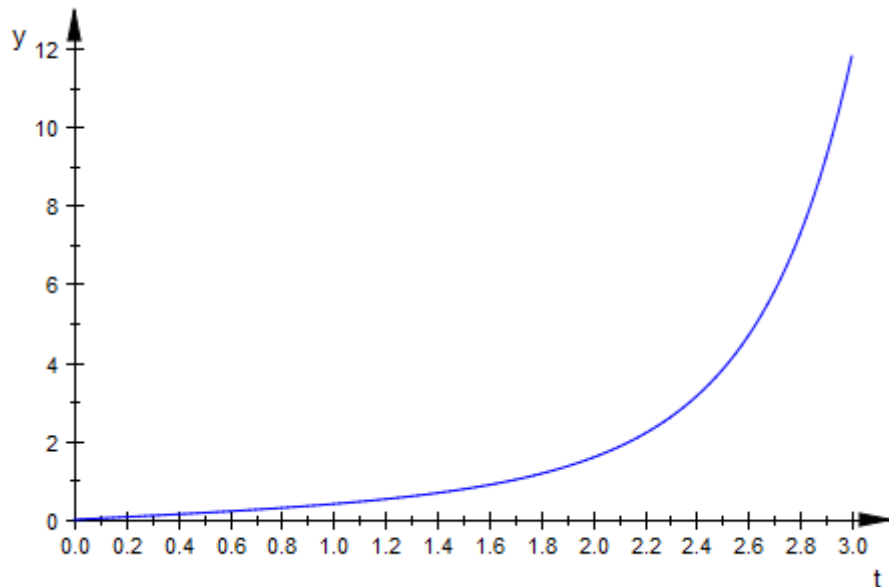
```
[0.3983192206, 0.5495737569]
```

```
[11.79725153, 30.00571043]
```

```
[0.002624699038, 0.3333436668]
```

Plot the numeric solution using the function generated by `numeric::odesolve2`. The function `numApprox` returns a list  $[y(t), y'(t)]$ . When plotting the solution  $y(t)$ , use brackets to extract the first entry of the solution list:

```
plotfunc2d(numApprox(t)[1], t = 0..3)
```



Use `numeric::odesolve2` to find numeric approximations for the following system of ODEs:

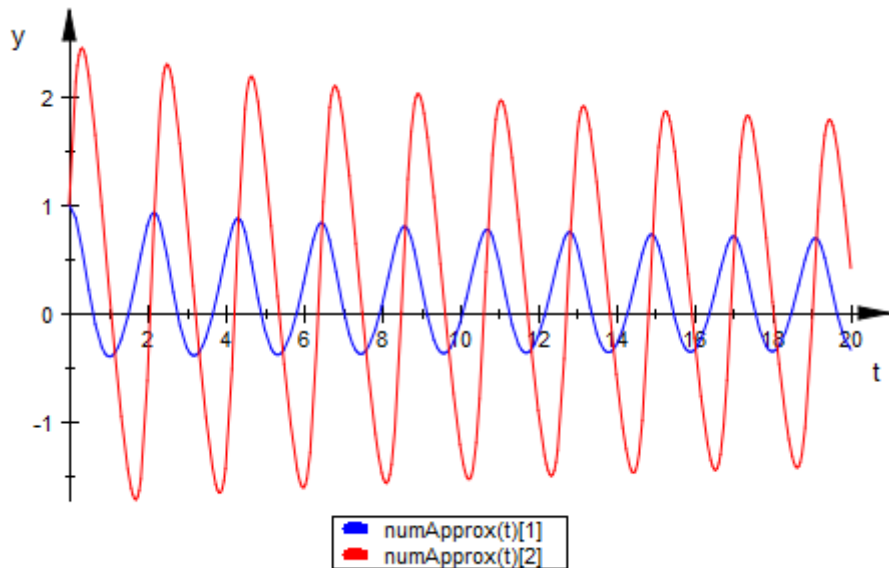
```
IVP := {x'(t) = -y(t) + x(t)^2,
        y'(t) = 10*x(t) - y(t)^2,
        x(0) = 1, y(0) = 1}:
fields := [x(t), y(t)]:
ODE := numeric::ode2vectorfield(IVP, fields):
```

```
numApprox := numeric::odesolve2(ODE)
```

```
proc numApprox(t) ... end
```

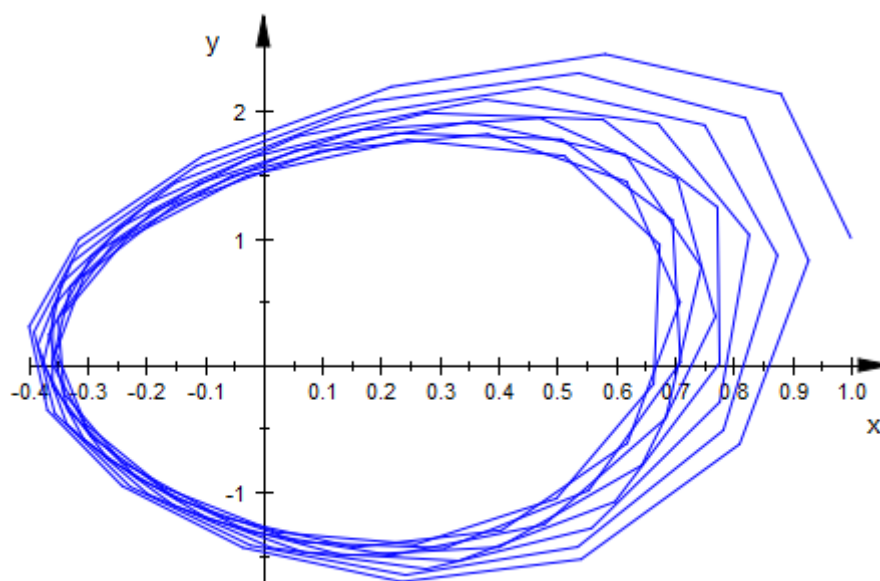
Plot the numeric solutions for  $x(t)$  and  $y(t)$  in one graph:

```
plotfunc2d(numApprox(t)[1], numApprox(t)[2], t = 0..20)
```



Use the `plot::Curve2d` plotting function to generate a parametric plot of the numeric solution:

```
curve := plot::Curve2d([numApprox(t)[1], numApprox(t)[2]],
                       t = 0..20):
plot(curve)
```



Alternatively, use `plot::Ode2d` or `plot::Ode3d`.

## Use General Simplification Functions

### In this section...

“When to Use General Simplifiers” on page 3-92

“Choose simplify or Simplify” on page 3-93

“Use Options to Control Simplification Algorithms” on page 3-93

### When to Use General Simplifiers

Simplification of mathematical expression is not a clearly defined subject. There is no universal idea as to which form of an expression is simplest. The form of a mathematical expression that is simplest for one problem turns out to be complicated or even unsuitable for another problem. For example, the following two mathematical expressions present the same polynomial in different forms:

$$(x + 1)(x - 2)(x + 3)(x - 4),$$

$$x^4 - 2x^3 - 13x^2 + 14x + 24.$$

The first form clearly shows the roots of this polynomial. This form is simpler for working with the roots. The second form serves best when you want to the coefficients of the polynomial.

If the problem you want to solve requires a particular form of an expression, the best approach is to choose the appropriate simplification function. See Choosing Simplification Functions.

Besides specific simplifiers, MuPAD offers two general ones:

- `simplify` searches for a simpler form by rewriting the terms of an expression. This function uses an internal set of rules for rewriting an expression. You cannot modify this set of rules.
- `Simplify` performs more extensive search for a simple form of an expression. For some expressions, this function can be slower, but more flexible and more powerful than `simplify`. `Simplify` uses a wider set of rules to search for a simpler form of an expression. `Simplify` lets you

extend the set of simplification rules and also accepts a number of options allowing you more control over the simplification algorithm.

If you do not need a particular form of expressions (expanded, factored, or expressed in particular terms), use `simplify` and `Simplify` to shorten mathematical expressions. For example, use these functions to find a shorter form for a final result. The general simplifiers also can help you in verifying the result.

## Choose `simplify` or `Simplify`

Use the `simplify` command to simplify elementary expressions such as:

```
simplify((x - 1)*(x + 1)*(x^2 + x + 1)*(x^2 + 1)*
(x^2 - x + 1)*(x^4 - x^2 + 1));
simplify(cos(x)^(-2) - tan(x)^2)
```

$$x^{12} - 1$$

$$1$$

For elementary expressions, `simplify` is an effective and fast simplifier. For more complicated expressions, `simplify` might be less effective. The returned form of the following expression can be shortened further. `Simplify` returns a simpler form:

```
f := (cos(x)^2 - sin(x)^2)/(sin(x)*cos(x));
simplify(f), Simplify(f)
```

$$\frac{2 \cos(x)^2 - 1}{\cos(x) \sin(x)}, 2 \cot(2x)$$

## Use Options to Control Simplification Algorithms

By default, in order to find the simplest form of an expression, the `Simplify` function takes one simplification step. This function can simplify some expressions further by taking more simplifications steps:

```
F := exp((cos(x)^2 - sin(x)^2)*sin(2*x)*(exp(2*x)
-2*exp(x) + 1)/(exp(2*x) - 1)):
Simplify(F)
```

$$e^{-\frac{\sin(4x) - \sin(4x)e^x}{2e^x + 2}}$$

You can change the number of internal simplification steps through the option `Steps`. This option is not available for `simplify`:

```
Simplify(F, Steps = 250)
```

$$e^{\frac{\sin(4x)(e^x - 1)}{2(e^x + 1)}}$$

By default, the general simplifiers return only one form of an expression—the form that MuPAD considers to be simplest. To return all forms found by `Simplify`, use the option `All`:

```
Simplify((x - 1)*(x + 1)*(x^2 + x + 1)*(x^2 + 1)
*(x^2 - x + 1)*(x^4 - x^2 + 1), All)
```

$$\left[ x^{12} - 1, (x^2 + 1)(x - 1)(x + 1)\sigma_2\sigma_1\sigma_3, (x - 1)(x + 1)\sigma_2\sigma_1\sigma_3 \right. \\ \left. + x^2(x - 1)(x + 1)\sigma_2\sigma_1\sigma_3 \right]$$

where

$$\sigma_1 = x^4 - x^2 + 1$$

$$\sigma_2 = x^2 - x + 1$$

$$\sigma_3 = x^2 + x + 1$$



While transforming an expression, MuPAD simplifiers keep all forms of an expression mathematically equivalent to the initial expression. For example, the simplifiers do not combine logarithms. The rule for combining logarithms does not hold for arbitrary complex arguments and, therefore, combining logarithms can be incorrect for some parameters:

```
Simplify(ln(x + 2) - ln(x^2 + 4*x + 4))
```

```
ln(x+2) - ln((x+2)^2)
```

Potential division by zero is the only exception of this rule:

```
Simplify(x*(x + 1)/x)
```

```
x+1
```

To apply more simplification rules that are not generally correct, but which can return simpler results, use the option `IgnoreAnalyticConstraints`. This option is available for both `simplify` and `Simplify`. For example, simplifying an expression with `IgnoreAnalyticConstraints`, you get the result with combined logarithms:

```
Simplify(ln(x + 2) - ln(x^2 + 4*x + 4), IgnoreAnalyticConstraints)
```

```
-ln(x+2)
```

For the list of all options available for the general simplifiers, see `simplify` and `Simplify`.

## Choose Simplification Functions

In this section...
“Collect Terms with Same Powers” on page 3-97
“Combine Terms of Same Algebraic Structures” on page 3-98
“Expand Expressions” on page 3-99
“Factor Expressions” on page 3-100
“Compute Normal Forms of Expressions” on page 3-101
“Compute Partial Fraction Decompositions of Expressions” on page 3-102
“Simplify Radicals in Arithmetic Expressions” on page 3-103
“Extract Real and Imaginary Parts of Complex Expressions” on page 3-103
“Rewrite Expressions in Terms of Other Functions” on page 3-104

Most mathematical expressions can be represented in different, but mathematically equivalent forms. Some of these forms might look simpler, for example, they can be visibly shorter. However, you might prefer other forms of the same expression for numeric computations. There is no general rule as to which form of an expression is the simplest. When solving a particular problem, you can choose the simplest form of an expression for this problem.

Besides the general simplification functions `simplify` and `Simplify`, MuPAD provides a set of functions for transforming mathematical expressions to particular forms. The following table helps you choose the function for transforming your expression to the appropriate form. To see a short description of a function and a set of examples, click the link in the left column. To see the detailed help page for a function, click the function name in the right column.

Type of Transformation	Function
Collect terms with the same powers	<code>collect</code>
Combine terms of the same algebraic structure	<code>combine</code>
Expand an expression	<code>expand</code>
Factor an expression	<code>factor</code>

Type of Transformation	Function
Normalize an expression	normal
Compute a partial fraction decomposition	partfrac
Simplify radicals in an expression	radsimp
Separate the real and imaginary parts of a complex expression	rectform
Rewrite an expression in terms of a specified target function	rewrite

## Collect Terms with Same Powers

If a mathematical expression contains terms with the same powers of a specified variable or expression, the `collect` function reorganizes the expression grouping such terms. When calling `collect`, specify the variables or expressions that the function should consider as unknowns. The `collect` function regards an original expression as a polynomial in the specified unknowns and groups the coefficients with equal powers:

```
collect(x*y^4 + x*z + 2*x^3 + x^2*y*z + 3*x^3*y^4*z^2
        + y*z^2 + 5*x*y*z, x)
```

$$(3y^4z^2 + 2)x^3 + (yz)x^2 + (y^4 + 5zy + z)x + yz^2$$

`collect` can consider an expression as the specified unknown. For example, group the terms of the following trigonometric expression with the equal powers of  $\sin(x)$  and  $\cos(x)$ :

```
f := cos(x)^4*sin(x) + cos(x)^2*sin(x)^3
    + cos(x)^2*sin(x)^2 - sin(x)^4:
collect(f, sin(x))
```

$$-\sin(x)^4 + \cos(x)^2 \sin(x)^3 + \cos(x)^2 \sin(x)^2 + \cos(x)^4 \sin(x)$$

```
collect(f, cos(x))
```

$$\sin(x) \cos(x)^4 + (\sin(x)^3 + \sin(x)^2) \cos(x)^2 - \sin(x)^4$$

The `collect` function also can accept several unknowns for collecting terms. If you have several unknowns, pass them to `collect` as a list:

```
collect(a^2*sin(x) - cos(x)^2*sin(x)^3 + cos(x)^2
      + a - a^2*sin(x)^4, [a, cos(x)])
```

$$(\sin(x) - \sin(x)^4) a^2 + a + (1 - \sin(x)^3) \cos(x)^2$$

### Combine Terms of Same Algebraic Structures

MuPAD also provides a function for combining subexpressions of an original expression. The `combine` function uses mathematical identities for the functions you indicate. For example, combine the trigonometric expression:

```
combine(2*sin(x)*cos(x), sincos)
```

$$\sin(2x)$$

If you do not specify a target function, `combine` uses the identities for powers wherever these identities are valid:

- $a^b a^c = a^{b+c}$
- $a^c b^c = (ab)^c$ , if  $c$  is an integer
- $(a^b)^c = a^{bc}$ , if  $c$  is an integer

For example, by default the function combines the following square roots:

```
combine(sqrt(2)*sqrt(x))
```

$$\sqrt{2x}$$

The function does not combine these square roots because the identity is not valid for negative values of variables:

```
combine(sqrt(x)*sqrt(y))
```

$$\sqrt{x} \sqrt{y}$$

As target functions, `combine` accepts `arctan`, `exp`, `gamma`, `ln`, `sincos`, and other functions. For the complete list of target functions, see the `combine` help page.

## Expand Expressions

For elementary expressions, the `expand` function transforms the original expression by multiplying sums of products:

```
expand((x + 1)*(x + 2)*(x + 3))
```

$$x^3 + 6x^2 + 11x + 6$$

`expand` also uses mathematical identities between the functions:

```
expand(sin(5*x))
```

$$16 \sin(x) \cos(x)^4 - 12 \sin(x) \cos(x)^2 + \sin(x)$$

`expand` works recursively for all subexpressions:

```
expand((sin(3*x) + 1)*(cos(2*x) - 1))
```

$$2 \sin(x) + 2 \cos(x)^2 - 10 \cos(x)^2 \sin(x) + 8 \cos(x)^4 \sin(x) - 2$$

To prevent the expansion of particular subexpressions, pass these subexpressions to `expand` as arguments:

```
expand((sin(3*x) + 1)*(cos(2*x) - 1), sin(3*x))
```

$$2 \cos(x)^2 - 2 \sin(3x) + 2 \sin(3x) \cos(x)^2 - 2$$

To prevent the expansion of all trigonometric subexpressions in this example, use the option `ArithmeticOnly`:

```
expand((sin(3*x) + 1)*(cos(2*x) - 1), ArithmeticOnly)
```

```
cos(2 x) - sin(3 x) + cos(2 x) sin(3 x) - 1
```

## Factor Expressions

To present an expression as a product of sums, try the `factor` function. The factored form of the following polynomial is visibly shorter than the original one. The factored form also shows that this polynomial has only one root  $x = -5$ :

```
factor(x^10 + 50*x^9 + 1125*x^8 + 15000*x^7 + 131250*x^6
      + 787500*x^5 + 3281250*x^4 + 9375000*x^3
      + 17578125*x^2 + 19531250*x + 9765625)
```

```
(x + 5)^10
```

For sums of rational expressions, `factor` first computes a common denominator, and then factors both the numerator and denominator:

```
f := (x^3 + 3*y^2)/(x^2 - y^2) + 3:
f = factor(f)
```

$$\frac{x^3 + 3y^2}{x^2 - y^2} + 3 = \frac{x^2(x+3)}{(x-y)(x+y)}$$

The function also can factor expressions other than polynomials and rational functions. Internally, MuPAD converts such expressions into polynomials or rational function by substituting subexpressions with identifiers. After factoring the expression with temporary identifiers, MuPAD restores the original subexpressions:

```
factor((ln(x)^2 - 1)/(cos(x)^2 - sin(x)^2))
```

$$\frac{(\ln(x) - 1) (\ln(x) + 1)}{(\cos(x) - \sin(x)) (\cos(x) + \sin(x))}$$

By default, `factor` searches for polynomial factors with rational numbers. The function does not factor the expression into a product containing  $\sqrt{2}$ :

```
factor(x^2 - 2)
```

$$x^2 - 2$$

To add the constant  $\sqrt{2}$  to the numbers used in factoring, use the option `Adjoin`:

```
factor(x^2 - 2, Adjoin = [sqrt(2)])
```

$$(x - \sqrt{2}) (x + \sqrt{2})$$

## Compute Normal Forms of Expressions

The `normal` function represents the original rational expression as a single rational term with expanded numerator and denominator. The greatest common divisor of the numerator and denominator of the returned expression is 1:

```
f := (x^3 + 3*y^2)/(x^2 - y^2) + 3:
f = normal(f)
```

$$\frac{x^3 + 3y^2}{x^2 - y^2} + 3 = \frac{x^3 + 3x^2}{x^2 - y^2}$$

`normal` cancels common factors that appear in numerator and denominator:

```
f := x^2/(x + y) - y^2/(x + y):
f = normal(f)
```

$$\frac{x^2}{x+y} - \frac{y^2}{x+y} = x - y$$

The `normal` function also handles expressions other than polynomials and rational functions. Internally, MuPAD converts such expressions into polynomials or rational functions by substituting subexpressions with identifiers. After normalizing the expression with temporary identifiers, MuPAD restores the original subexpressions:

```
f := (exp(2*x) - exp(2*y))/(exp(3*x) - exp(3*y)):
f = normal(f)
```

$$\frac{e^{2x} - e^{2y}}{e^{3x} - e^{3y}} = \frac{e^x + e^y}{e^{2x} + e^{2y} + e^x e^y}$$

## Compute Partial Fraction Decompositions of Expressions

The `partfrac` function returns a rational expression in the form of a sum of a polynomial and rational terms. In each rational term, the degree of the numerator is smaller than the degree of the denominator. For some expressions, `partfrac` returns visibly simpler forms, for example:

```
partfrac((x^6 + 15*x^5 + 94*x^4 + 316*x^3 + 599*x^2
+ 602*x + 247)/(x^6 + 14*x^5 + 80*x^4
+ 238*x^3 + 387*x^2 + 324*x + 108), x)
```

$$\frac{1}{x+1} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^3} + 1$$

The denominators in rational terms represent the factored common denominator of the original expression:

```
factor(x^6 + 14*x^5 + 80*x^4 + 238*x^3 + 387*x^2 + 324*x + 108)

(x+1)(x+2)^2(x+3)^3
```



## Simplify Radicals in Arithmetic Expressions

As an alternative to the general simplifiers `simplify` and `Simplify`, use the `radsimp` function to simplify arithmetic expressions involving square roots or other radicals. For example, simplify the following numeric expression:

```
f := 3*sqrt(7)/(sqrt(7) - 2);
radsimp(f)
```

$$\frac{3\sqrt{7}}{\sqrt{7}-2}$$

$$2\sqrt{7}+7$$

## Extract Real and Imaginary Parts of Complex Expressions

When working with complex numbers, you might need to separate the real and imaginary part of a symbolic expression. To extract the real and imaginary part of a complex number, you use the `Re` and `Im` functions. For symbolic expressions, these functions return:

```
Re(tan(x)), Im(tan(x))
```

```
ℜ(tan(x)), ℑ(tan(x))
```

Use the `rectform` function to split a symbolic expression into its real and imaginary parts:

```
y := rectform(tan(x))
```

$$\frac{\sin(2 \Re(x))}{\cosh(2 \Im(x)) + \cos(2 \Re(x))} + \frac{\sinh(2 \Im(x))}{\cosh(2 \Im(x)) + \cos(2 \Re(x))} i$$

To extract the real and imaginary parts of `y`, use the `Re` and `Im` functions:

```
Re(y); Im(y)
```

$$\frac{\sin(2 \Re(x))}{\cosh(2 \Im(x)) + \cos(2 \Re(x))}$$

$$\frac{\sinh(2 \Im(x))}{\cosh(2 \Im(x)) + \cos(2 \Re(x))}$$

## Rewrite Expressions in Terms of Other Functions

To present an expression in terms of a particular function, use the `rewrite` command. The command uses mathematical identities between functions. For example, rewrite an expression containing trigonometric functions in terms of a particular trigonometric function:

```
sin(x) = rewrite(sin(x), tan);
cos(x) = rewrite(cos(x), tan);
sin(2*x) + cos(3*x)^2 = rewrite(sin(2*x) + cos(3*x)^2, tan)
```

$$\sin(x) = \frac{2 \tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)^2 + 1}$$

$$\cos(x) = -\frac{\tan\left(\frac{x}{2}\right)^2 - 1}{\tan\left(\frac{x}{2}\right)^2 + 1}$$

$$\cos(3x)^2 + \sin(2x) = \frac{\left(\tan\left(\frac{3x}{2}\right)^2 - 1\right)^2}{\left(\tan\left(\frac{3x}{2}\right)^2 + 1\right)^2} + \frac{2 \tan(x)}{\tan(x)^2 + 1}$$

Use `rewrite` to express the trigonometric and hyperbolic functions in terms of the exponential function:

```
sin(x) = rewrite(sin(x), exp);
cos(x) = rewrite(cos(x), exp);
sinh(x) = rewrite(sinh(x), exp);
```

`cosh(x) = rewrite(cosh(x), exp)`

$$\sin(x) = \frac{e^{-xi}i - e^{xi}i}{2}$$

$$\cos(x) = \frac{e^{-xi} + e^{xi}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^{-x} + e^x}{2}$$

The command also expresses inverse hyperbolic functions in terms of logarithms:

`arcsinh(x) = rewrite(arcsinh(x), ln);`

`arccosh(x) = rewrite(arccosh(x), ln)`

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

As target functions, `rewrite` accepts: direct and inverse trigonometric functions, direct and inverse hyperbolic functions, `diff`, `D`, `erf`, `exp`, `fact`, `gamma`, `harmonic`, `piecewise`, and more. See the `rewrite` help page for the complete list of target functions.

## If You Want to Simplify Results Further

### In this section...

“Increase the Number of Simplification Steps” on page 3-106

“Apply Several Simplification Functions” on page 3-107

“Use Options” on page 3-108

“Use Assumptions” on page 3-109

### Increase the Number of Simplification Steps

When simplifying a complicated expression, MuPAD can return results that would benefit from further simplifications. Suppose you want to simplify the following expression:

$$f := (- (4 * \pi * 3^{(1/2)}) / 3 + \pi * (-1)^{(3/4)} * 2^{(1/2)} * 3^{(1/2)} * (- (2 * i) / 3 - 2/3)) * (\cos(3 * \arccos(x)))$$

$$- \cos(3 \arccos(x)) \left( \frac{4 \pi \sqrt{3}}{3} + \pi (-1)^{3/4} \sqrt{2} \sqrt{3} \left( \frac{2}{3} + \frac{2i}{3} \right) \right)$$

First, try calling the general simplifier `simplify`:

`simplify(f)`

$$- \pi \cos(3 \arccos(x)) \left( \frac{4 \sqrt{3}}{3} + (-1)^{3/4} \sqrt{6} \left( \frac{2}{3} + \frac{2i}{3} \right) \right)$$

The returned expression has an even shorter representation. To simplify this result further, call the `Simplify` command:

`Simplify(f)`

$$- \frac{2 \pi \sqrt{3} x \left( 2 + (-1)^{3/4} \sqrt{2} (1+i) \right) (4x^2 - 3)}{3}$$

You can simplify the result even more by increasing the number of steps:

```
Simplify(f, Steps = 150)
```

```
0
```

## Apply Several Simplification Functions

To transform a very complicated expression or an expression that should appear in a particular form, you might need to apply several simplification functions. When you transform an expression to a particular form by using a special transformation function, the function can return results that are not fully simplified. Suppose you want to rewrite the trigonometric expression in terms of exponential functions. The `rewrite` command replaces the `sin` and `cos` functions by exponential functions, but does not simplify the result:

```
f := rewrite(sin(x)/cos(x), exp)
```

$$\frac{\frac{e^{-xi}}{2} - \frac{e^{xi}}{2}}{\frac{e^{-xi}}{2} + \frac{e^{xi}}{2}}$$

To simplify the resulting expression, call the `simplify` command:

```
simplify(f)
```

$$-\frac{e^{2xi} - 1}{e^{2xi} + 1}$$

The more powerful `Simplify` function converts this expression back to its trigonometric form:

```
Simplify(f)
```

```
tan(x)
```

## Use Options

When transforming expressions, the MuPAD simplifiers apply the rules valid for the entire complex plane. For example, try to simplify this expression containing logarithms:

$$h := \ln(x + 1)/2 - \ln(1 - 1/x)/2 - \ln(1 - x)/2 + \ln(1/x + 1)/2$$

$$\frac{\ln(x+1)}{2} - \frac{\ln\left(1 - \frac{1}{x}\right)}{2} - \frac{\ln(1-x)}{2} + \frac{\ln\left(\frac{1}{x}+1\right)}{2}$$

By default, the simplifier does not combine logarithms because this operation is only valid for particular real numbers. For complex numbers, combining logarithms is not generally valid:

`Simplify(h)`

$$\frac{\ln(x+1)}{2} - \frac{\ln\left(1 - \frac{1}{x}\right)}{2} - \frac{\ln(1-x)}{2} + \frac{\ln\left(\frac{1}{x}+1\right)}{2}$$

If you solve a problem that does not require application of strict mathematical rules, try using the `IgnoreAnalyticConstraints` option. With this option, the simplifier uses a set of mathematical rules that are not generally correct. For example, if you use the `IgnoreAnalyticConstraints` option, the simplifier returns:

`Simplify(h, IgnoreAnalyticConstraints)`

$$\ln(x+1) - \frac{\ln(x-1)}{2} - \frac{\ln(1-x)}{2}$$

The results obtained with the option `IgnoreAnalyticConstraints` are most useful for many in engineering and physics problems. Note that when you use this option, the simplifiers do not guarantee the equivalence of the original and simplified expressions for the entire complex plane.

## Use Assumptions

When transforming an expression, the simplification functions apply the rules valid for the entire plane of complex numbers. By default, MuPAD does not assume any additional mathematical properties on the identifiers. For example, the identity  $\ln(e^x) = x$  is not generally valid for all complex numbers:

```
simplify(ln(exp(x)))
```

$$\ln(e^x)$$

When you work with real numbers, the simplification functions can also use the rules valid for real numbers. Use the `assume` or `assuming` command to specify that a variable  $x$  represents a real number. The `assume` command creates a permanent assumption. The `assuming` command creates a temporary assumption, which is valid during a single command. The simplifier applies the appropriate rule and returns the expected result:

```
simplify(ln(exp(x))) assuming x in R_
```

$$x$$

When you simplify the following expression, the returned expression is shorter than the original one. However, you can further simplify the returned expression:

```
f := ln(- 2*sin(-(sin(x)*I)/2)^2 + sin(-sin(x)*I)*I + 1);
Simplify(f)
```

$$\ln\left(-2 \sin\left(\frac{\sin(x) i}{2}\right)^2 - \sin(\sin(x) i) i + 1\right)$$

$$\ln\left(2 \sinh\left(\frac{\sin(x)}{2}\right)^2 + \sinh(\sin(x)) + 1\right)$$

Increasing the number of steps simplifies the expression further:

```
Simplify(f, Steps = 300)
```

```
ln(cosh(sin(x)) + sinh(sin(x)))
```

If you want to get a simplified result for real  $x$ , assume that  $x$  is real:

```
assume(x, Type::Real);  
Simplify(f, Steps = 300)
```

```
sin(x)
```

To remove an assumption, use the `unassume` command:

```
unassume(x);  
is(x, Type::Real)
```

```
UNKNOWN
```

When assuming any additional mathematical property for a variable (such as assuming that  $x$  is real), make sure that your problem does not require solutions to be valid for all complex numbers. Be especially careful if your initial expression contains complex numbers.

For more information about assumptions, see “Properties and Assumptions”.



## Convert Expressions Involving Special Functions

### In this section...

“Simplify Special Functions Automatically” on page 3-111

“Use General Simplifiers to Reduce Special Functions” on page 3-111

“Expand Expressions Involving Special Functions” on page 3-113

“Verify Solutions Involving Special Functions” on page 3-113

### Simplify Special Functions Automatically

MuPAD provides many special functions commonly used in engineering and science. MuPAD uses standard mathematical notations for special functions. If you do not recognize a notation, see Mathematical Notations Uses in Typeset Mode.

Particular parameter choices can simplify special functions. Often MuPAD handles such simplifications automatically. For example, the following parameters reduce hypergeometric functions to elementary functions:

```
hypergeom([], [], z);
hypergeom([1], [], z);
hypergeom([a], [], z)
```

$$e^z$$

$$-\frac{1}{z-1}$$

$$\frac{1}{(1-z)^a}$$

### Use General Simplifiers to Reduce Special Functions

MuPAD does not automatically simplify some functions. For example, it does not automatically simplify the Meijer G special function:

```
meijerG([], [], [[1], []], z)
```

$$G_{0,1}^{1,0} \left( 1 \mid z \right)$$

The general simplification functions, `simplify` and `Simplify`, represent this expression in terms of elementary functions:

```
simplify(meijerG([], [], [[1], []], z))
```

$$z e^{-z}$$

MuPAD also does not use automatic simplifications for many expressions involving special functions. Suppose you get an expression containing the Fresnel sine integral function:

```
2*fresnelS(z) + fresnelS(-z)
```

$$S(-z) + 2 S(z)$$

To apply the reflection rule  $\text{fresnelS}(-z) = -\text{fresnelS}(z)$  and simplify this expression, explicitly call one of the general simplifiers:

```
simplify(2*fresnelS(z) + fresnelS(-z))
```

$$S(z)$$

Particular values of parameters can reduce more general special functions to expressions containing simpler special functions. For example, reduce `meijerG` to the hypergeometric functions:

```
Simplify(meijerG([[1/3, 1/3, 3/2], []], [[0], [-2/3, 4/3]], z))
```

$$3 \sqrt{\pi} {}_2F_1 \left( -\frac{1}{2}, \frac{2}{3}; \frac{5}{3}; -z \right) - 2 \sqrt{\pi} \sqrt{z+1}$$

The following choice of parameters expresses `meijerG` in terms of the Bessel functions:

```
simplify(meijerG([], [], [[1], [1]], z))
```

$$z J_0(2\sqrt{z})$$

## Expand Expressions Involving Special Functions

MuPAD supports expansions of expressions containing special functions. The resulting expressions can involve the original or additional special functions, or both. For example, the `expand` command expresses the beta function by gamma functions:

```
beta(x + 1, y) = expand(beta(x + 1, y))
```

$$\beta(y, x + 1) = \frac{x \Gamma(x) \Gamma(y)}{x \Gamma(x + y) + y \Gamma(x + y)}$$

When you expand the gamma function, MuPAD expresses it in terms of gamma functions:

```
gamma(5*x + 1) = expand(gamma(5*x + 1))
```

$$\Gamma(5x + 1) = \frac{5^{5x} \sqrt{5} x \Gamma\left(x + \frac{1}{5}\right) \Gamma\left(x + \frac{2}{5}\right) \Gamma\left(x + \frac{3}{5}\right) \Gamma\left(x + \frac{4}{5}\right) \Gamma(x)}{4 \pi^2}$$

## Verify Solutions Involving Special Functions

When solving equations (especially ordinary differential equations), you often get the results in terms of special functions. For example, consider the following differential equation:

```
eq := diff(y(x), x, x) - x^3*y(x) = 0
```

$$\frac{\partial^2}{\partial x^2} y(x) - x^3 y(x) = 0$$

For this ODE, the solver returns the result in terms of the Bessel functions:

```
S := solve(ode(eq, y(x)))
```

$$\left\{ C2 \sqrt{x} I_{\frac{1}{5}}\left(\frac{2x^{5/2}}{5}\right) + C3 \sqrt{x} K_{\frac{1}{5}}\left(\frac{2x^{5/2}}{5}\right) \right\}$$

To verify correctness of the returned solution, try substituting it into the original equation by using `evalAt` or its shortcut `|`. You get the following long and complicated result that still contains the Bessel special functions. MuPAD does not automatically simplify this result:

```
eq | y(x) = S[1]
```

$$\begin{aligned}
& C2 \sqrt{x} \left( \frac{\sigma_3}{2x} - x^{3/2} \left( \frac{2\sigma_5}{x} - x^{3/2} I_{\frac{1}{5}}(\sigma_6) \right) + \frac{I_{\frac{1}{5}}(\sigma_6)}{2x^2} + \frac{3\sqrt{x}\sigma_5}{2} \right) - \frac{C2 I_{\frac{1}{5}}(\sigma_6)}{\sigma_1} - \frac{C3 K_{\frac{1}{5}}(\sigma_6)}{\sigma_1} - \frac{C2 \sigma_3}{\sqrt{x}} \\
& - \frac{C3 \sigma_2}{\sqrt{x}} - x^3 \left( C2 \sqrt{x} I_{\frac{1}{5}}(\sigma_6) + C3 \sqrt{x} K_{\frac{1}{5}}(\sigma_6) \right) \\
& + C3 \sqrt{x} \left( \frac{\sigma_2}{2x} + x^{3/2} \left( \frac{2\sigma_4}{x} + x^{3/2} K_{\frac{1}{5}}(\sigma_6) \right) + \frac{K_{\frac{1}{5}}(\sigma_6)}{2x^2} - \frac{3\sqrt{x}\sigma_4}{2} \right) = 0
\end{aligned}$$

where

$$\sigma_1 = 4x^{3/2}$$

$$\sigma_2 = \frac{K_{\frac{1}{5}}(\sigma_6)}{2x} + x^{3/2} \sigma_4$$

$$\sigma_3 = \frac{I_{\frac{1}{5}}(\sigma_6)}{2x} - x^{3/2} \sigma_5$$

$$\sigma_4 = K_{-\frac{4}{5}}(\sigma_6)$$

$$\sigma_5 = I_{-\frac{4}{5}}(\sigma_6)$$

$$\sigma_6 = \frac{2x^{5/2}}{5}$$

Simplifying this expression proves the correctness of the solution:

```
simplify(eq | y(x) = S[1])
```

```
TRUE
```

The `testeq` command serves best for verifying correctness of the solutions. The command automatically simplifies expressions on both sides of the equation:

```
testeq(eq | y(x) = S[1])
```

```
TRUE
```

For more information see [Testing Results](#).

## When to Use Assumptions

By default, MuPAD assumes that all symbolic parameters and variables represent complex numbers. If you perform computations that involve unknowns with natural restrictions, you can set assumptions on these unknowns. For example, when solving an equation where one of the parameters represents real numbers, request the solver to consider this parameter as a real number. Use assumptions to limit the number of solutions to those necessary and to improve code performance.

When solving an equation, inequality, or a system, you can use assumptions on parameters of the equation and assumptions on the variables you solve for. Setting assumptions on parameters and variables affects performance of the solver in different ways:

- **Assumptions on parameters** tend to narrow the area in which the solver tries to find the solutions, thereby improving performance of the solver. If you can identify mathematical properties of the parameters in your equation, inequality, or system, use these properties to set as many assumptions on parameters as possible.
- **Assumptions on variables** can narrow the returned results. MuPAD applies assumptions on variables after the solver finds the solutions. The solver verifies the results against the assumptions and returns only those solutions that agree with the assumptions. Adding this extra task can slow down the solver. Use assumptions on variables sparsely.

Alternatively, you can simplify already returned complicated results by using assumptions. See [Using Assumptions](#).

There are two types of assumptions you can set:

- **Permanent assumptions** hold true for all calculations MuPAD performs after you set the assumptions. If you want MuPAD to stop using a permanent assumption, you must explicitly delete the assumption. Permanent assumptions serve best when you know that an object holds its property throughout the solution process. For more information see [Using Permanent Assumptions](#).
- **Temporary assumptions** hold true only for the particular evaluation where you set them. For example, if you use a temporary assumption while

solving a single equation, the solver applies this assumption only to solve this particular equation. Using temporary assumptions to solve problems works best when an object holds its property only during particular calculations. Temporary assumptions also help you to keep the object name free and reuse it during the solution process. For more information see [Using Temporary Assumptions](#).



## Use Permanent Assumptions

### In this section...

“Set Permanent Assumptions” on page 3-119

“Add Permanent Assumptions” on page 3-122

“Clear Permanent Assumptions” on page 3-123

### Set Permanent Assumptions

Permanent assumptions work best for the mathematical properties that hold true throughout your computations. Suppose, you want to calculate the time during which a free-falling object drops from the height  $h$ . The kinematic equation for the free fall motion is  $h = gt^2$ , where  $g$  is the free fall acceleration. Using this equation, calculate the time during which the object falls from a certain height. Without assumptions, you get the complete solution for all possible values of parameters including complex values:

```
t = solve(h = g*t^2/2, t)
```

$$t = \begin{cases} \left\{ \frac{\sqrt{2}\sqrt{h}}{\sqrt{g}}, -\frac{\sqrt{2}\sqrt{h}}{\sqrt{g}} \right\} & \text{if } g \neq 0 \\ \mathbb{C} & \text{if } g = 0 \wedge h = 0 \\ \emptyset & \text{if } g = 0 \wedge h \neq 0 \end{cases}$$

If you do not consider the special case where no gravitational forces exist, you can safely assume that the gravitational acceleration is positive. This assumption removes the special zero-gravity cases from the solution:

```
assume(g > 0);
t = solve(h = g*t^2/2, t)
```

$$t = \left\{ \frac{\sqrt{2}\sqrt{h}}{\sqrt{g}}, -\frac{\sqrt{2}\sqrt{h}}{\sqrt{g}} \right\}$$

The variable  $h$  in the equation represents the height from which the object falls. If you do not consider that someone initially throws the object upward and that the object reflects from the ground, the height  $h$  is always positive. Therefore, you can assume that both gravitational acceleration  $g$  and height  $h$  are positive:

```
assume(g > 0 and h > 0);
t = solve(h = g*t^2/2, t)
```

$$t = \left\{ \frac{\sqrt{2} \sqrt{h}}{\sqrt{g}}, -\frac{\sqrt{2} \sqrt{h}}{\sqrt{g}} \right\}$$

Assuming that the time of the drop is a positive value, you get the expected result. When you set assumptions on variables, the solver compares the obtained solutions with the specified assumptions. This additional task can slow down the solver:

```
assume(g > 0 and h > 0 and t > 0);
t := solve(h = g*t^2/2, t)
```

$$\left\{ \frac{\sqrt{2} \sqrt{h}}{\sqrt{g}} \right\}$$

The solver returns the solutions as a set, even if the set contains only one element. To access the elements of a solution set, use square brackets or the `op` command:

```
time = t[1]
```

$$\text{time} = \frac{\sqrt{2} \sqrt{h}}{\sqrt{g}}$$

Clear the variable  $t$  for further computations:

```
delete t
```

If you set several assumptions for the same object, each new assumption overwrites the previous one:

```
assume(h in R_);
assume(h <> 0);
is(h in R_), is(h <> 0)
```

UNKNOWN, TRUE

If you want to keep the previous assumption while adding a new one, see Adding Assumptions.

The `assume` command cannot solve assumptions in the form of equations and does not assign values to the variables:

```
assume(g + 5 = 14.8 and 2*t = 14);
h = g*t^2/2
```

$$h = \frac{g t^2}{2}$$

When you set an assumption in the form of an inequality, both sides of the inequality must represent real values. Inequalities with complex numbers are invalid because the field of complex numbers is not an ordered field. For example, if you try to use the following assumption, MuPAD returns an error:

```
assume(t > 2*I)
```

Error: Assumptions are inconsistent. [property::\_assume]

You can use complex values in an assumption written in the form of an equation:

```
assume(t = 2*PI*I)
```

## Add Permanent Assumptions

When you set an assumption on an object, MuPAD replaces the previous assumptions on that object with the new assumption:

```
assume(x in Z_);  
assume(x in R_);  
is(x in Z_), is(x in R_)
```

UNKNOWN, TRUE

To add a new assumption without removing the previous assumptions, use the `assumeAlso` command:

```
assume(x in Z_);  
assumeAlso(x in R_);  
is(x in Z_), is(x in R_)
```

TRUE, TRUE

Also, you can set multiple assumptions in one function call by using the logical operators. For example, set two assumptions on  $x$ :

```
assume(x in Z_ and x in R_);  
is(x in Z_), is(x in R_)
```

TRUE, TRUE

When adding assumptions, always check that a new assumption does not contradict the existing assumption. MuPAD does not guarantee to detect conflicting assumptions. For example, assume that  $y$  is simultaneously nonzero, real and an imaginary value. `Type::Imaginary` refers to all complex numbers lying on the imaginary axis. When you set these assumptions, MuPAD does not issue any warning and does not error:

```
assume(y <> 0);  
assumeAlso(y in R_);  
assumeAlso(y, Type::Imaginary)
```

---

**Note** Do not set conflicting assumptions because they can lead to unpredictable and inconsistent results.

---

To check if the assumption still holds true, use the `is` command. For example, MuPAD drops the assumption that the variable `y` represents imaginary numbers because this assumption conflicts with the combination of the previous two assumptions:

```
is(y <> 0), is(y in R_), is(y, Type::Imaginary)

TRUE, TRUE, FALSE
```

If you set conflicting assumptions, the order in which you set them does not always determine which assumption MuPAD accepts:

```
assume(y <> 0);
assumeAlso(y, Type::Imaginary);
assumeAlso(y in Z_);
is(y <> 0), is(y, Type::Imaginary), is(y in Z_)

TRUE, FALSE, TRUE
```

## Clear Permanent Assumptions

Permanent assumptions hold for all further calculations. Before using a parameter in other calculations, check which properties MuPAD assumes to be valid for this parameter. The `property::showprops` command returns all assumptions specified for the parameter:

```
assume(g in R_);
assume(h in Z_);
assume(t > 0);
property::showprops(g),
property::showprops(h),
property::showprops(t)
```

$[g \in \mathbb{R}], [h \in \mathbb{Z}], [0 < t]$ 

The `unassume` command clears a particular object from all assumptions:

```
unassume(g);
unassume(h);
unassume(t);
property::showprops(g),
property::showprops(h),
property::showprops(t)
```

 $[], [], []$ 

To delete the value of a parameter and clear all assumptions set for this parameter, use the `delete` command:

```
delete g, h, t
```

For example, assign the value  $\frac{g t^2}{2}$  to the variable `h` and assume that  $h > 0$ :

```
h := g*t^2/2;
assume(g > 0 and h > 0);
property::showprops(h); h
```

 $\left[ 0 < g, 0 < \frac{g t^2}{2} \right]$ 
 $\frac{g t^2}{2}$ 

The `unassume` command clears the assumption, but does not remove the value of the variable:

```
unassume(h > 0);
property::showprops(h); h
```

```
[]
```

$$\frac{gt^2}{2}$$

The delete command clears the assumption and the value:

```
delete h;  
property::showprops(h); h
```

```
[]
```

```
h
```

## Use Temporary Assumptions

### In this section...

“Create Temporary Assumptions” on page 3-126

“Assign Temporary Values to Parameters” on page 3-128

“Interactions Between Temporary and Permanent Assumptions” on page 3-129

“Use Temporary Assumptions on Top of Permanent Assumptions” on page 3-130

### Create Temporary Assumptions

Use temporary assumptions to specify that an object holds mathematical properties for a particular calculation. Temporary assumptions also help you narrow a general solution and get specific solutions. For example, the following equation describes linear motion with constant acceleration:

$r = r_0 + v_0 t + \frac{at^2}{2}$ . Here,  $r$  is the distance the object travels,  $r_0$  is the initial distance,  $v_0$  is the initial velocity,  $a$  is the constant acceleration, and  $t$  is the time of travel. If you know all other parameters and want to calculate the time that the object was moving, solve the equation for the variable  $t$ :

```
t = solve(r = r_0 + v_0*t + a*t^2/2, t)
```



$$t = \begin{cases} \left\{ -\frac{v_0 - \sigma_1}{a}, -\frac{v_0 + \sigma_1}{a} \right\} & \text{if } a \neq 0 \\ \left\{ \frac{r - r_0}{v_0} \right\} & \text{if } a = 0 \wedge v_0 \neq 0 \\ \mathbb{C} & \text{if } r = r_0 \wedge a = 0 \wedge v_0 = 0 \\ \emptyset & \text{if } r \neq r_0 \wedge a = 0 \wedge v_0 = 0 \end{cases}$$

where

$$\sigma_1 = \sqrt{v_0^2 + 2ar - 2ar_0}$$

Suppose, you want to keep the general solution for all possible cases of the linear motion with constant acceleration. You also want to derive several special cases of this motion and get particular solutions for these cases. For example, one of the objects you consider moves with constant velocity. Derive the solution for this object from the general solution for the time of the motion by assuming the acceleration  $a = 0$ :

```
t = solve(r = r_0 + v_0*t + a*t^2/2, t)
      assuming a = 0 and r > r_0 and v_0 > 0
```

$$t = \left\{ \frac{r - r_0}{v_0} \right\}$$

The assumption  $a = 0$  holds true only for this particular call to solve. The assumption does not affect other calculations:

```
is(a = 0)
```

UNKNOWN

If you set an assumption in the form of an inequality, both sides of an inequality should represent real values. Inequalities with complex numbers

are invalid because the field of complex numbers is not an ordered field. For example, if you try to use the following assumption, MuPAD returns an error:

```
y + 1 assuming y > 2*I
```

```
Error: Assumptions are inconsistent. [property::_assume]
```

You can use complex values in assumptions presented in forms of equations:

```
y + 1 assuming y = 2*I
```

```
1+2i
```

## Assign Temporary Values to Parameters

To solve the linear motion equation for particular values of the parameters, assign the values to the parameters:

```
r := 4:
r_0 := 0:
v_0 := 3:
a := 2:
t = solve(r = r_0 + v_0*t + a*t^2/2, t) assuming t > 0
```

```
t = {1}
```

If you use assignments, MuPAD evaluates variables to their values in all further computations:

```
r, r_0, v_0, a
```

```
4, 0, 3, 2
```

To be able to reuse the variables in further computations, use the `delete` command:

```
delete r, r_0, v_0, a
```

Using assumptions, you can temporarily assign values to the parameters. For example, solve the equation for the following values:

```
t = solve(r = r_0 + v_0*t + a*t^2/2, t)
assuming r = 4 and r_0 = 0 and v_0 = 3 and a = 2 and t > 0
```

```
t = {1}
```

The variables remain free for further calculations because temporary assumptions do not hold true:

```
r, r_0, v_0, a, t
```

```
r, r_0, v_0, a, t
```

If assumptions contain linear equations with one variable, MuPAD solves these equations, inserts the solutions into the expression, and then evaluates the expression:

```
r = r_0 + v_0*t + a*t^2/2 assuming a + 5 = 5
                               and 2*v_0 + 4 = 14
                               and t = 3 and r_0 = 0
```

```
r = 15
```

## Interactions Between Temporary and Permanent Assumptions

The assuming command temporarily overwrites all permanent assumptions set on an object:

```
assume(z in R_);
z assuming z = -2*I
```

```
-2i
```

After evaluating the statement with a temporary assumption, MuPAD reinstates the permanent assumption:

```
is(z in R_)
```

```
TRUE
```

See how to use temporary assumptions in combination with permanent assumptions in Using Temporary Assumptions on Top of Permanent Assumptions.

## Use Temporary Assumptions on Top of Permanent Assumptions

Suppose you set permanent assumptions on a MuPAD object. If you evaluate the object with a temporary assumption set by the `assuming` command, MuPAD ignores the permanent assumptions in this evaluation:

```
assume(x in R_);
solve(x^3 + x = 0, x) assuming (x <> 0)
```

```
{-i i}
```

To use permanent assumptions and a temporary assumption together, add the temporary assumption using the `assumingAlso` command:

```
assume(x in R_);
solve(x^3 + x = 0, x) assumingAlso (x <> 0)
```

```
∅
```

When you use temporary assumptions on top of the permanent ones, always check that the assumptions do not contradict each other. Contradicting assumptions can lead to inconsistent and unpredictable results. In some cases, MuPAD detects conflicting assumptions and issues the following error:

```
assume(x < 0);
x assumingAlso (x > 0);
```

Error: Assumptions are inconsistent. [property::\_assume]

MuPAD does not guarantee to detect contradicting assumptions:

```
assume(x, Type::Even);  
x assumingAlso (x + 1, Type::Even)
```

$x$

## Choose Differentiation Function

MuPAD provides two functions for differentiation. The choice of the function depends on which type of object you want to differentiate. To differentiate mathematical expressions, use the `diff` command. For example:

```
diff(cos(x), x);  
diff(x^3, x)
```

$-\sin(x)$

$3x^2$

To differentiate a function or functional expression, use `D` or its shortcut `'`. Using this command, you can differentiate any standard mathematical function or your custom created function. For example:

```
D(cos);  
f := x -> x*sin(x);  
f'
```

$-\sin$

$x \rightarrow \sin(x) + x \cos(x)$

## Differentiate Expressions

For differentiating an expression, use the `diff` command. Specify the expression you want to differentiate, and the differentiation variable. Specifying the differentiation variable is important even if your expression contains only one variable. For example, find the derivative of an expression with one variable:

```
diff(x^2 + sqrt(sin(x)), x)
```

$$2x + \frac{\cos(x)}{2\sqrt{\sin(x)}}$$

---

**Note** If you do not specify differentiation variable, `diff(expr)` returns the expression `expr`.

---

Find first-order partial derivatives of a multivariable expression by specifying differentiation variables:

```
diff(sin(x*cos(x*y)), x);
diff(sin(x*cos(x*y)), y)
```

$$\cos(x \cos(x y)) (\cos(x y) - x y \sin(x y))$$

$$-x^2 \sin(x y) \cos(x \cos(x y))$$

To take second and higher order derivatives, you can use nested calls to the `diff` function. More efficiently, use only one `diff` command and specify variables for each differentiation step. Calling `diff` only once is shorter and also can improve performance because MuPAD internally converts nested calls to `diff` into a single call with multiple arguments:

```
diff(diff(sqrt(sin(x)), x), x);
diff(sqrt(sin(x)), x, x)
```

$$-\frac{\sqrt{\sin(x)}}{2} - \frac{\cos(x)^2}{4 \sin(x)^{3/2}}$$

$$-\frac{\sqrt{\sin(x)}}{2} - \frac{\cos(x)^2}{4 \sin(x)^{3/2}}$$

When computing higher order derivatives with respect to one variable, use the sequence operator as a shortcut:

```
diff(sqrt(sin(x)), x $ 3) = diff(sqrt(sin(x)), x, x, x)
```

$$\frac{\cos(x)}{4 \sqrt{\sin(x)}} + \frac{3 \cos(x)^3}{8 \sin(x)^{5/2}} = \frac{\cos(x)}{4 \sqrt{\sin(x)}} + \frac{3 \cos(x)^3}{8 \sin(x)^{5/2}}$$

To compute mixed derivatives, specify differentiation variables for each step:

```
diff(x*cos(x*y), y, x)
```

$$-2 x \sin(x y) - x^2 y \cos(x y)$$

---

**Note** To improve performance, MuPAD assumes that all mixed derivatives commute. For example,  $\frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y)$ .

---

This assumption suffices for most of engineering and scientific problems.



## Differentiate Functions

To compute derivatives of functions, use the differential operator  $D$ . This operator differentiates both standard mathematical functions and your own functions created in MuPAD. For example, find the first derivatives of the following standard mathematical functions implemented in MuPAD:

$D(\sin)$ ,  $D(\exp)$ ,  $D(\cosh)$ ,  $D(\sqrt{\cdot})$ ,  $D(\text{heaviside})$

$\cos$ ,  $\exp$ ,  $\sinh$ ,  $\frac{1}{2\sqrt{\text{id}}}$ ,  $\text{dirac}$

Create your own function with one variable and compute a derivative of this function:

$f := x \rightarrow x^3$   
 $D(f)$

$x \rightarrow 3x^2$

Alternatively, use  $'$  as a shortcut for the differential operator  $D$ :

$f := x \rightarrow \sin(x)/x^2$   
 $f'$   
 $f'(x)$

$x \rightarrow \frac{\cos(x)}{x^2} - \frac{2 \sin(x)}{x^3}$

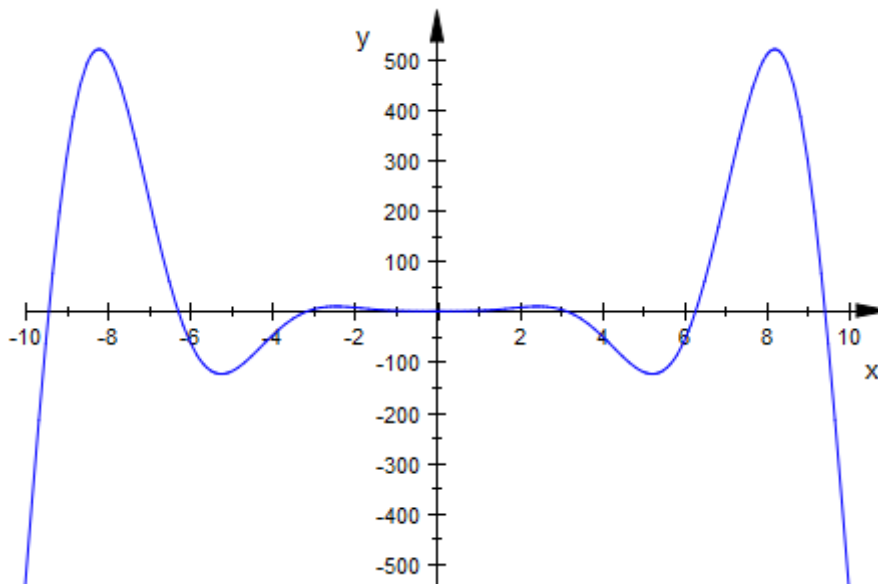
$\frac{\cos(x)}{x^2} - \frac{2 \sin(x)}{x^3}$

Computing the first derivatives of a function lets you find its local extrema (minima and maxima). For example, create this function and plot it on the interval  $-10 < x < 10$ :

$F := x \rightarrow x^3 \sin(x)$ ;

```
plot(F, x = -10..10)
```

```
x → x3 sin(x)
```



Find the local extrema of  $F$  on the interval  $-10 < x < 10$ . If the point is a local extremum (either minimum or maximum), the first derivative of the function at that point equals 0. Therefore, to find the local extrema of  $F$ , solve the equation  $F'(x) = 0$ . Use the `AllRealRoots` option to return more than one solution.

```
extrema := numeric::solve(F'(x) = 0, x = -10..10, AllRealRoots)
```

```
{-8.204531363, -5.232938454, -2.455643863, 0.0, 2.455643863, 5.232938454, 8.204531363}
```

Now, compute the corresponding values of  $F$ . For example, compute  $F$  for the third element,  $-2.455643863$ , in the solution set:

```
F(extrema[3])
```

9.379492487

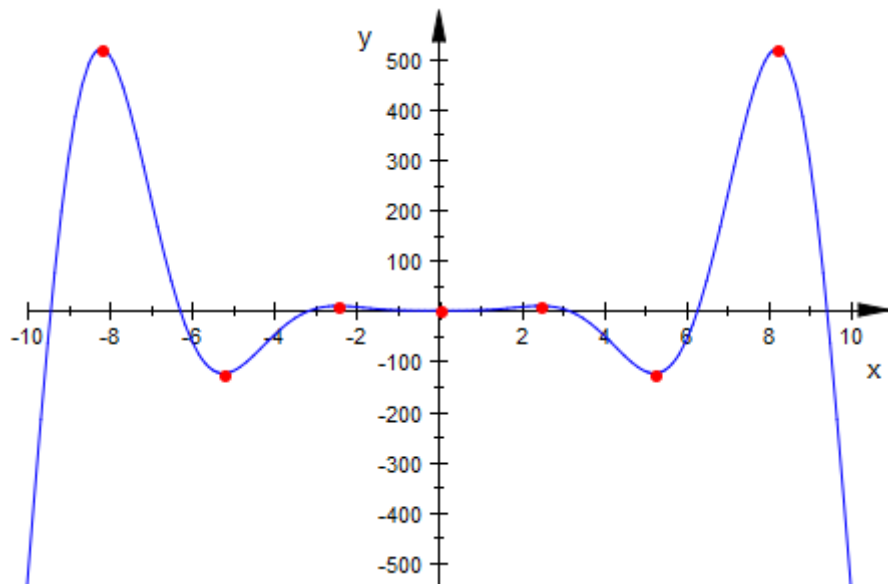
To compute the values of  $F$  for all local minima and maxima, use the following command. Here,  $\$$  is used to evaluate  $F$  for every element of the extrema set.

```
points := {[x, F(x)] $ x in extrema}
```

```
{[-8.204531363, 518.6949936], [-5.232938454, -124.3166806],  
[-2.455643863, 9.379492487], [0.0, 0.0], [2.455643863, 9.379492487],  
[5.232938454, -124.3166806], [8.204531363, 518.6949936]}
```

Plot function  $F$  with extrema points:

```
plot(F, points, x = -10..10)
```



To compute a derivative of a multivariable function, specify the differentiation variable. The operator  $D$  does not accept the variable names. Instead of providing a variable name, provide its index. For example, integrate the

following function with respect to its first variable  $x$ . Then integrate the function with respect to its second variable  $y$ :

```
f := (x, y) -> x^2 + y^3:
D([1], f);
D([2], f)
```

$$(x, y) \rightarrow 2x$$

$$(x, y) \rightarrow 3y^2$$

The list of indices accepted by the operator  $D$  refers to the order in which you provided the variables when creating a function:

```
f := (x, y) -> x^2 + y^3:
D([1], f);
f := (y, x) -> x^2 + y^3:
D([1], f)
```

$$(x, y) \rightarrow 2x$$

$$(y, x) \rightarrow 3y^2$$

To find second- and higher-order partial derivatives of a function, use the same index two or more times. For example, compute the second-order partial derivatives with respect to  $x$  and with respect to  $y$ :

```
f := (x, y) -> x^3*sin(y):
D([1, 1], f);
D([2, 2], f)
```

$$(x, y) \rightarrow 6x \sin(y)$$

$$(x, y) \rightarrow -x^3 \sin(y)$$

To compute second- and higher-order derivatives with respect to several variables (mixed derivatives), provide a list of indices of differentiation variables:

```
f := (x, y) -> x^3*sin(y):  
D([1, 2], f);
```

$$(x, y) \rightarrow 3x^2 \cos(y)$$

---

**Note** To improve performance, MuPAD assumes that all mixed derivatives commute. For example,  $D_{12}(f) = D_{21}(f)$ . This assumption suffices for most engineering and scientific problems.

---

## Compute Indefinite Integrals

To integrate a mathematical expression  $f$  means to find an expression  $F$  such that the first derivative of  $F$  is  $f$ . The expression  $F$  is an antiderivative of  $f$ . Integration is a more complicated task than differentiation. In contrast to differentiation, there is no general algorithm for computing integrals of an arbitrary expression. When you differentiate an expression, the result is often represented in terms of the same or less complicated functions. When you integrate an expression, the result often involves much more complicated functions than those you use in the original expression. For example, if the original expression consists of elementary functions, you can get the result in terms of elementary functions:

```
int(x + 1/(x^2), x)
```

$$\frac{x^2}{2} - \frac{1}{x}$$

The following integrand also consists of standard trigonometric functions, but here the integrator cannot return the result in terms of elementary functions. The antiderivative involves a special function:

```
int(sin(x)/x, x)
```

$\text{Si}(x)$

When you compute an indefinite integral, MuPAD implicitly assumes that the integration variable is real. The result of integration is valid for all real numbers, but can be invalid for complex numbers. You also can define properties of the integration variables by using the `assume` function. The properties you specify can interfere with the assumption that the integration variable is real. If MuPAD cannot integrate an expression using your assumption, the `int` function issues a warning. Use the `intlib::printWarnings` function to switch the warnings on and off. For example, switch on the warnings:

```
intlib::printWarnings(TRUE):
```

Suppose you want to integrate the following expression under the assumption that the integration variable is positive. This assumption does not conflict with the assumption that the variable is real. The `int` command uses your assumption:

```
f := abs(x):
int(f, x) assuming x > 0
```

$$\frac{x^2}{2}$$

Integrate this expression under the assumption that  $x$  is an integer. MuPAD cannot integrate the expression over a discrete subset of the real numbers. The `int` command issues a warning, and then integrates over the field of real numbers:

```
int(f, x) assuming x in Z_
```

Warning: Cannot integrate when 'x' has property 'Z\_'. The assumption

$$\frac{x^2 \operatorname{sign}(x)}{2}$$

For a discrete set of values of the integration variable, compute a sum instead of computing an integral. See “Summation” for details.

Now integrate under the assumption that  $x$  is imaginary. The `int` command cannot compute the integral of the expression over imaginary numbers. It issues a warning and integrates the expression over the domain of complex numbers:

```
assume(x, Type::Imaginary);
int(f, x)
```

Warning: Cannot integrate when 'x' has property '{x\*I | x in R\_}'. Th

$$\begin{cases} 0 & \text{if } x = 0 \\ \frac{x^2 \operatorname{sign}(x)}{2} & \text{if } x \notin \mathbb{R} \end{cases}$$

For more information about the assumptions, see “Properties and Assumptions”. Before you proceed with other computations, clear the assumption on the variable  $x$ :

```
unassume(x):
```

Also, disable the warnings:

```
intlib::printWarnings(FALSE):
```



## Compute Definite Integrals

For definite integration, the `int` command restricts the integration variable `x` to the given range of integration.

```
int(sin(ln(x)), x = 0..5)
```

$$\frac{5 \sin(\ln(5))}{2} - \frac{5 \cos(\ln(5))}{2}$$

If the `int` command determines that an integral does not converge, it returns the special value `undefined`:

```
int(exp(x*I), x = 1..infinity)
```

`undefined`

When the `int` command cannot compute an integral and also cannot prove that the integral does not converge, it returns an unresolved integral:

```
int(sin(cos(x)), x = 0..10)
```

$$\int_0^{10} \sin(\cos(x)) \, dx$$

For definite integrals, the `int` command restricts the integration to the specified interval. If you use the `assume` function to set properties on the integration variable, `int` temporarily overwrites these properties and integrates over the specified interval. To display warnings, set the value of `intlilb::printWarnings` to `TRUE`:

```
intlilb::printWarnings(TRUE):
assume(x > 0):
int(x, x = 1 .. 2);
int(x, x = -3 .. 1)
```

Warning: The assumption that 'x' has property '[1, 2]' instead of giv

$$\frac{3}{2}$$

Warning: The assumption that 'x' has property '[-3, 1]' instead of gi

$$-4$$

After computing an integral, MuPAD restores the assumptions set for integration variable. If you do not want the assumptions to affect further computations, use the `unassume` function:

```
unassume(x)
```

MuPAD also makes implicit assumptions on the specified interval. Suppose, you use the integration range as  $[a, b]$ . The system assumes that both  $a$  and  $b$  represent real numbers, and that  $a \leq b$  unless you clearly specify otherwise. If you set the value of `intlib::printWarnings` to `TRUE`, MuPAD displays the warning about using implicit assumptions:

```
int(heaviside(x - a), x = a..b)
```

Warning: Cannot decide if 'a <= b' is true, will temporarily assume i

$$-\text{heaviside}(b - a)(a - b)$$

To avoid this implicit assumption, specify that  $a > b$  or  $a < b$

```
int(heaviside(x - a), x = a..b) assuming a > b
```

$$0$$

```
int heaviside(x - a), x = a..b assuming a < b
```

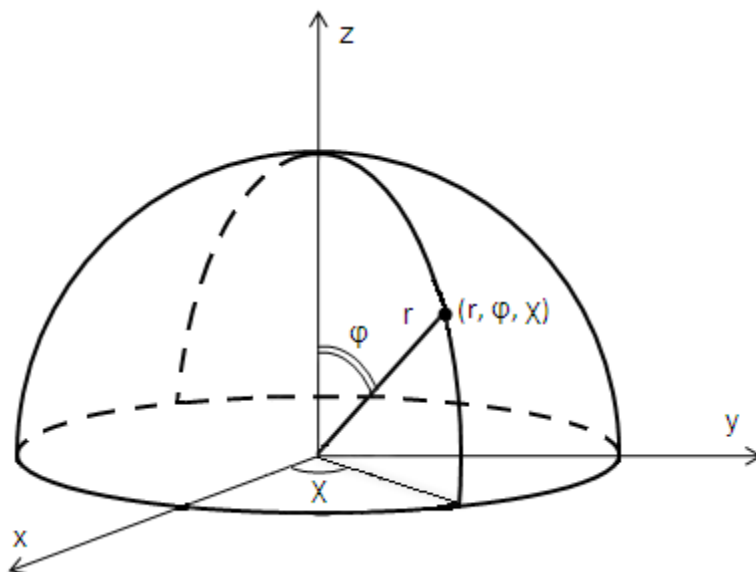
$b - a$

For further computations, disable the warnings:

```
intlib::printWarnings(FALSE):
```

## Compute Multiple Integrals

To compute multiple integrals, use nested calls to `int`. For example, compute the surface area and the volume of a sphere using spherical coordinates.



Compute the expression for the surface area of a sphere. The distance from the center of a sphere to the surface remains constant,  $r = R$ . The angle  $\varphi$  changes its value from 0 to  $\pi$ . The angle  $\chi$  changes its value from 0 to  $2\pi$ :

```
int(int(R^2*sin(phi), phi = 0..PI), chi = 0..2*PI)
```

$$4 \pi R^2$$

Compute the expression for the volume of a sphere. The angles accept the same values, but the distance from the center of any point inside the sphere ranges from 0 to  $R$ . To find the volume, compute the following triple integral:

```
int(int(int(r^2*sin(phi), r = 0..R), phi = 0..PI), chi = 0..2*PI)
```

$$\frac{4 \pi R^3}{3}$$

## Apply Standard Integration Methods Directly

### In this section...

“Integration by Parts” on page 3-148

“Change of Variable” on page 3-149

### Integration by Parts

Integration by parts is one of the common methods for computing integrals. Using this method, you rewrite the original integral in terms of an expression containing a simpler integral. Integration by parts for indefinite integrals uses the definition:

$$\int u'(x) v(x) dx = u(x) v(x) - \int u(x) v'(x) dx$$

For definite integrals, integration by parts is defined as follows:

$$\int_a^b u'(x) v(x) dx = u(b) v(b) - u(a) v(a) - \int_a^b u(x) v'(x) dx$$

Internally, MuPAD uses integration by parts along with other integration methods. To use this method explicitly, call the `intlib::byparts` function. If you want to integrate an expression by parts, keep the original integral unevaluated. By default, `int` returns evaluated integrals. Use the `hold` or `freeze` commands to prevent evaluation of the integral:

```
f := freeze(int)(exp(a*x)*cos(I*x*b), x)
```

$$\int e^{ax} \cos(b x i) dx$$

Call `intlib::byparts` and specify the part of an expression you want to integrate. For example, specify  $u'(x) = e^{ax}$ :

```
f_int := intlib::byparts(f, exp(a*x))
```

$$-\int \left( -\frac{b e^{ax} \sin(b x i)}{a} \right) dx + \frac{e^{ax} \cos(b x i)}{a}$$

To evaluate the resulting integral, use the `eval` command:

```
eval(f_int)
```

$$\frac{e^{ax} \cos(b x i)}{a} + \frac{b e^{ax} \left( a \left( \frac{e^{bx i}}{2} - \frac{e^{-bx i}}{2} \right) - b \left( \frac{e^{bx} + e^{-bx}}{2} \right) i \right) i}{a(a^2 - b^2)}$$

If the resulting expression is too long, try using the `simplify` or `Simplify` function:

```
Simplify(%)
```

$$\frac{e^{ax-bx} (a+b+a e^{2bx} - b e^{2bx})}{2(a^2 - b^2)}$$

## Change of Variable

Change of variable is also one of the common methods for computing integrals. For explicit use of this method, MuPAD provides the `intlib::changevar` function. When changing an integration variable, you need to keep the integral unevaluated. By default, `int` returns evaluated integrals. Use the `hold` or `freeze` commands to prevent evaluation of the integral:

```
f := intlib::changevar(hold(int)(sin(exp(x))), x, t = exp(x), t)
```

$$\int \frac{\sin(t)}{t} dt$$

To evaluate the resulting integral, use the `eval` command:

`eval(f)`

`Si(t)`

The change of variable method also works for computing definite integrals:

```
f := intlib::changevar(hold(int)(x/sqrt(1 - x^2),  
                        x = a..b), t = x^2, t)
```

$$\int_a^{b^2} \frac{1}{2\sqrt{1-t}} dt$$



## Get Simpler Results

When computing integrals, MuPAD applies strict mathematical rules. For example, integrate the following expression:

```
int(arcsin(sin(x)), x)
```

$$x \arcsin(\sin(x)) - \frac{x^2 \operatorname{sign}(\cos(x))}{2}$$

If you want a simple practical solution, try the `IgnoreAnalyticConstraints` option. With this option, MuPAD uses a set of simplified mathematical rules that are not generally correct. The returned results might be shorter and more useful, for example:

```
int(arcsin(sin(x)), x, IgnoreAnalyticConstraints)
```

$$\frac{x^2}{2}$$

## If an Integral Is Undefined

If one of the following conditions is true, a definite integral  $\int_a^b f(x) dx$  might not exist in a strict mathematical sense:

- If the interior of the integration interval  $(a, b)$  contains poles of the integrand  $f(x)$ .
- If  $a = -\infty$  or  $b = \infty$  or both.

If  $f(x)$  changes sign at all poles in  $(a, b)$ , the so-called infinite parts of the integral to the left and to the right of a pole can cancel each other. In this case, use the `PrincipalValue` option to find a weaker form of a definite integral called the Cauchy principal value. For example, this integral is not defined because it has a pole at  $x = 0$ :

```
int(1/x, x = -1..1)
```

```
undefined
```

To compute the Cauchy principal value, call `int` with the option `PrincipalValue`:

```
int(1/x, x = -1..1, PrincipalValue)
```

```
0
```

If an expression can be integrated in a strict mathematical sense, and such an integral exists, the Cauchy principal value coincides with the integral:

```
int(x^2, x = -1..1) = int(x^2, x = -1..1, PrincipalValue)
```

```
 $\frac{2}{3} = \frac{2}{3}$ 
```

## If MuPAD Cannot Compute an Integral

### In this section...

“Approximate Indefinite Integrals” on page 3-153

“Approximate Definite Integrals” on page 3-154

If the `int` command cannot compute a closed form of an integral, MuPAD returns an unresolved integral:

```
int(sin(sinh(x)), x)
```

$$\int \sin(\sinh(x)) \, dx$$

If MuPAD cannot compute an integral of an expression, one of the following reasons may apply:

- The antiderivative does not exist in a closed form.
- The antiderivative exists, but MuPAD cannot find it.

Try to approximate these integrals by using one of the following methods:

- For indefinite integrals, use series expansions. Use this method to approximate an integral around a particular value of the variable.
- For definite integrals, use numeric approximations.

### Approximate Indefinite Integrals

If `int` cannot compute an indefinite integral in a closed form, it returns an unresolved integral:

```
F := int(cos(x)/sqrt(1 + x^2), x)
```

$$\int \frac{\cos(x)}{\sqrt{x^2 + 1}} \, dx$$

To approximate the result around some point, use the `series` function. For example, approximate the integral around  $x = 0$ :

```
series(F, x = 0)
```

$$x - \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

If you know in advance that the integral cannot be found in a closed form, skip calculating the symbolic form of the integral. To use the system more efficiently, call the `series` command to expand the integrand, and then integrate the result:

```
int(series(cos(x)/sqrt(1 + x^2), x = 0), x)
```

$$x - \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

## Approximate Definite Integrals

If `int` cannot compute a definite integral in a closed form, it returns an unresolved integral:

```
F := int(cos(x)/sqrt(1 + x^2), x = 0..10)
```

$$\int_0^{10} \frac{\cos(x)}{\sqrt{x^2 + 1}} dx$$

To approximate the result numerically, use the `float` function:

```
float(F)
```

```
0.375706283
```

If you know in advance that the integral cannot be found in a closed form, skip calculating the symbolic form of the integral. Use the system more

efficiently by calling the `numeric::int` function. This command applies numeric integration methods from the beginning:

```
numeric::int(cos(x)/sqrt(1 + x^2), x = 0..10)
```

```
0.375706283
```

## Compute Symbolic Sums

### In this section...

“Indefinite Sums” on page 3-156

“Definite Sums” on page 3-157

“Sums Over Roots of a Polynomial” on page 3-158

Suppose you have an expression with a set of discrete values of a variable. Computing a sum of this expression over the set of variables is called summation. The variable over which you compute the sum is called the summation index. The function you get as a result of a symbolic summation is called antidifference. MuPAD implicitly assumes that the summation index uses only integer values. For continuous values of a variable, summation naturally turns to integration. Similarly to integration, you can compute indefinite and definite sums including sums over roots of polynomials.

### Indefinite Sums

The function  $f(i) = \sum_i x_i$  is called the indefinite sum of  $x_i$  over  $i$ , if the following identity holds for all values of  $i$ :

$$f(i+1) - f(i) = x_i$$

When you compute an indefinite sum, the result often involves much more complicated functions than those you use in the original expression. If the original expression consists of elementary functions, you can get the result in terms of elementary functions:

$$\text{sum}(x^2/(x^2 - 1), x)$$

$$x - \frac{1}{2(x-1)} - \frac{1}{2x}$$

Although the following expression consists of elementary functions, the result involves a special function:

```
sum(x/(x^2 + 1), x)
```

$$\frac{\psi(x-i)}{2} + \frac{\psi(x+i)}{2}$$

## Definite Sums

When computing an indefinite sum, the `sum` command implicitly assumes that the integration index runs through all integer numbers. Definite summation lets you specify the range of the summation index. For example, specify the summation index range using a symbolic parameter:

```
sum(x/(x^2 + 1), x = a..10*a)
```

$$-\frac{\psi(a-i)}{2} - \frac{\psi(a+i)}{2} + \frac{\psi(10a+1-i)}{2} + \frac{\psi(10a+1+i)}{2}$$

`sum` also computes definite sums with infinite boundaries:

```
sum(x^n/n!, n = 0..infinity);
sum((-1)^n*x^(2*n + 1)/(2*n + 1)!, n = 0..infinity)
```

$$e^x$$

$$\sin(x)$$

To find a sum over two variables, use nested calls to `sum`:

```
sum(sum(x^n/n!, n = 0..infinity), x = a..100*a)
```

$$\frac{e^{100a+1} - e^a}{e - 1}$$

If your sum has a small finite number of terms, use the `_plus` command instead of `sum`. The `sum` command is slower than `_plus`:

```
_plus(x/(x^2 + 1) $ x = 0..10)
```

$$\frac{3829008689}{1693047850}$$

To compute a sum for a large finite number of terms, use the `sum` command:

```
sum(x/(x^2 + 1), x = 1..10^10)
```

$$-\frac{\psi(1-i)}{2} - \frac{\psi(1+i)}{2} + \frac{\psi(10000000001-i)}{2} + \frac{\psi(10000000001+i)}{2}$$

If the result of a finite summation contains more than 1000 terms, the `sum` command returns an unexpanded symbolic sum. If you want to display all the terms explicitly, use the `expand` function. To get the expanded result in the following example, delete the colon at the end of the example:

```
S := sum(exp(x)/(x^2 + 1), x = a..a + 1000);
expand(S):
```

$$\sum_{x=a}^{a+1000} \frac{e^x}{x^2 + 1}$$

## Sums Over Roots of a Polynomial

The `sum` command also computes sums for which the summation index runs over all roots of a polynomial. To specify all roots of a polynomial, use `RootOf`:

```
sum(i^10, i = RootOf(a*X^10 + b*X^8 + c*X^5 + 1, X))
```

$$-\frac{10 a^4 - 5 a^3 c^2 + 2 b^5}{a^5}$$



## Approximate Sums Numerically

If the `sum` command cannot compute a sum, MuPAD returns an unresolved sum. For example, try to compute the following sum:

```
sum(exp(x)^(-x), x = 0..infinity)
```

$$\sum_{x=0}^{\infty} \frac{1}{(e^x)^x}$$

The reasons MuPAD cannot compute the closed form of a particular sum are the same as the reasons for not computing an integral:

- The antidifference does not exist in a closed form.
- The antidifference exists, but MuPAD cannot find it.
- MuPAD can find the antidifference on a larger computer, but runs out of time or memory on the available machine.

If MuPAD cannot compute a definite sum, try to approximate it numerically:

```
S := sum(exp(x)^(-x), x = 0..infinity);
float(S)
```

$$\sum_{x=0}^{\infty} \frac{1}{(e^x)^x}$$

1.386318602

If you know in advance that the antidifference cannot be computed in a closed form, skip trying to calculate this sum symbolically. For such expressions, call the `numeric::sum` function to perform numeric summation directly. Trying to calculate a symbolic sum, and then approximating it numerically can be much slower than applying numeric summation from the beginning:

```
numeric::sum(exp(x)^(-x), x = 0..infinity)
```

1.386318602

## Compute Taylor Series for Univariate Expressions

Taylor series expansions serve for approximating an arbitrary expression by a polynomial expression around some value of a variable. Taylor series expansions approximate expressions for which the derivatives up to infinite order exist around a particular value  $x_0$  of a variable  $x$ :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} + \frac{f''(x_0)}{2!} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

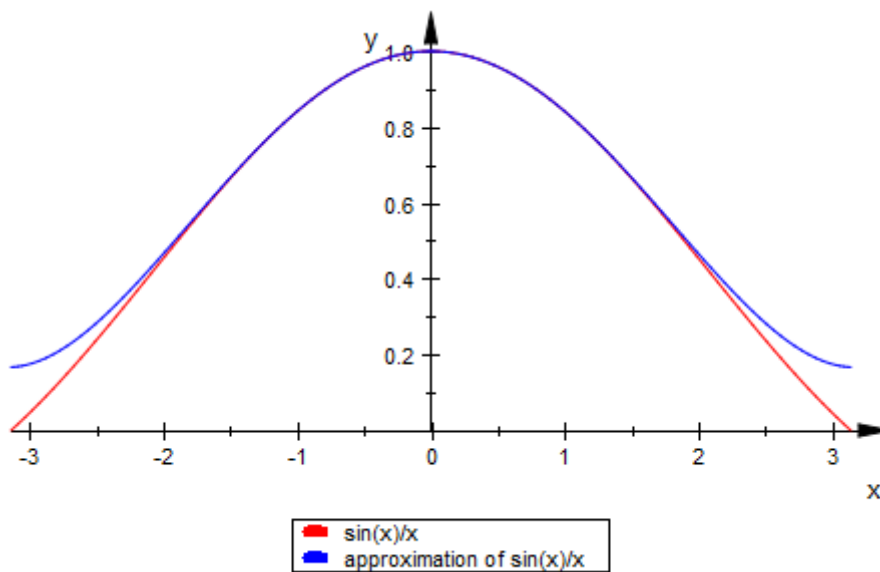
To compute Taylor series expansion, use the `taylor` command. For example, approximate the expression  $\sin(x)/x$  around  $x = 0$ :

```
exact := sin(x)/x:
approx := taylor(sin(x)/x, x)
```

$$1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)$$

Plot the exact expression and its Taylor series expansion in the same coordinate system. The Taylor series expansion approximates the expression near  $x = 0$ , but visibly deviates from  $\sin(x)/x$  for larger  $|x|$ :

```
plot(
  plot::Function2d(exact, x = -PI..PI,
    Legend = "sin(x)/x",
    Color = RGB::Red),
  plot::Function2d(approx, x = -PI..PI,
    Legend = "approximation of sin(x)/x")
)
```



Accuracy of an approximation depends on the proximity to the expansion point and on the number of terms used in the series expansion. See how to specify the number of terms in [Controlling the Number of Terms in Series Expansions](#).

Taylor series expansions around  $x = 0$  are also called Maclaurin series expansions. Approximate the expressions by Maclaurin series:

```
taylor(exp(x), x);
taylor(sin(x), x);
taylor(cos(x)/(1 - x), x)
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{13x^4}{24} + \frac{13x^5}{24} + O(x^6)$$

The Maclaurin series expansion does not exist for the following expression. MuPAD returns an error:

```
taylor(arccot(x), x)
```

Error: Cannot compute a Taylor expansion of 'arccot(x)'. Try 'series'

You can represent the following expression by a Taylor series around  $x = 1$ . To compute the series expansion around a nonzero value of a variable, specify the value. For example, compute the Taylor series expansions around  $x = 1$  for the following expressions:

```
taylor(ln(x), x = 1);
taylor(arccot(x), x = 1)
```

$$x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + O((x-1)^7)$$

$$\frac{\pi}{4} - \frac{x-1}{2} + \frac{(x-1)^2}{4} - \frac{(x-1)^3}{12} + \frac{(x-1)^5}{40} + O((x-1)^6)$$

The `taylor` command returns results in the form of Taylor series including the order term  $O$ . To convert the results to a regular polynomial expression without the  $O$ -term, use the `expr` command:

```
s := taylor(sin(x)/exp(x), x);
expr(s)
```

$$x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} + O(x^7)$$

$$\frac{x^6}{90} - \frac{x^5}{30} + \frac{x^3}{3} - x^2 + x$$

## Compute Taylor Series for Multivariate Expressions

To compute a series expansion for a multivariate expression, use the `mtaylor` command. When expanding multivariate expressions, list all variables and their values to specify the expansion point. By default, `mtaylor` computes the series expansion around the point where the values of all variables are equal to zero:

```
mtaylor(exp(x*y), [x, y])
```

$$\frac{x^2 y^2}{2} + x y + 1$$

Specify all nonzero values of the variables at the point where you want to compute the series expansion. Besides numbers and symbolic parameters, `mtaylor` also accepts real positive infinity *infinity* and real negative infinity  $-\infty$ :

```
mtaylor((1 - x)^(1/2)/exp(1/y), [x, y = infinity])
```

$$\begin{aligned} & \frac{x^2}{8y} - \frac{x}{2} - \frac{x^2}{16y^2} + \frac{x^3}{16y} + \frac{x^2}{48y^3} - \frac{x^3}{32y^2} + \frac{5x^4}{128y} + \frac{x}{2y} - \frac{x}{4y^2} + \frac{x}{12y^3} - \frac{x}{48y^4} - \frac{x^2}{8} - \frac{x^3}{16} \\ & - \frac{5x^4}{128} - \frac{7x^5}{256} - \frac{1}{y} + \frac{1}{2y^2} - \frac{1}{6y^3} + \frac{1}{24y^4} - \frac{1}{120y^5} + 1 \end{aligned}$$

## Control Number of Terms in Series Expansions

Taylor series expansions approximate an arbitrary expression with a polynomial. The number of terms in a series expansion determines the accuracy of the approximation. The number of terms in a series expansion depends on the truncation order of the expansion. By default, MuPAD computes the first six terms of series expansions:

```
taylor(exp(x), x)
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)$$

The number of terms includes the terms with coefficients equal to zero. For example, the Taylor series expansion of  $\cos(x)$  includes the terms  $0x$ ,  $0x^3$ , and  $0x^5$ . MuPAD computes these terms, but does not display them:

```
taylor(cos(x), x)
```

$$1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

Suppose, you want to approximate an exponential function with the polynomial expression around  $x = 0$ . Use the third parameter in `taylor` to specify the order of series expansion. For example, compute the series expansions `approx1` specifying the truncation order 3. Compare the result with the series expansion computed for the default order:

```
exact := exp(x):
approx1 := taylor(exp(x), x, 3);
approx2 := taylor(exp(x), x)
```

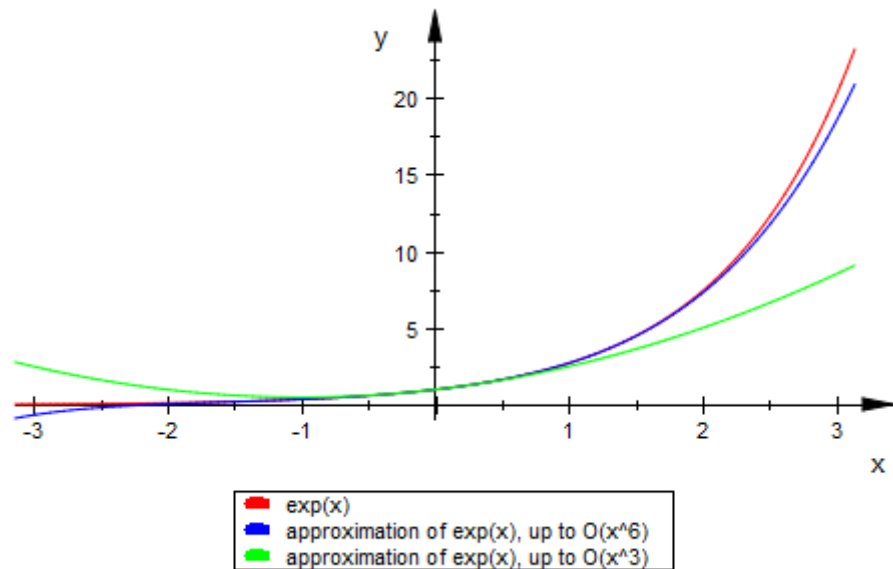
$$1 + x + \frac{x^2}{2} + O(x^3)$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)$$



Plot the exact expression, `exact`, and its Taylor series expansions, `approx1` and `approx2`, in the same coordinate system. The series expansion with more terms approximates the expression  $\exp(x)$  better:

```
plot(
  plot::Function2d(exact, x = -PI..PI,
    Legend = "exp(x)",
    Color = RGB::Red),
  plot::Function2d(approx2, x = -PI..PI,
    Legend = "approximation of exp(x), up to O(x^6)",
    Color = RGB::Blue),
  plot::Function2d(approx1, x = -PI..PI,
    Legend = "approximation of exp(x), up to O(x^3)",
    Color = RGB::Green)
)
```



There are two ways to change the truncation order for series expansions:

- Locally by passing the truncation order as the third parameter to `taylor`.  
By using this parameter, you specify the truncation order for a particular

series expansion. All other series expansions use the default order. The parameter is available for the following commands: `taylor`, `mtaylor`, and `series`. For more information, see the help pages for these commands.

- Globally by using the environment variable `ORDER`. When you change this variable, all series expansions use the new truncation order.

To change the truncation order for a particular series expansion, pass the new order as a third parameter to `taylor`:

```
taylor(exp(x), x, 10)
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + O(x^{10})$$

To change the default truncation order for all series expansions, modify the environment variable `ORDER`:

```
ORDER := 7:
taylor(exp(x), x)
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + O(x^7)$$

The following computations use the new value of `ORDER`:

```
taylor(sqrt(1 - x), x)
```

$$1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \frac{21x^6}{1024} + O(x^7)$$

To restore the default value of `ORDER`, use the `delete` command:

```
delete ORDER:
taylor(sqrt(1 - x), x)
```

$$1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} + O(x^6)$$

## O-term (The Landau Symbol)

When computing the series expansions, you get expressions with one of the terms represented as the Landau symbol  $O$ . This term shows the truncation order (error order) of the computed series expansion.

MuPAD automatically simplifies the  $O$ -term:

$$\begin{aligned} &O(10*x^7 + 20*x^8), \\ &O(x^2 + x + 1), \\ &O(x^3*y^2 + x^2*y^2) \end{aligned}$$

$$O(x^7), O(1), O(x^2 y^2)$$

When evaluating expressions with order terms, the system uses standard arithmetical operations for these terms:

$$\begin{aligned} &O(x^3)/O(x^2), \\ &O(x^3) + O(x^2), \\ &O(x^3)*O(x^2) \end{aligned}$$

$$O(x), O(x^2), O(x^5)$$

## Compute Generalized Series

The Taylor series expansion is the most common way to approximate an expression by a polynomial. However, not all expressions can be represented by Taylor series. For example, you cannot compute a Taylor series expansion for the following expression around  $x = 2$ :

```
taylor(1/(x^3 - 8), x = 2)
```

```
Error: Cannot compute a Taylor expansion of '1/(x^3 - 8)'. Try 'series'
```

If a Taylor series expansion does not exist for your expression, try to compute other power series. MuPAD provides the function `series` for computing power series. When you call `series`, MuPAD tries to compute the following power series:

- Taylor series
- Laurent series
- Puiseux series. For more information, see `Series::Puiseux`.
- Generalized series expansion of  $f$  around  $x = x_0$ . For more information, see `Series::gseries`.

As soon as `series` computes any type of power series, it does not continue computing other types of series, but stops and returns the result. For example, for this expression it returns a Laurent series:

```
S := series(1/(x^3 - 8), x = 2);
testtype(S, Type::Series(Laurent))
```

$$\frac{1}{12(x-2)} - \frac{1}{24} + \frac{x-2}{72} - \frac{(x-2)^2}{288} + \frac{(x-2)^3}{1728} + O((x-2)^5)$$

```
TRUE
```

When computing series expansions, MuPAD returns only those results that are valid for all complex values of the expansion variable in some neighborhood of the expansion point. If you need the expansion to be valid only for real numbers, use the option `Real`. For example, when you compute the series expansion of the following expression for complex numbers, `series` returns:

```
series(sign(x^2*sin(x)), x = 0)
```

$$\text{sign}(x^2 \sin(x)) + O(x^6)$$

When you compute the series expansion for real numbers, `series` returns a simplified result:

```
series(sign(x^2*sin(x)), x = 0, Real)
```

$$\text{sign}(x)^3 + O(x^6)$$

Along the real axis, compute series expansions for this expression when  $x$  approaches the value 0 from the left and from the right sides:

```
series(sign(x^2*sin(x)), x = 0, Left);
series(sign(x^2*sin(x)), x = 0, Right)
```

$$-1 + O(x^6)$$

$$1 + O(x^6)$$

## Compute Bidirectional Limits

Suppose, you have a function  $f(x)$ . The value  $C$  is a limit of the function  $f(x)$  at  $x = x_0$ :

$$C = \lim_{x \rightarrow x_0} f(x)$$

MuPAD provides the `limit` command for computing limits. When computing a limit for a variable approaching 0, you can omit specifying  $x_0$ . By default, the `limit` command assumes  $x_0 = 0$ :

```
limit(sin(x)/x, x = 0);
limit((1 - cos(x))/x, x)
```

1

0

---

**Note** Avoid computing limits for floating-point arguments.

---

If you use floating-point numbers as the parameters of `limit`, the round-off error can completely change the result. For example, a small error in the following example with the floating-point parameter changes the result from a rational number to the floating-point infinity:

```
limit((sin(x) - x)/x^3, x = 0);
limit((1.000001*sin(x) - x)/x^3, x = 0)
```

$-\frac{1}{6}$

RD\_INF

## Compute Right and Left Limits

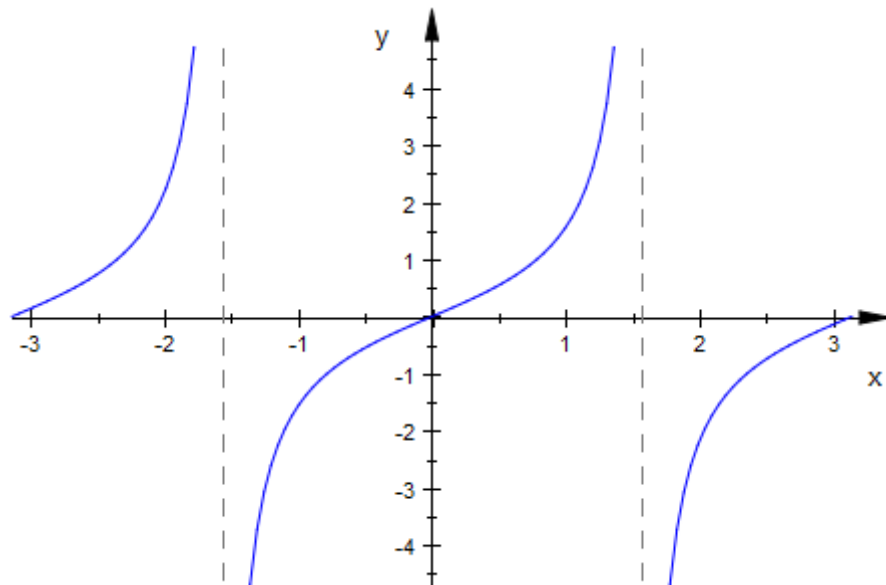
If a limit of a function at a particular value of a variable does not exist, the `limit` command returns `undefined`. For example,  $\tan(x)$  does not have a bidirectional limit at  $x = \frac{\pi}{2}$ :

```
limit(tan(x), x = PI/2)
```

`undefined`

The plot of this function shows that the function can have two different limits as the variable  $x$  approaches the value  $\frac{\pi}{2}$  from the left and from the right:

```
plot(tan(x), x = -PI..PI)
```



To compute one-sided limits of a function, use the options `Left` and `Right`:

```
limit(tan(x), x = PI/2, Left);
```

```
limit(tan(x), x = PI/2, Right)
```

```
∞
```

```
- ∞
```

If the function has a bidirectional limit at some point, one-sided limits are equal at this point. They also are equal to the bidirectional limit at this point:

```
Left = limit(abs(tan(x)), x = PI/2, Left);  
Right = limit(abs(tan(x)), x = PI/2, Right);  
Bidirectional = limit(abs(tan(x)), x = PI/2)
```

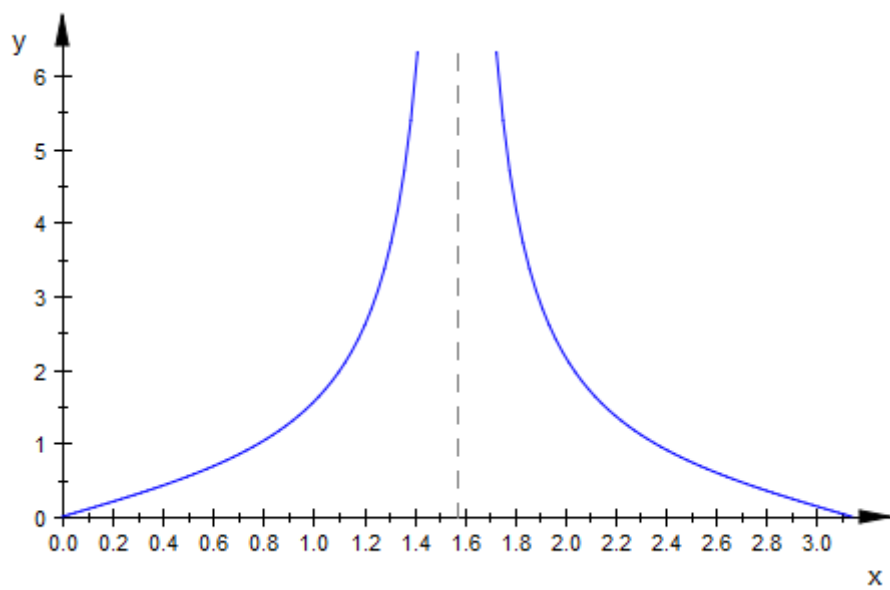
```
Left = ∞
```

```
Right = ∞
```

```
Bidirectional = ∞
```

```
plot(abs(tan(x)), x = 0..PI)
```





## If Limits Do Not Exist

If the `limit` command cannot compute a limit of a function at a particular point and also cannot prove that the limit is not defined at this point, the command returns an unresolved limit:

```
limit(gamma(1/x)*cos(sin(1/x)), x = 0)
```

$$\lim_{x \rightarrow 0} \Gamma\left(\frac{1}{x}\right) \cos\left(\sin\left(\frac{1}{x}\right)\right)$$

If `limit` can prove that the limit is undefined at a particular point, then it returns `undefined`:

```
limit(exp(x)*cos(1/x), x = 0)
```

```
undefined
```

The function  $\exp(x) \cos(1/x)$  also does not have one-sided limits at  $x = 0$ :

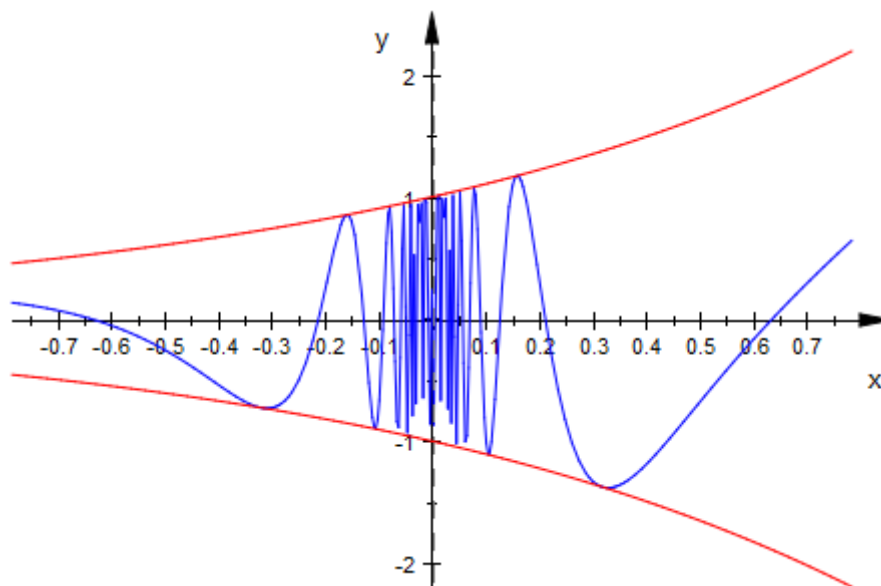
```
limit(exp(x)*cos(1/x), x = 0, Left);
limit(exp(x)*cos(1/x), x = 0, Right)
```

```
undefined
```

```
undefined
```

The plot shows that as  $\exp(x) \cos(1/x)$  approaches  $x = 0$ , the function oscillates between  $-e^x$  and  $e^x$ :

```
p1 := plot::Function2d(exp(x)*cos(1/x), x = -PI/4..PI/4):
p2 := plot::Function2d(exp(x), x = -PI/4..PI/4, Color = RGB::Red):
p3 := plot::Function2d(-exp(x), x = -PI/4..PI/4, Color = RGB::Red):
plot(p1, p2, p3)
```



To get the interval of all possible accumulation points of the function  $\exp(x) \cos(1/x)$  near the singularity  $x = 0$ , use the option `Intervals`:

```
limit(exp(x)*cos(1/x), x = 0, Intervals)
```

```
[-1, 1]
```

## Create Matrices

MuPAD supports creating and operating on vectors and multidimensional matrices. Vectors and matrices in MuPAD can contain arbitrary MuPAD objects: numbers, variables, arithmetical expressions, and so on. The simplest way to create a matrix is to use the `matrix` command:

```
matrix([[1, 2, 3], [4, 5, 6]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

When creating a matrix, you can explicitly specify its dimensions. If you specify matrix dimensions, you can use a flat list to specify all elements of a matrix. The `matrix` command takes the entries from the flat list and generates a matrix, row by row. For example, create the following 2×3 matrix:

```
matrix(2, 3, [1, 2, 3, 4, 5, 6])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Using the same list of elements, create the following 3×2 matrix:

```
matrix(3, 2, [1, 2, 3, 4, 5, 6])
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

If you specify matrix dimensions, and then enter rows or columns shorter than the declared dimensions, MuPAD pads the matrix with zero elements:

```
matrix(3, 3, [[1, 2, 3], [4, 5, 6]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

```
matrix(3, 3, [[1, 2, 3], [4]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To create a matrix of zeros, specify matrix dimensions and omit specifying elements:

```
matrix(3, 2)
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If you use a flat list, MuPAD cannot determine where to put zeros and, therefore, issues an error:

```
matrix(3, 3, [1, 2, 3, 4])
```

Error: The number of list entries does not match matrix row dimension

If you specify matrix dimensions, and then enter rows or columns longer than the declared dimensions, MuPAD also issues an error:

```
A := matrix(2, 3, [[1, 2, 3], [4, 5, 6], [7, 8, 9]])
```

Error: The number of list entries does not match matrix row dimension

The matrix command creates an object of the type Dom::Matrix():

```
A := matrix([[1, 2, 3], [4, 5, 6]]):
type(A)
```

Dom::Matrix()

## Create Vectors

Vectors in MuPAD do not form a separate data type. As matrices, vectors belong to the type `Dom::Matrix()`. To create a row or a column vector, use the `matrix` command and specify one of the dimensions to be 1. For example, create a row vector that contains five elements:

```
matrix(1, 5, [1, 2, 3, 4, 5])
```

$$(1\ 2\ 3\ 4\ 5)$$

Now, create a column vector of five elements:

```
matrix(5, 1, [1, 2, 3, 4, 5])
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

If you do not specify the dimensions, the `matrix` command creates a column vector:

```
matrix([x, y, z])
```

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

## Create Special Matrices

MuPAD provides functions for creating special types of matrices such as identity, diagonal, Hilbert, Toeplitz, and other matrices. For example, create the 3×3 identity matrix:

```
matrix::identity(3)
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, create the 3×3 diagonal matrix with the number 5 on the diagonal:

```
matrix(3, 3, 5, Diagonal)
```

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To create a matrix containing variables or arithmetical expressions, always use a list to specify matrix elements. For example, when creating the 3×3 diagonal matrix with the variable  $x$  on its main diagonal, specify the diagonal elements in a list  $[x, x, x]$ . As a shortcut for creating this list, you can use the sequence generator  $\$$ :

```
matrix(3, 3, [x $ 3], Diagonal)
```

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

To create special matrices such as Hilbert, Toeplitz, Pascal, or Vandermonde matrices, use the appropriate function of the `linalg` library. For example, to create the 4×4 Hilbert matrix, use the `linalg::hilbert` function:

```
linalg::hilbert(3)
```



$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

To create a matrix of random numbers, use the `linalg::randomMatrix` function. For example, create a matrix of random integer numbers:

```
linalg::randomMatrix(3, 4, Dom::Integer)
```

$$\begin{pmatrix} 824 & -65 & -814 & -741 \\ -979 & -764 & 216 & 663 \\ 880 & 916 & 617 & -535 \end{pmatrix}$$

Now, create a matrix that has random rational numbers on the main diagonal and zeros everywhere else:

```
linalg::randomMatrix(3, 3, Diagonal, Dom::Rational)
```

$$\begin{pmatrix} -\frac{245}{597} & 0 & 0 \\ 0 & \frac{747}{79} & 0 \\ 0 & 0 & -\frac{535}{477} \end{pmatrix}$$

## Access and Modify Matrix Elements

### In this section...

“Use Loops to Modify Matrix Elements” on page 3-184

“Use Functions to Modify Matrix Elements” on page 3-185

MuPAD lets you access and change each individual element of a vector or a matrix. For example, create the  $3 \times 4$  matrix of zeros:

```
A := matrix(3, 4)
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To access any element of the matrix, use square brackets to specify indices. For example, assign the value 22 to the second element of the second row of A:

```
A[2, 2] := 22:
```

Now, assign the value 23 to the third element of the second row of A:

```
A[2, 3] := 23:
```

Display the modified matrix A:

```
A
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 22 & 23 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

### Use Loops to Modify Matrix Elements

When changing values of the elements of a matrix, you can use loops. For example, use the for loop to define each element of A as a product of its row and column indices:

```
A := matrix(3, 4):  
for i from 1 to 3 do  
  for j from 1 to 4 do  
    A[i, j] := i*j  
  end_for  
end_for:  
A
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

## Use Functions to Modify Matrix Elements

Another way to specify a large number of matrix entries efficiently is to create and use a function. For example, define each element as a sum of its row and column indices:

```
f := (i, j)->(i + j):  
A := matrix(3, 4, f)
```

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

## Create Matrices over Particular Rings

The `matrix` command creates a matrix over `Dom::ExpressionField()`. The components of such matrices can be arbitrary arithmetical expressions. Alternatively, you can create your own matrix constructor and use it to create matrices with elements in a particular ring. When operating on such matrices, most of the MuPAD functions perform computations over the specified ring. One exception is the `numeric` library functions. The call `Dom::Matrix(R)` creates the constructor for matrices of arbitrary dimensions with the elements in the ring `R`. To specify the ring `R`, you can use the predefined rings and fields such as `Dom::Integer` or `Dom::IntegerMod(n)` or others from the `Dom` library. For example, define the constructor that creates matrices over the ring of integer numbers:

```
constructor := Dom::Matrix(Dom::Integer)
```

```
Dom::Matrix(Dom::Integer)
```

Use that constructor to produce matrices with integer elements:

```
A := constructor(3, 3, [[1, 2, 3], [2, 3, 1], [3, 1, 2]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

When you use the constructor to create a matrix, you must ensure that all matrix elements belong to the ring or can be converted to the elements in that ring. Otherwise, the constructor issues an error and does not create a matrix:

```
constructor(3, 3, [[1/3, 2, 3], [2, 3, 1], [3, 1, 2]])
```

```
Error: Cannot define a matrix over 'Dom::Integer'. [(Dom::Matrix(Dom::
```

If you use a constructor to create a matrix over a particular ring, you cannot use that matrix in operations that create matrices with elements outside

of the ring. For example, you cannot compute the inverse of the matrix A because the inverse matrix contains noninteger numbers:

1/A

FAIL

Now, create the matrix containing the same elements as A, but use the constructor for matrices with rational numbers:

```
constructorRational := Dom::Matrix(Dom::Rational):
B := constructorRational(3, 3, [[1, 2, 3], [2, 3, 1], [3, 1, 2]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

Compute the inverse of the matrix B:

1/B

$$\begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix}$$

## Use Sparse and Dense Matrices

When you use matrices in MuPAD computations, both computational efficiency and memory use can depend on whether the matrix is sparse or dense. Sparse matrices contain a large number of zero-valued elements. The internal storage of matrices in MuPAD is optimized for sparse data. MuPAD saves the nonzero elements and their indices. When you use sparse matrices, MuPAD assumes that all unspecified elements are zeros. When operating on large sparse matrices, consider the following methods for better performance:

- To create matrices, use the `matrix` function or the constructor `Dom::Matrix()` whenever possible. Both the constructor and the function create matrices over the ring of arbitrary MuPAD expressions `Dom::ExpressionField()`.
- When solving systems of equations represented by sparse matrices, avoid computing inverse matrices. Instead, use `linalg::matlinsolve` to find exact symbolic solutions or `numeric::matlinsolve` to find numeric approximations.
- Avoid creating large empty matrices, and then using indexed assignments for nonzero values. Indexed assignments in MuPAD are expensive operations. Specifying the elements at the same time when you create a matrix is more efficient. For example, the command

```
matrix(10, 10, [-1, 2, -1], Banded):
```

is more efficient than

```
A := matrix(10, 10):
for i from 1 to 10 do
  A[i, i] := 2:
end_for:
for i from 1 to 9 do
  A[i, i + 1] := -1:
  A[i + 1, i] := -1:
end_for:
```

Dense matrices contain only a few or no zero-valued elements. MuPAD provides a special matrix domain for dense matrices. To create such matrices, use the `densematrix` function, which is a shortcut for the constructor

`Dom::DenseMatrix()`. You also can use the constructor itself. For matrices of the `Dom::DenseMatrix()` domain, indexed reading and writing is faster than for matrices of the `Dom::Matrix()` domain.

## Compute with Matrices

### In this section...

“Basic Arithmetic Operations” on page 3-190

“More Operations Available for Matrices” on page 3-191

### Basic Arithmetic Operations

When performing basic arithmetic operations on matrices, use the standard arithmetic operators. For example, add, subtract, multiply, and divide the following two matrices by using the standard +, -, \*, and / operators:

```
A := matrix([[a, b], [c, d]]):
B := matrix([[1, 2], [3, 4]]):
A + B, A - B, A*B, A/B
```

$$\begin{pmatrix} a+1 & b+2 \\ c+3 & d+4 \end{pmatrix}, \begin{pmatrix} a-1 & b-2 \\ c-3 & d-4 \end{pmatrix}, \begin{pmatrix} a+3 & b+2 \\ c+3 & d+4 \end{pmatrix}, \begin{pmatrix} \frac{3b}{2} - 2a & a - \frac{b}{2} \\ \frac{3d}{2} - 2c & c - \frac{d}{2} \end{pmatrix}$$

To perform basic operations on a matrix and a number, use the same operators. When you multiply a matrix by a number, MuPAD multiplies all elements of a matrix by that number:

```
5*A
```

$$\begin{pmatrix} 5a & 5b \\ 5c & 5d \end{pmatrix}$$

When you add a number to a matrix, MuPAD multiplies the number by an identity matrix, and then adds the result to the original matrix:

```
A + 5
```



$$\begin{pmatrix} a+5 & b \\ c & d+5 \end{pmatrix}$$

---

**Note** MATLAB adds a number to each element of a matrix. MuPAD adds a number only to the diagonal elements of a matrix.

---

You can combine matrices with the same number of rows by using the concatenation operator (.):

A.B

$$\begin{pmatrix} a & b & 1 & 2 \\ c & d & 3 & 4 \end{pmatrix}$$

## More Operations Available for Matrices

Besides standard arithmetic operations, many other MuPAD functions are available for computations involving matrices and vectors. To check whether a particular function accepts matrices as parameters, see the “Parameters” section of the function help page. The following functions can operate on matrices:

- The `conjugate` function computes the conjugate of each complex element of a matrix:

```
A := matrix([[1, 2 + 3*I], [1 - I, 2*I]]):
conjugate(A)
```

$$\begin{pmatrix} 1 & 2-3i \\ 1+i & -2i \end{pmatrix}$$

- The `int` and `diff` functions compute the derivative and the integral of each element of a matrix:

```
A := matrix(2, 2, [x, x^2, x^3, x^4]):
int(A, x), diff(A, x)
```

$$\begin{pmatrix} \frac{x^2}{2} & \frac{x^3}{3} \\ \frac{x^4}{4} & \frac{x^5}{5} \end{pmatrix}, \begin{pmatrix} 1 & 2x \\ 3x^2 & 4x^3 \end{pmatrix}$$

- The `expand` function expands each element of a matrix:

```
A := matrix(2, 2, [x, (x + 1)^2, x*(x - 1), x*(x + 4)]):
expand(A)
```

$$\begin{pmatrix} x & x^2 + 2x + 1 \\ x^2 - x & x^2 + 4x \end{pmatrix}$$

- The `map` function applies the specified function to all operands of each element of a matrix:

```
A := matrix(3, 3, [1, 2, 3], Diagonal):
B := map(A, sin)
```

$$\begin{pmatrix} \sin(1) & 0 & 0 \\ 0 & \sin(2) & 0 \\ 0 & 0 & \sin(3) \end{pmatrix}$$

- The `float` function converts each element of a matrix or numerical subexpressions of each element of a matrix to floating-point numbers:

```
float(B)
```

$$\begin{pmatrix} 0.8414709848 & 0 & 0 \\ 0 & 0.9092974268 & 0 \\ 0 & 0 & 0.1411200081 \end{pmatrix}$$

- The `evalAt` function (and its shortcut `|`) substitutes the specified object by another specified object, and then evaluates each element of a matrix:

```
A := matrix(2, 2, [x, x^2, x^3, x^4]):
```

$A|x = 2$

$$\begin{pmatrix} 2 & 4 \\ 8 & 16 \end{pmatrix}$$

- The `subs` function returns a copy of a matrix in which the specified object replaces all instances of another specified object. The function does not evaluate the elements of a matrix after substitution:

```
A := matrix(2, 2, [x, x^2, x^3, x^4]):
subs(A, x = exp(y))
```

$$\begin{pmatrix} e^y & e^{2y} \\ e^{3y} & e^{4y} \end{pmatrix}$$

- The `has` function determines whether a matrix contains the specified object:

```
A := matrix(2, 2, [x, x^2, x^3, x^4]):
has(A, x^3), has(A, x^5)
```

TRUE, FALSE

- The `iszero` function checks whether all elements of a matrix are zeros:

```
A := matrix(2, 2):
iszero(A)
```

TRUE

```
A[1, 1] := 1:
iszero(A)
```

FALSE

- The `norm` function computes the infinity norm (row sum norm) of a matrix:

```
A := matrix(2, 2, [a_1_1, a_1_2, a_2_1, a_2_2]):  
norm(A)
```

$$\max(|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|)$$

- The zip(A, B, f) function combines matrices A and B into a matrix C so that  $C_{ij} = f(A_{ij}, B_{ij})$ :

```
A := matrix(2, 2, [a, b, c, d]):  
B := matrix(2, 2, [10, 20, 30, 40]):  
zip(A, B, _power)
```

$$\begin{pmatrix} a^{10} & b^{20} \\ c^{30} & d^{40} \end{pmatrix}$$

## Compute Determinants and Traces of Square Matrices

MuPAD provides the functions for performing many special operations on matrices. You can compute the dimensions of a matrix, swap or delete columns and rows, or transpose a matrix. For square matrices, you can compute determinants and traces.

To compute the determinant of a square matrix, use the `det` function. For example, compute the determinant of the following 2×2 matrix:

```
A := matrix(2, 2, [a, b, c, d]):
det(A)
```

$ad - bc$

Now, compute the determinant of the 12×12 Hilbert matrix:

```
det(linalg::hilbert(12))
```

$$\frac{1}{379106579436304517151885479034796391880188687864118464104324304732160000000000}$$

To compute a sum of the diagonal elements of a square matrix (the trace of a matrix), use the `linalg::tr` function. For example, the trace of the matrix A is:

```
A := matrix(2, 2, [a, b, c, d]):
linalg::tr(A)
```

$a + d$

Now, compute the trace of the 12×12 Hilbert matrix:

```
H := linalg::hilbert(12):
linalg::tr(H)
```

$$\frac{744355888}{334639305}$$

## Invert Matrices

To find the inverse of a matrix, enter  $1/A$  or  $A^{(-1)}$ :

```
A := matrix([[a, b], [c, d]]):  
B := matrix([[1, 2], [3, 4]]):  
1/A; B^(-1)
```

$$\begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

When MuPAD cannot compute the inverse of a matrix, it returns FAIL:

```
C := matrix([[1, 1], [1, 1]]): 1/C  
  
FAIL
```

## Transpose Matrices

To transpose a matrix, use the transpose command:

```
A := matrix([[1, 2, 3], [4, 5, 6]]):  
transpose(A)
```

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

## Swap and Delete Rows and Columns

The MuPAD linalg library provides the functions for interchanging or deleting rows and columns of a matrix. For example, to swap two rows of a matrix, use `linalg::swapRow`. To swap two columns, use `linalg::swapCol`:

```
OriginalMatrix := linalg::pascal(3);  
SwapRows := linalg::swapRow(OriginalMatrix, 1, 2);  
SwapColumns := linalg::swapCol(OriginalMatrix, 1, 2)
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{pmatrix}$$

To delete a row or a column, use `linalg::delRow` or `linalg::delCol`, respectively:

```
OriginalMatrix := linalg::pascal(3);  
DeleteRows := linalg::delRow(OriginalMatrix, 3);  
DeleteColumns := linalg::delCol(OriginalMatrix, 3)
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

To delete a block of rows or columns simultaneously, use the same functions as for one row or column. Specify the range of rows or columns that you want to delete:

```
OriginalMatrix := linalg::pascal(3);  
DeleteRows := linalg::delRow(OriginalMatrix, 2..3);  
DeleteColumns := linalg::delCol(OriginalMatrix, 2..3)
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

$$(1 \ 1 \ 1)$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

## Compute Dimensions of a Matrix

To find the dimensions of a matrix, use the `linalg::matdim` command. For example, concatenate the following matrices and compute the dimensions of the resulting matrix:

```
A := matrix([[a, b], [c, d]]):  
B := matrix([[1, 2, 3], [4, 5, 6]]):  
linalg::matdim(A.B.A.A.B)
```

```
[2, 12]
```

## Compute Reduced Row Echelon Form

For the reduced row echelon form of a matrix, the following conditions are valid:

- The rows with all zero elements are at the bottom.
- The pivot (leading coefficient) of each nonzero row always occurs to the right of the leading coefficient of the row above.
- If the component ring of the original matrix is a field, the reduced row echelon form is unique, and each pivot is 1.

To find the reduced row echelon form of a matrix, use the `linalg::gaussJordan` function. This function performs Gauss Jordan elimination on a matrix:

```
A := matrix([[1, 2, 3, 4], [5, 6, 7, 8],  
            [9, 10, 11, 12], [13, 14, 15, 16]]):  
linalg::gaussJordan(A)
```

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## Compute Rank of a Matrix

The rank of a matrix is the number of independent rows of a matrix. For a matrix in its reduced row echelon form, the rank is the number of nonzero rows. To compute the rank of a matrix, use the `linalg::rank` function. For example, compute the rank of the following square matrix:

```
A := matrix([[1, 2, 3, 4], [5, 6, 7, 8],  
            [9, 10, 11, 12], [13, 14, 15, 16]]):  
linalg::rank(A)
```

2

Now, compute the reduced row echelon form and the rank of the following 3×4 matrix:

```
OriginalMatrix := matrix([[1, 2, 3], [5, 6, 7],  
                          [9, 10, 11], [13, 14, 15]]):  
RREF = linalg::gaussJordan(OriginalMatrix);  
Rank = linalg::rank(OriginalMatrix)
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{pmatrix}$$

$$\text{RREF} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank = 2

## Compute Bases for Null Spaces of Matrices

All vectors  $\vec{x}$  such that  $A\vec{x} = \vec{0}$  form the null space of the matrix  $A$ . The basis of a null space is a list  $B$  of linearly independent vectors, such that the equation  $A\vec{x} = \vec{0}$  is valid if and only if  $\vec{x}$  is a linear combination of the vectors in  $B$ . To find a basis for the null space of a matrix, use the `linalg::nullspace` function. For example, compute the basis for the null space of the square matrix  $A$ :

```
A := matrix([[1, 2, 3, 4], [5, 6, 7, 8],
             [9, 10, 11, 12], [13, 14, 15, 16]]):
linalg::nullspace(A)
```

$$\left[ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right]$$

Now, compute the basis for the null space of the following  $3 \times 4$  matrix:

```
B := matrix([[1, 2, 3], [5, 6, 7],
             [9, 10, 11], [13, 14, 15]]):
linalg::nullspace(B)
```

$$\left[ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right]$$

## Find Eigenvalues and Eigenvectors

Linear transformations are operations that matrices perform on vectors. An eigenvalue and eigenvector of a square matrix  $A$  are, respectively, a scalar  $\lambda$  and a nonzero vector  $\vec{v}$  such that

$$A \vec{v} = \lambda \vec{v}.$$

Typically, if a matrix changes the length of a vector, but does not change its direction, the vector is called an eigenvector of the matrix. The scaling factor is the eigenvalue associated with this eigenvector.

MuPAD provides the functions for computing eigenvalues and eigenvectors. For example, create the following square matrix:

```
A := matrix([[1, 2, 3], [4, 5, 6], [1, 2, 3]])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

To compute the eigenvalues of the matrix  $A$ , use the `linalg::eigenvalues` function:

```
linalg::eigenvalues(A)
```

```
{0, 9}
```

The `linalg::eigenvalues` function returns a set of eigenvalues. A set in MuPAD cannot contain duplicate elements. Therefore, if a matrix has eigenvalues with multiplicities greater than 1, MuPAD automatically removes duplicate eigenvalues. If you want the `linalg::eigenvalues` function to return eigenvalues along with their multiplicities, use the `Multiple` option. For example, zero is a double eigenvalue of the matrix  $A$ :

```
linalg::eigenvalues(A, Multiple)
```

```
[[0, 2], [9, 1]]
```

To compute the eigenvectors of a matrix, use the `linalg::eigenvectors` function. The function returns eigenvectors along with corresponding eigenvalues and their multiplicities:

```
linalg::eigenvectors(A)
```

```
[[ [0, 2], [ (1), (-2) ] ], [ 9, 1, [ (1), (5/2) ] ] ]
```

The `linalg::eigenvalues` function computes eigenvalues of a matrix by finding the roots of the characteristic polynomial of that matrix. There is no general method for solving polynomial equations of orders higher than 4. When trying to compute eigenvalues of a large matrix, the solver can return complicated solutions or solutions in the form of `RootOf`. Also, the solver can fail to find any solutions for some matrices. For example, create the 6 × 6 Pascal matrix:

```
P := linalg::pascal(6)
```

```
( 1 1 1 1 1 1
  1 2 3 4 5 6
  1 3 6 10 15 21
  1 4 10 20 35 56
  1 5 15 35 70 126
  1 6 21 56 126 252 )
```

For that matrix, MuPAD finds eigenvalues in the form of `RootOf`:

```
eigenvalues := linalg::eigenvalues(P)
```

```
RootOf(z6 - 351 z5 + 6084 z4 - 13869 z3 + 6084 z2 - 351 z + 1, z)
```

You can find floating-point approximation of the result by using the `float` command:

`float(eigenvalues)`

`{0.003004389575, 0.06429432079, 0.4893388287, 2.04357378, 15.55347327, 332.8463154}`

For more information about approximating eigenvalues and eigenvectors numerically, see [Numeric Eigenvalues and Eigenvectors](#).



## Find Jordan Canonical Form of a Matrix

The *Jordan canonical form* of a square matrix is a block matrix in which each block is a Jordan block. A *Jordan block* is a square matrix with an eigenvalue of the original matrix on the main diagonal. A block also can contain 1s on its first superdiagonal. Each Jordan block corresponds to a particular eigenvalue. Single eigenvalues produce  $1 \times 1$  Jordan blocks. If an  $n \times n$  square matrix has  $n$  linearly independent eigenvectors, the Jordan form of that matrix is a diagonal matrix with the eigenvalues on the main diagonal. For example, create the  $3 \times 3$  Pascal matrix P:

```
P := linalg::pascal(3)
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

The Jordan canonical form of the matrix P is a diagonal matrix with the eigenvalues on its main diagonal:

```
linalg::eigenvalues(P);  
linalg::jordanForm(P)
```

$$\{1, 4 - \sqrt{15}, \sqrt{15} + 4\}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 - \sqrt{15} & 0 \\ 0 & 0 & \sqrt{15} + 4 \end{pmatrix}$$

To find the Jordan canonical form of a matrix along with the nonsingular similarity transformation matrix T that transforms the original matrix to its Jordan form, use the All option:

```
[J, T] := linalg::jordanForm(P, All)
```

$$\left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 - \sqrt{15} & 0 \\ 0 & 0 & \sqrt{15} + 4 \end{pmatrix}, \begin{pmatrix} -2 & \frac{\sqrt{15}}{5} + 1 & 1 - \frac{\sqrt{15}}{5} \\ -1 & -\frac{2\sqrt{15}}{5} - 1 & \frac{2\sqrt{15}}{5} - 1 \\ 1 & 1 & 1 \end{pmatrix} \right]$$

You can restore the original matrix from its Jordan form and the similarity transformation:

```
simplify(T*J*T^(-1))
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

You cannot transform to a diagonal form matrices for which the number of linearly independent eigenvectors is less than the matrix dimensions. For example, the following matrix has the triple eigenvalue 2. The Jordan block corresponding to that eigenvalue has 1s on its first superdiagonal:

```
A := matrix([[ -6, 11, -15, -11], [11, -14, 22, 16],
             [ -6, 7, -7, -7], [24, -32, 43, 34]]):
linalg::jordanForm(A)
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The 4×4 matrix A has a triple eigenvalue 2 and only two eigenvectors:

```
linalg::eigenvectors(A)
```

$$\left[ \left[ 1, 1, \left[ \begin{pmatrix} -\frac{1}{\omega} \\ \frac{1}{\omega} \\ -\frac{1}{\omega} \\ 1 \end{pmatrix} \right] \right], \left[ 2, 3, \left[ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right]$$

## Compute Matrix Exponentials

You can use the `exp` function to compute matrix exponentials  $e^A = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$ . For example, compute the matrix exponential for the following square matrix, and then simplify the result:

```
A := matrix([[0, 1], [-1, 0]]):  
simplify(exp(A*t))
```

$$\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

Now, compute the matrix exponential for the following matrix. Simplify the result:

```
B := matrix(2, 2, [0, 2, 2, 0]):  
simplify(exp(B*x))
```

$$\begin{pmatrix} \cosh(2x) & \sinh(2x) \\ \sinh(2x) & \cosh(2x) \end{pmatrix}$$

## Compute Cholesky Factorization

The Cholesky factorization expresses a complex Hermitian (self-adjoint) positive definite matrix as a product of a lower triangular matrix  $L$  and its Hermitian transpose  $L^H$ :  $A = LL^H$ . The Hermitian transpose of a matrix is the complex conjugate of the transpose of that matrix. For real and symmetric matrices, the transpose coincides with the Hermitian transpose. Thus, the Cholesky factorization of a real symmetric positive definite matrix is  $A = LL^T$ , where  $L^T$  is the transpose of  $L$ .

Create the 5×5 Pascal matrix. This matrix is symmetric and positive definite:

```
P := linalg::pascal(5)
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$$

To compute the Cholesky decomposition of a matrix, use the `linalg::factorCholesky` function. The result is the following lower triangular matrix:

```
L := linalg::factorCholesky(P)
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

The product of the triangular matrix  $L$  and its transpose gives the original matrix  $P$ :

```
testeq(P = L*linalg::transpose(L))
```

```
TRUE
```

---

**Note** MuPAD returns a lower triangular matrix as a result of Cholesky factorization. MATLAB returns an upper triangular matrix, which is the transpose of the result returned by MuPAD.

---

When MuPAD cannot determine whether a matrix is Hermitian, `linalg::factorCholesky` throws an error:

```
A := matrix([[a, b, c], [b, c, a], [c, a, b]]):  
linalg::factorCholesky(A)
```

```
Error: A Hermitian matrix is expected. [linalg::factorCholesky]
```

If you know that the matrix is Hermitian, you can suppress this error. The `NoCheck` option suppresses the error and lets the `linalg::factorCholesky` function continue the computation:

```
linalg::factorCholesky(A, NoCheck)
```

$$\begin{pmatrix} \sqrt{a} & 0 & 0 \\ \frac{b}{\sqrt{a}} & \frac{\sigma_3}{\sqrt{|a|}} & 0 \\ \frac{c}{\sqrt{a}} - \frac{\sigma_2}{b\sqrt{|a|}\sigma_3} & \frac{\sqrt{b\sigma_1|a||b|^2 - \sigma_1|b|^2|c|^2 - |\sigma_2|^2}}{\sqrt{\sigma_1}\sqrt{|a|}|b|}} & \end{pmatrix}$$

where

$$\sigma_1 = ||b|^2 - c|a|$$

$$\sigma_2 = c|b|^2 - a|b|a$$

$$\sigma_3 = \sqrt{c|a| - |b|^2}$$

If you know that the matrix is real and symmetric, use the `Real` option to avoid complex conjugates (implied by  $|a| = a^*\bar{a}$ ) in the result:

`linalg::factorCholesky(A, NoCheck, Real)`

$$\begin{pmatrix} \sqrt{a} & 0 & 0 \\ \frac{b}{\sqrt{a}} & \sqrt{\frac{ac-b^2}{a}} & 0 \\ \frac{c}{\sqrt{a}} - \frac{bc-a^2}{a\sqrt{\frac{ac-b^2}{a}}} & \sqrt{\frac{-a^3-3abc+b^3+c^3}{ac-b^2}} & \end{pmatrix}$$

## Compute LU Factorization

The LU factorization expresses an  $m \times n$  matrix  $A$  as follows:  $P \cdot A = L \cdot U$ . Here  $L$  is an  $m \times m$  lower triangular matrix that contains 1s on the main diagonal,  $U$  is an  $m \times n$  matrix upper triangular matrix, and  $P$  is a permutation matrix. To compute the LU decomposition of a matrix, use the `linalg::factorLU` function. For example, compute the LU decomposition of the following square matrix:

```
A := matrix([[0, 0, 1], [1, 2, 3], [0, 1, 2]]):
[L, U, p] := linalg::factorLU(A)
```

$$\left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, [2, 3, 1] \right]$$

Instead of returning the permutation matrix  $P$ , MuPAD returns the list  $p$  with numbers corresponding to row exchanges in the matrix  $A$ . For an  $n \times n$  matrix, the list  $p$  represents the following permutation matrix with indices  $i$  and  $j$  ranging from 1 to  $n$ :

$$P_{ij} = \delta_{p_i j} = \begin{cases} 1 & \text{if } j = p_i \\ 0 & \text{if } j \neq p_i \end{cases}$$

Using this expression, restore the permutation matrix  $P$  from the list  $p$ :

```
P := matrix(3, 3):
for i from 1 to 3 do
  P[i, p[i]] := 1
end_for:
P
```

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



More efficiently, compute the result of applying the permutation matrix to A without restoring the permutation matrix itself:

```
PA := matrix(3, 3):
for i from 1 to 3 do
  PA[i, 1..3] := A[p[i], 1..3]
end_for:
PA
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

The product of the lower triangular matrix L and the upper triangular matrix U is the original matrix A with the rows interchanged according to the permutation matrix P:

```
testeq(PA = L*U)
```

TRUE

Now, compute the LU decomposition for the 3 × 2 matrix B:

```
B := matrix([[1, 2], [3, 4], [5, 6]]):
[L, U, p] := linalg::factorLU(B)
```

$$\left[ \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}, [1, 2, 3] \right]$$

The permutation matrix for this LU factorization shows that the order of the rows does not change. Therefore, the product of the lower triangular matrix L and the upper triangular matrix U gives the original matrix A:

```
testeq(B = L*U)
```

TRUE

## Compute QR Factorization

The QR factorization expresses an  $m \times n$  matrix  $A$  as follows:  $A = Q \cdot R$ . Here  $Q$  is an  $m \times m$  unitary matrix, and  $R$  is an  $m \times n$  upper triangular matrix. If the components of  $A$  are real numbers,  $Q$  is an orthogonal matrix. To compute the QR decomposition of a matrix, use the `linalg::factorQR` function. For example, compute the QR decomposition of the  $3 \times 3$  Pascal matrix:

```
P := linalg::pascal(3):
[Q, R] := linalg::factorQR(P)
```

$$\left[ \left( \begin{array}{ccc} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{array} \right), \left( \begin{array}{ccc} \sqrt{3} & 2\sqrt{3} & \frac{10\sqrt{3}}{3} \\ 0 & \sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{6} \end{array} \right) \right]$$

The product of  $Q$  and  $R$  gives the original  $3 \times 3$  Pascal matrix:

```
testeq(P = Q*R)
```

TRUE

Also, you can perform the QR factorization for matrices that contain complex values. In this case, the matrix  $Q$  is unitary:

```
B := matrix([[I, -1], [1, I]]):
[Q, R] := linalg::factorQR(B)
```

$$\left[ \left( \begin{array}{cc} \frac{\sqrt{2}i}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}i}{2} \end{array} \right), \left( \begin{array}{cc} \sqrt{2} & \sqrt{2}i \\ 0 & 0 \end{array} \right) \right]$$

Again, the product of  $Q$  and  $R$  gives the original matrix  $B$ :

```
testeq(B = Q*R)
```

TRUE

## Compute Determinant Numerically

To compute the determinant of a square matrix numerically, use the `numeric::det` function. For example, compute the determinant of the  $5 \times 5$  Pascal matrix:

```
numeric::det(linalg::pascal(5))
```

```
1.0
```

When you use numeric functions, the result can be extremely sensitive to roundoff errors. For example, the determinant of a Pascal matrix of any size is 1. However, if you use the `numeric::det` function to compute the determinant of a  $25 \times 25$  Pascal matrix, you get the following incorrect result:

```
numeric::det(linalg::pascal(15))
```

```
1.000043957
```

When computing determinants numerically, you can use the `HardwareFloats` and `SoftwareFloats` options to employ the hardware or software float arithmetic, respectively. (You can use the short names for these options: `Hard` and `Soft`.)

When you use the `HardwareFloats` option, MuPAD converts all input data to hardware floating-point numbers, and passes the data for processing by compiled C code outside of the MuPAD session. Then, the results get back into the MuPAD session. Hardware arithmetic often allows you to perform computations much faster than software arithmetic, which uses the MuPAD kernel for performing computations.

The precision of hardware arithmetic is limited to about 15 digits. By default, the `numeric::det` function uses the `HardwareFloats` option. The function switches to software arithmetic under one or more of the following conditions:

- You use the `SoftwareFloats` option or the `MinorExpansion` option explicitly.
- The current value of `DIGITS` is larger than 15.

- The input data or computed data involves numbers that are larger than  $10^{308}$  or smaller than  $10^{-308}$ . Hardware floats cannot represent such numbers.

The precision of hardware and software arithmetic can differ. Therefore, the results obtained with the `HardwareFloats` and `SoftwareFloats` options also can differ. For example, compute the determinant of the  $25 \times 25$  Pascal matrix using each of the options. Both numeric results are several orders larger than the correct answer because of the roundoff errors. However, the result obtained using software arithmetic is several orders closer to the correct answer:

```
P := linalg::pascal(25):
detP := det(P):
float(detP);
numeric::det(P, SoftwareFloats);
numeric::det(P, HardwareFloats)
```

1.0

1669339685.0

$1.039468525 \cdot 10^{15}$

Another example of numerically ill-conditioned matrices is the Hilbert matrices. For example, create the  $20 \times 20$  Hilbert matrix, compute its determinant symbolically, and then approximate the result numerically:

```
H := linalg::hilbert(15):
deth := det(H):
float(deth)
```

$1.058542743 \cdot 10^{-124}$

Now, use the `numeric::det` function to compute the determinant numerically:

```
numeric::det(H)
```

$-3.822215463 \cdot 10^{-121}$

The numeric result obtained with the `SoftwareFloats` option is closer to the correct result:

```
numeric::det(linalg::hilbert(15), SoftwareFloats)
```

$3.277553006 \cdot 10^{-123}$

To prevent the conversion of input data to floating-point numbers while using the `numeric::det` function, use the `Symbolic` option. This option allows you to compute the determinant exactly (without roundoff errors). For matrices over particular rings and fields, the determinant computed by `numeric::det` with the `Symbolic` option can differ from the determinant computed by `det`. The reason is that `det` performs computations over the component domain of the input matrix. The `numeric::det` function with the `Symbolic` option always performs computations over the field of arbitrary MuPAD expressions. For example, create the following matrix over the domain `Dom::IntegerMod(5)`:

```
A := Dom::Matrix(Dom::IntegerMod(5))([[1, 2], [3, 4]])
```

$$\begin{pmatrix} 1 \bmod 5 & 2 \bmod 5 \\ 3 \bmod 5 & 4 \bmod 5 \end{pmatrix}$$

The `det` function computes the determinant over the component domain `Dom::IntegerMod(5)`:

```
det(A)
```

$3 \bmod 5$

The `numeric::det` function with the `Symbolic` option computes the determinant of the following matrix instead of the original matrix `A`:

```
expr(A)
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The determinant of this matrix is an integer number:

```
numeric::det(A, Symbolic)
```

$-2$

The `numeric::det` function switches to the `Symbolic` option under one or more of the following conditions:

- You use the `Symbolic` option explicitly.
- The input data contains symbolic objects.

## Compute Eigenvalues and Eigenvectors Numerically

When computing eigenvalues and eigenvectors of some matrices symbolically, you can get a long result in a form that is not suitable for further computations. For example, the `linalg::eigenvectors` function returns the following results for eigenvalues and eigenvectors of the  $3 \times 3$  Hilbert matrix:

```
H := linalg::hilbert(3):  
eigen := linalg::eigenvectors(H)
```



$$\left[ \left[ \left[ \sigma_5, 1, \begin{pmatrix} -\frac{6559}{3375\sigma_8} - \frac{48\sigma_8}{5} + 8\sigma_5^2 - \frac{2113}{450} \\ \frac{45913}{13500\sigma_8} + \frac{84\sigma_8}{5} - \frac{32\sigma_5^2}{3} + \frac{5066}{675} \\ 1 \end{pmatrix} \right] \right], \left[ \frac{23}{45} - \frac{\sigma_8}{2} - \frac{6559}{64800\sigma_8} - \sigma_6, 1, \right.$$

$$\left. \left[ \begin{pmatrix} \sigma_4 + \frac{6559}{6750\sigma_8} + \frac{24\sigma_8}{5} + 8\sigma_2 - \frac{2113}{450} \\ -\sigma_3 - \frac{45913}{27000\sigma_8} - \frac{42\sigma_8}{5} - \frac{32\sigma_2}{3} + \frac{5066}{675} \\ 1 \end{pmatrix} \right] \right], \left[ \frac{23}{45} - \frac{\sigma_8}{2} - \frac{6559}{64800\sigma_8} + \sigma_6, 1, \right.$$

$$\left. \left[ \begin{pmatrix} -\sigma_4 + \frac{6559}{6750\sigma_8} + \frac{24\sigma_8}{5} + 8\sigma_1 - \frac{2113}{450} \\ \sigma_3 - \frac{45913}{27000\sigma_8} - \frac{42\sigma_8}{5} - \frac{32\sigma_1}{3} + \frac{5066}{675} \\ 1 \end{pmatrix} \right] \right] \right]$$

where

$$\sigma_1 = \left( \sigma_6 - \frac{6559}{64800\sigma_8} - \frac{\sigma_8}{2} + \frac{23}{45} \right)^2$$

$$\sigma_2 = \left( \sigma_6 + \frac{6559}{64800\sigma_8} + \frac{\sigma_8}{2} - \frac{23}{45} \right)^2$$

$$\sigma_3 = \frac{42\sqrt{3}\sigma_7 i}{5}$$

$$\sigma_4 = \frac{24\sqrt{3}\sigma_7 i}{5}$$

$$\sigma_5 = \frac{6559}{32400\sigma_8} + \sigma_8 + \frac{23}{45}$$

$$\sigma_6 = \frac{\sqrt{3}\sigma_7 i}{2}$$

Numeric approximation of the result returned by the symbolic `linalg::eigenvectors` function gives a shorter answer that contains complex numbers:

`float(eigen)`

$$\left[ \left[ \left[ 1.408318927 - 1.084202172 \cdot 10^{-19} i, 1.0, \left[ \left( \begin{array}{c} 2.558147015 - 1.575682167 \cdot 10^{-18} i \\ 1.422413022 + 1.522668397 \cdot 10^{-18} i \\ 1.0 \end{array} \right) \right] \right] \right], \right. \\ \left. \left[ \left[ 0.002687340356 + \sigma_1, 1.0, \left[ \left( \begin{array}{c} 0.1853704181 - \sigma_3 \\ -1.036411196 + \sigma_2 \\ 1.0 \end{array} \right) \right] \right] \right], \right. \\ \left. \left[ \left[ 0.1223270659 + \sigma_1, 1.0, \left[ \left( \begin{array}{c} -0.8435174328 - \sigma_3 \\ 0.8139981738 + \sigma_2 \\ 1.0 \end{array} \right) \right] \right] \right] \right]$$

where

$$\sigma_1 = 5.421010862 \cdot 10^{-20} i$$

$$\sigma_2 = 8.67361738 \cdot 10^{-19} i$$

$$\sigma_3 = 4.33680869 \cdot 10^{-19} i$$

If you need simple (though approximate) eigenvalues and eigenvectors of the Hilbert matrix in further computations, use numeric methods from the beginning. To approximate eigenvalues and eigenvectors of a matrix numerically, use the `numeric::eigenvectors` function. The function returns

eigenvalues, eigenvectors, and residues (estimated errors for the numerical eigenvalues):

```
[eigenvalues, eigenvectors, residues] := numeric::eigenvalues(H)
```

$$\left[ \begin{array}{c} [1.408318927, 0.1223270659, 0.002687340356], \\ \left( \begin{array}{ccc} 0.827044927 & -0.5474484307 & -0.1276593297 \\ 0.4598639044 & 0.5282902351 & 0.7137468858 \\ 0.3232984352 & 0.6490066589 & -0.6886715317 \end{array} \right), \\ [4.317754261 \cdot 10^{-16}, 3.213231541 \cdot 10^{-16}, 7.71731031 \cdot 10^{-17}] \end{array} \right]$$

Small residue values indicate that roundoff errors do not significantly affect the results. To suppress the computation of the residues, use the `NoResidues` option:

```
numeric::eigenvalues(H, NoResidues)
```

$$\left[ \begin{array}{c} [1.408318927, 0.1223270659, 0.002687340356], \\ \left( \begin{array}{ccc} 0.827044927 & -0.5474484307 & -0.1276593297 \\ 0.4598639044 & 0.5282902351 & 0.7137468858 \\ 0.3232984352 & 0.6490066589 & -0.6886715317 \end{array} \right), \text{NIL} \end{array} \right]$$

If you want to compute only eigenvalues of a matrix, use the `numeric::eigenvalues` function:

```
numeric::eigenvalues(H)
```

[1.408318927, 0.1223270659, 0.002687340356]

When computing eigenvalues and eigenvectors numerically, you can use the `HardwareFloats` and `SoftwareFloats` options to employ hardware or software float arithmetic, respectively. For information about these options, see the Numeric Determinant section. For more details, see the `numeric::eigenvectors` and `numeric::eigenvalues` help pages.

## Compute Factorizations Numerically

### In this section...

“Cholesky Decomposition” on page 3-227

“LU Decomposition” on page 3-228

“QR Decomposition” on page 3-231

“Singular Value Decomposition” on page 3-233

For numeric factorization functions, you can use the `HardwareFloats`, `SoftwareFloats` and `Symbolic` options. For information about these options, see the Numeric Determinant section. For more details, see the help pages of the numeric functions provided for each particular factorization function.

### Cholesky Decomposition

The Cholesky decomposition of a Hermitian (self-adjoint) positive definite matrix is a product of a lower triangular matrix  $L$  and its Hermitian transpose  $L^H$ :  $A = LL^H$ . The Hermitian transpose of a matrix is the complex conjugate of its transpose. For real symmetric positive definite matrices,  $A = LL^T$ , where  $L^T$  is the transpose of  $L$ .

You can perform the Cholesky factorization of complex matrices symbolically by using `linalg::factorCholesky` or numerically by using `numeric::factorCholesky`. For example, create the following 2×2 matrix:

```
A := matrix([[sin(1), I], [-I, exp(2)]])
```

$$\begin{pmatrix} \sin(1) & i \\ -i & e^2 \end{pmatrix}$$

First, compute the Cholesky decomposition of that matrix by using the symbolic `linalg::factorCholesky` function:

```
L := linalg::factorCholesky(A)
```

$$\begin{pmatrix} \sqrt{\sin(1)} & 0 \\ -\frac{i}{\sqrt{\sin(1)}} & \sqrt{e^2 - \frac{1}{\sin(1)}} \end{pmatrix}$$

Now, compute the Cholesky decomposition by using the symbolic `numeric::factorCholesky` function:

```
L := numeric::factorCholesky(A)
```

$$\begin{pmatrix} 0.917317276 & 0 \\ -1.090135361 i & 2.490112647 \end{pmatrix}$$

When using `numeric::factorCholesky`, you can prevent the conversion of data to floating-point numbers and, therefore, get the symbolic result by using the `Symbolic` option:

```
L := numeric::factorCholesky(A, Symbolic)
```

$$\begin{pmatrix} \sqrt{\sin(1)} & 0 \\ -\frac{i}{\sqrt{\sin(1)}} & \sqrt{e^2 - \frac{1}{\sin(1)}} \end{pmatrix}$$

The product of the triangular matrix `L` and its Hermitian transpose gives the original matrix `A`:

```
teste(A = L*htranspose(L))
```

```
TRUE
```

## LU Decomposition

LU factorization expresses an  $m \times n$  matrix `A` as a product of a lower triangular matrix `L` and an upper triangular matrix `U`:  $A = LU$ . Also, LU decomposition can involve a row permutation matrix:  $PA = LU$ . To compute the LU

decomposition of a matrix numerically, use the `numeric::factorLU` function. For example, create the following 4×4 Toeplitz matrix:

```
T := linalg::toeplitz(4, [1, 2, 3, 4, 5])
```

$$\begin{pmatrix} 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Use the `numeric::factorLU` function to compute the LU decomposition of the matrix T numerically:

```
[L, U, p] := numeric::factorLU(T)
```

$$\left[ \begin{pmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0.3333333333 & 0.6666666667 & 1.0 & 0 \\ 0.6666666667 & 0.3333333333 & 0 & 1.0 \end{pmatrix}, \begin{pmatrix} 3.0 & 4.0 & 5.0 & 0 \\ 0 & 1.0 & 2.0 & 3.0 \\ 0 & 0 & 0 & 2.0 \\ 0 & 0 & 0 & 4.0 \end{pmatrix}, [1, 4, 3, 2] \right]$$

Instead of returning the permutation matrix P, MuPAD returns the list p with numbers corresponding to row exchanges in a matrix T. For an  $n \times n$  matrix, the list p represents the following permutation matrix with indices i and j ranging from 1 to n:

$$P_{ij} = \delta_{p_i j} = \begin{cases} 1 & \text{if } j = p_i \\ 0 & \text{if } j \neq p_i \end{cases}$$

Using this expression, restore the permutation matrix P from the list p:

```
P := matrix(4, 4):
for i from 1 to 4 do
  P[i, p[i]] := 1
end_for:
P
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

More efficiently, compute the result of applying the permutation matrix to A without restoring the permutation matrix itself:

```
PT := matrix(4, 4):
for i from 1 to 4 do
  PT[i, 1..4] := T[p[i], 1..4]
end_for:
PT
```

$$\begin{pmatrix} 3 & 4 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

Within floating-point precision, the product of the lower triangular matrix L and the upper triangular matrix U is the original matrix A with the rows interchanged according to the permutation matrix P:

```
float(PT) = L*U
```

$$\begin{pmatrix} 3.0 & 4.0 & 5.0 & 0 \\ 0 & 1.0 & 2.0 & 3.0 \\ 1.0 & 2.0 & 3.0 & 4.0 \\ 2.0 & 3.0 & 4.0 & 5.0 \end{pmatrix} = \begin{pmatrix} 3.0 & 4.0 & 5.0 & 0 \\ 0 & 1.0 & 2.0 & 3.0 \\ 1.0 & 2.0 & 3.0 & 4.0 \\ 2.0 & 3.0 & 4.0 & 5.0 \end{pmatrix}$$

The symbolic LU factorization function uses a different pivoting strategy than the numeric LU factorization function. Therefore, the symbolic function can return different results for the same matrix:

```
linalg::factorLU(T)
```



$$\left[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 5 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 5 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -12 \end{pmatrix}, [1, 2, 3, 4] \right]$$

## QR Decomposition

The QR factorization expresses an  $m \times n$  matrix  $A$  as follows:  $A = Q \cdot R$ . Here  $Q$  is an  $m \times m$  unitary matrix, and  $R$  is an  $m \times n$  upper triangular matrix. If the components of  $A$  are real numbers,  $Q$  is an orthogonal matrix. To compute the QR decomposition of a matrix numerically, use the `numeric::factorQR` function. For example, create the following  $3 \times 3$  Vandermonde matrix:

```
V := linalg::vandermonde([2, PI, 1/3])
```

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & \pi & \pi^2 \\ 1 & \frac{1}{3} & \frac{1}{9} \end{pmatrix}$$

When computing the QR decomposition of that matrix symbolically, you get the following long result:

```
[Q, R] := linalg::factorQR(V)
```

$$\left[ \begin{array}{c} \left( \begin{array}{ccc} \frac{\sqrt{3}}{3} - \frac{\frac{\pi}{3} - \frac{11}{9}}{\sigma_1} & -\frac{5(13\pi - 24\pi^2 + 9\pi^3 - 2)}{6\sigma_2\sigma_3} \\ \frac{\sqrt{3}}{3} & \frac{\frac{2\pi}{3} - \frac{7}{9}}{\sigma_1} & \frac{25(3\pi^2 - 7\pi + 2)}{6\sigma_2\sigma_3} \\ \frac{\sqrt{3}}{3} & -\frac{\frac{\pi}{3} + \frac{4}{9}}{\sigma_1} & \frac{5(16\pi - 13\pi^2 + 3\pi^3 - 4)}{2\sigma_2\sigma_3} \end{array} \right) \\ \left( \begin{array}{ccc} \sqrt{3} & \frac{\sqrt{3}(\pi + \frac{7}{3})}{3} & \frac{\sqrt{3}(\pi^2 + \frac{37}{9})}{3} \\ 0 & \sigma_1 & -\frac{\frac{37\pi}{27} + \frac{7\pi^2}{9} - \frac{2\pi^3}{3} - \frac{392}{81}}{\sigma_1} \\ 0 & 0 & \sigma_2 \end{array} \right) \end{array} \right]$$

where

$$\sigma_1 = \sqrt{\frac{2\pi^2}{3} - \frac{14\pi}{9} + \frac{62}{27}}$$

$$\sigma_2 = \sqrt{\frac{25(61\pi^2 - 28\pi - 42\pi^3 + 9\pi^4 + 4)}{18\sigma_3}}$$

$$\sigma_3 = 9\pi^2 - 21\pi + 31$$

To get a shorter answer, approximate this result by floating-point numbers:

`[Q, R] := float([Q, R])`

$$\left[ \begin{array}{c} \left( \begin{array}{ccc} 0.5773502692 & 0.08763169824 & -0.8117803595 \\ 0.5773502692 & 0.6592065645 & 0.4817814566 \\ 0.5773502692 & -0.7468382628 & 0.3299989029 \end{array} \right) \\ \left( \begin{array}{ccc} 1.732050808 & 3.160949992 & 8.071769864 \\ 0 & 1.997275809 & 6.773652774 \\ 0 & 0 & 1.544537491 \end{array} \right) \end{array} \right]$$

You can get the same result faster by calling the numeric factorization function from the beginning:

```
[Q, R] := numeric::factorQR(V)
```

$$\left[ \begin{array}{ccc} \left( \begin{array}{ccc} 0.5773502692 & 0.08763169824 & -0.8117803595 \\ 0.5773502692 & 0.6592065645 & 0.4817814566 \\ 0.5773502692 & -0.7468382628 & 0.3299989029 \end{array} \right) \\ \left( \begin{array}{ccc} 1.732050808 & 3.160949992 & 8.071769864 \\ 0 & 1.997275809 & 6.773652774 \\ 0 & 0 & 1.544537491 \end{array} \right) \end{array} \right]$$

Within floating-point precision, the product of Q and R gives the original 3×3 Vandermonde matrix V:

```
float(V) = Q*R
```

$$\begin{pmatrix} 1.0 & 2.0 & 4.0 \\ 1.0 & 3.141592654 & 9.869604401 \\ 1.0 & 0.3333333333 & 0.1111111111 \end{pmatrix} = \begin{pmatrix} 1.0 & 2.0 & 4.0 \\ 1.0 & 3.141592654 & 9.869604401 \\ 1.0 & 0.3333333333 & 0.1111111111 \end{pmatrix}$$

## Singular Value Decomposition

The singular value decomposition expresses an  $m \times n$  matrix A as follows:  $A = USV^H$ . Here S is an  $m \times n$  diagonal matrix with singular values of A on its diagonal. The columns of the  $m \times m$  matrix U are the left singular vectors for corresponding singular values. The columns of the  $n \times n$  matrix V are the right singular vectors for corresponding singular values.  $V^H$  is the Hermitian transpose (the complex conjugate of the transpose) of V.

To compute the singular value decomposition of a matrix, use the `numeric::svd` or `numeric::singularvectors` function. These two functions are equivalent. For example, compute the singular value decomposition of the following matrix:

```
A := matrix([[9, 4], [6, 8], [2, 7]]):
svd := numeric::svd(A)
```

$$\left[ \begin{pmatrix} 0.6104993556 & 0.7174382767 & 0.3355187863 \\ 0.6645912432 & -0.2336056545 & -0.7097512787 \\ 0.4308236486 & -0.6562855456 & 0.6194192978 \end{pmatrix}, [14.9359164, 5.188294644], \right. \\ \left. \begin{pmatrix} 0.6925379522 & 0.7213814419 \\ 0.7213814419 & -0.6925379522 \end{pmatrix}, [8.704148513 \cdot 10^{-14}, 8.704148513 \cdot 10^{-14}, \right. \\ \left. 6.039716306 \cdot 10^{-31}], [4.352074257 \cdot 10^{-14}, 4.352074257 \cdot 10^{-14}] \right]$$

Instead of returning the diagonal matrix  $S$ , MuPAD returns the list  $d$  of the diagonal elements of that matrix:

```
d := svd[2]
[14.9359164, 5.188294644]
```

You can restore the matrix  $S$  from that list:

```
S := matrix(3, 2, d, Diagonal)

$$\begin{pmatrix} 14.9359164 & 0 \\ 0 & 5.188294644 \\ 0 & 0 \end{pmatrix}$$

```

The numeric::svd function also computes the residues  $res_U$  and  $res_V$  for the numeric singular vectors. The residues are the estimated errors for the numerical matrices  $U$  and  $V$ :

```
res_U = svd[4];
res_V = svd[5]
res_U = [8.704148513 · 10-14, 8.704148513 · 10-14, 6.039716306 · 10-31]
```

$$\text{res}_V = \left[ 4.352074257 \cdot 10^{-14}, 4.352074257 \cdot 10^{-14} \right]$$

Small residue values indicate that roundoff errors do not significantly affect the results. To suppress the computation of the residues, use the `NoResidues` option:

```
svd := numeric::svd(A, NoResidues)
```

$$\left[ \begin{pmatrix} 0.6104993556 & 0.7174382767 & 0.3355187863 \\ 0.6645912432 & -0.2336056545 & -0.7097512787 \\ 0.4308236486 & -0.6562855456 & 0.6194192978 \end{pmatrix}, [14.9359164, 5.188294644], \right. \\ \left. \begin{pmatrix} 0.6925379522 & 0.7213814419 \\ 0.7213814419 & -0.6925379522 \end{pmatrix}, \text{NIL}, \text{NIL} \right]$$

Within the floating-point precision, the product of  $U$ ,  $S$ , and the Hermitian transpose of  $V$  returns the original matrix  $A$ :

```
[U, d, V, res_U, res_V] := svd:
A = U*S*conjugate(transpose(V))
```

$$\begin{pmatrix} 9 & 4 \\ 6 & 8 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 9.0 & 4.0 \\ 6.0 & 8.0 \\ 2.0 & 7.0 \end{pmatrix}$$

## Mathematical Constants Available in MuPAD

In this section...
“Special Real Numbers” on page 3-236
“Infinities” on page 3-237
“Boolean Constants” on page 3-237
“Special Values” on page 3-237
“Special Sets” on page 3-238

### Special Real Numbers

MuPAD provides symbolic representations of the following commonly used special real numbers. You can perform exact computations that include the constants. Also, you can get floating-point approximations with the current precision DIGITS by using the float function.

E Euler number (exponential constant, base of natural logarithm)

$$e \approx 2.718281828$$

PI  $\pi \approx 3.141592653$

EULER Euler-Mascheroni constant

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \frac{1}{i} \right) - \ln(n) \approx 0.5772156649$$

CATALAN Catalan constant

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} \approx 0.9159655941$$

## Infinities

MuPAD provides the symbolic representations for real and complex infinities. Many functions accept infinities as their arguments. For example, you can use infinities when computing sums, limits, integrals, and so on. Also, you can perform arithmetical operations with infinities. MuPAD can return infinities as results of computations:

<code>infinity</code>	Real positive infinity
<code>complexInfinity</code>	Infinite point of the complex plane
<code>RD_INF</code>	Real positive infinity used in floating-point intervals
<code>RD_NINF</code>	Real negative infinity used in floating-point intervals

## Boolean Constants

MuPAD uses a three-state logic with the Boolean constants `TRUE`, `FALSE`, and `UNKNOWN`. You can use these constants in computations. MuPAD can return these constants as a result of computations.

## Special Values

The following objects in MuPAD represent special values. You can use these special values in computations. MuPAD can return these values as a result of computations:

<code>I</code>	Imaginary unit $\sqrt{-1}$ .
<code>NIL</code>	Null object of the domain <code>DOM_NIL</code> . MuPAD uses this object to indicate missing objects explicitly.

<code>null()</code>	Null (void) object of the domain <code>DOM_NULL</code> . This object does not produce any visible output. MuPAD removes this object from data structures (such as sequences, lists, sets, and so on).
<code>undefined</code>	Undefined value.
<code>RD_NAN</code>	Undefined value used in floating-point intervals. If you use typeset mode, MuPAD displays this value as <b>NaN</b> in output regions.
<code>FAIL</code>	Failure object of the domain <code>DOM_FAIL</code> .

## Special Sets

MuPAD provides the following predefined sets and lets you use them in computations. For example, you can use these predefined sets to compute intersections, differences, and unions, or to make assumptions. MuPAD can use these sets to return results of computations:

<code>C_</code>	The set $\mathbb{C}$ of complex numbers
<code>N_</code>	The set of positive integers: $\mathbb{Z} \cap [1, \infty)$
<code>Q_</code>	The set $\mathbb{Q}$ of rational numbers
<code>R_</code>	The set $\mathbb{R}$ of real numbers
<code>Z_</code>	The set $\mathbb{Z}$ of integers



## Special Functions Available in MuPAD

In this section...
“Dirac and Heaviside Functions” on page 3-239
“Gamma Functions” on page 3-239
“Zeta Function and Polylogarithms” on page 3-240
“Airy and Bessel Functions” on page 3-240
“Exponential and Trigonometric Integrals” on page 3-241
“Error Functions and Fresnel Functions” on page 3-241
“Hypergeometric, Meijer G, and Whittaker Functions” on page 3-241
“Elliptic Integrals” on page 3-242
“Lambert W Function (omega Function)” on page 3-242

### Dirac and Heaviside Functions

The following MuPAD functions represent the Dirac  $\delta$ -distribution and the Heaviside (step) function. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

<code>dirac</code>	Dirac $\delta$ -function
<code>heaviside</code>	Heaviside function

### Gamma Functions

MuPAD provides the following functions to represent the  $\beta$ -function,  $\Gamma$ -function, and other related special functions. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

<code>beta</code>	$\beta$ -function
<code>binomial</code>	Binomial expression $\binom{m}{n}$
<code>gamma</code>	$\Gamma$ -function

<code>igamma</code>	Incomplete $\Gamma$ -function
<code>lngamma</code>	Logarithmic $\Gamma$ -function
<code>psi</code>	Polygamma function

## Zeta Function and Polylogarithms

The following MuPAD functions represent the Riemann  $\zeta$ -function and the related dilogarithm and polylogarithm special functions. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

<code>dilog</code>	Dilogarithm function
<code>polylog</code>	Polylogarithm function
<code>zeta</code>	Riemann $\zeta$ -function

## Airy and Bessel Functions

The following MuPAD functions represent the Bessel and Airy special functions. You can use these functions as an input for your computations. MuPAD can return results in terms of these functions:

<code>airyAi</code>	Airy function Ai
<code>airyBi</code>	Airy function Bi
<code>besselI</code>	Modified Bessel function of the first kind
<code>besselJ</code>	Bessel function of the first kind
<code>besselK</code>	Modified Bessel function of the second kind
<code>besselY</code>	Bessel function of the second kind

## Exponential and Trigonometric Integrals

The following MuPAD functions represent exponential and trigonometric integrals. You can use these functions as an input for your computations. MuPAD can return results in terms of these functions:

Ci	Cosine integral
Chi	Hyperbolic cosine integral
Ei	Exponential integral
Si	Sine integral
Ssi	Shifted sine integral
Shi	Hyperbolic sine integral

## Error Functions and Fresnel Functions

The following MuPAD functions represent the error functions (integrals of Gaussian distribution) and Fresnel functions. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

erf	Error function
erfc	Complementary error function
erfi	Imaginary error function
inverf	Inverse of error function
inverfc	Inverse of complementary error function
fresnelC	Fresnel cosine integral function
fresnelS	Fresnel sine integral function

## Hypergeometric, Meijer G, and Whittaker Functions

The following MuPAD functions represent the hypergeometric function, the more general Meijer G function, and related functions. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

hypergeom	Hypergeometric function
kummerU	Confluent hypergeometric KummerU function
meijerG	Meijer G function
whittakerM	Whittaker's M function
whittakerW	Whittaker's W function

### Elliptic Integrals

The following MuPAD functions represent the elliptic integrals of different kinds. You can use these functions as input for your computations. MuPAD can return results in terms of these functions:

ellipticK	Complete elliptic integral of the first kind
ellipticCK	Complementary complete elliptic integral of the first kind
ellipticF	Incomplete elliptic integral of the first kind
ellipticE	Elliptic integral of the second kind
ellipticCE	Complementary complete elliptic integral of the second kind
ellipticPi	Elliptic integral of the third kind
ellipticCPi	Complementary complete elliptic integral of the third kind
ellipticNome	Elliptic nome

### Lambert W Function (omega Function)

The `lambertW` function represents the solutions of the equation  $ye^y = x$ . You can use the function as input for your computations. MuPAD can return results in terms of this function.

## Floating-Point Arguments and Function Sensitivity

In this section...
“Use Symbolic Computations When Possible” on page 3-244
“Increase Precision” on page 3-244
“Approximate Parameters and Approximate Results” on page 3-246
“Plot Special Functions” on page 3-247

Particular choices of parameters can reduce some special functions to simpler special functions, elementary functions, or numbers. Nevertheless, for most parameters MuPAD returns the symbolic notation of a special function. In such cases, you can approximate the value of a special function numerically. To approximate a special function numerically, use the `float` command or call the special function with floating-point arguments.

When approximating the value of a special function numerically, remember that floating-point results can be extremely sensitive to numeric precision. Also, floating-point results are prone to roundoff errors. The following approaches can help you recognize and avoid incorrect results:

- When possible, use symbolic computations. Switch to floating-point arithmetic only if you cannot obtain symbolic results. See “Use Symbolic Computations When Possible” on page 3-244.
- Numeric computations are sensitive to the `DIGITS` environment variable that determines the numeric working precision. Increase the precision of numeric computations, and check if the result changes significantly. See “Increase Precision” on page 3-244.
- Compute the value of a special function symbolically, and then approximate the result numerically. Also, compute the value of a special function using the floating-point parameters. Significant difference in these two results indicates that one or both approximations are incorrect. See “Approximate Parameters and Approximate Results” on page 3-246.
- Plot the function. See “Plot Special Functions” on page 3-247.

## Use Symbolic Computations When Possible

By default, MuPAD performs computations in exact symbolic form. For example, standard mathematical constants have their own symbolic representations in MuPAD. Using these representations, you can keep the exact value of the constant throughout your computations. You always can find a numeric approximation of a constant by using the `float` function:

```
pi := float(PI)
```

```
3.141592654
```

Avoid unnecessary conversions to floating-point numbers. A floating-point number approximates a constant; it is not the constant itself. Using this approximation, you can get incorrect results. For example, the `heaviside` special function returns different results for the sine of  $\pi$  and the sine of 10-digit floating-point approximation of  $\pi$ :

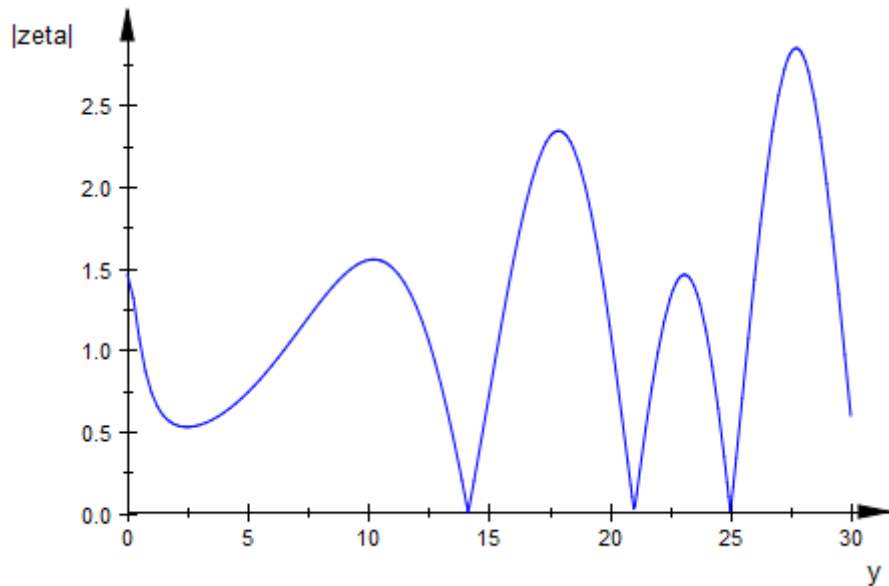
```
heaviside(sin(PI)), heaviside(sin(pi))
```

```
 $\frac{1}{2}$ , 0.0
```

## Increase Precision

The Riemann hypothesis states that all nontrivial zeros of the Riemann Zeta function  $\zeta(z)$  have the same real part  $\Re(z) = \frac{1}{2}$ . To locate possible zeros of the Zeta function, plot its absolute value  $\left| \zeta\left(\frac{1}{2} + iy\right) \right|$ . The following plot shows the first three nontrivial roots of the Zeta function  $\zeta\left(\frac{1}{2} + yi\right)$ :

```
plot(abs(zeta(1/2 + I*y)), y = 0..30,  
      AxesTitles = ["y", "|zeta|"])
```



Use the numeric solver to approximate the first three zeros of this Zeta function:

```
numeric::solve(zeta(1/2 + I*y), y = 13..15),
numeric::solve(zeta(1/2 + I*y), y = 20..22),
numeric::solve(zeta(1/2 + I*y), y = 24..26)
```

```
{14.13472514}, {21.02203964}, {25.01085758}
```

Now, consider the same function, but slightly increase the real part:

$\zeta\left(\frac{1000000001}{2000000000} + iy\right)$ . According to the Riemann hypothesis, this function does not have a zero for any real value  $y$ . By default, MuPAD uses 10 significant decimal digits for computations that involve floating-point numbers. When you use the `numeric::solve` solver with the default number of digits, the solver finds the following (nonexisting) zero of the Zeta function:

```
numeric::solve(zeta(1000000001/2000000000 + I*y), y = 14..15)
```

{14.13472514}

Increasing the numbers of digits shows that the result is incorrect. The Zeta function  $\zeta\left(\frac{1000000001}{2000000000} + iy\right)$  does not have a zero at  $14 < y < 15$ :

```
DIGITS:=15;
numeric::solve(zeta(1000000001/2000000000 + I*y), y = 14..15)
```

∅

```
delete DIGITS;
```

### Approximate Parameters and Approximate Results

Bessel functions with half integer indices return exact symbolic expressions. Approximating these expressions by floating-point numbers, you can get very unstable results. For example, the exact symbolic expression for the following Bessel function is:

```
B := besselJ(53/2, PI)
```

$$\left( 351 \sqrt{2} \left( \frac{119409675}{\pi^4} - \frac{20300}{\pi^2} - \frac{315241542000}{\pi^6} + \frac{445475704038750}{\pi^8} - \frac{366812794263762000}{\pi^{10}} \right. \right. \\ \left. \left. + \frac{182947881139051297500}{\pi^{12}} - \frac{55720697512636766610000}{\pi^{14}} + \frac{10174148683695239020903125}{\pi^{16}} \right. \right. \\ \left. \left. - \frac{1060253389142977540073062500}{\pi^{18}} + \frac{57306695683177936040949028125}{\pi^{20}} \right. \right. \\ \left. \left. - \frac{1331871030107060331702688875000}{\pi^{22}} + \frac{8490677816932509614604641578125}{\pi^{24}} + 1 \right) \right) / \pi^2$$

Use the float command to approximate this expression numerically:

```
float(B)
```

-2854.225191



Now, call the Bessel function with the floating-point parameter. Significant difference in these two approximations indicates that one or both results are incorrect:

```
besselJ(53/2, float(PI))
```

```
6.900145607 10-23
```

Increase the numeric working precision to obtain more accurate approximations:

```
DIGITS:= 45: float(B); besselJ(53/2, float(PI))
```

```
6.90014560772087465957327665452188601038351877 10-23
```

```
6.9001456069172794165785003948591349704494668 10-23
```

```
delete DIGITS;
```

Now you can see that using the floating-point parameter to compute the Bessel function produces the correct result (within working precision). Approximation of the exact symbolic expression for that Bessel function returns the wrong result because of numerical instability.

## Plot Special Functions

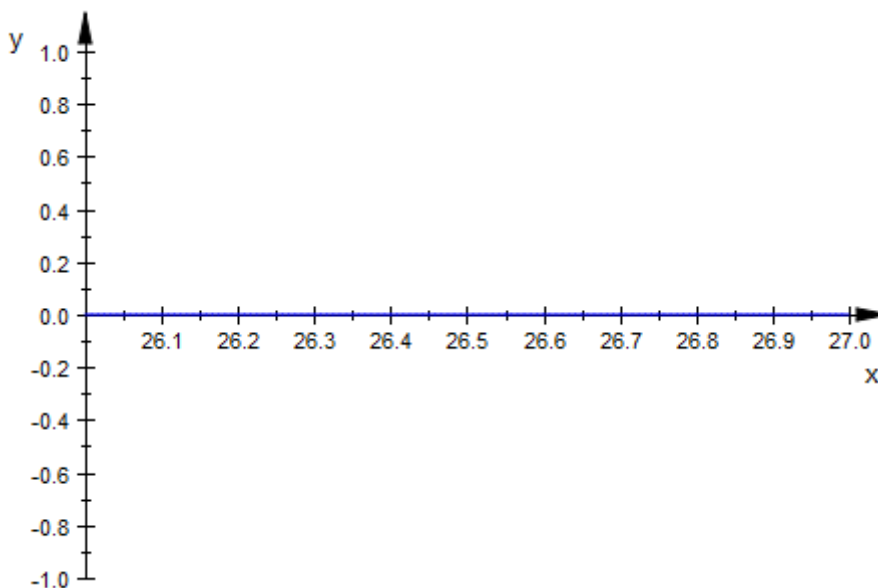
Plotting the function can help you recognize incorrect floating-point approximations. For example, the numeric approximation of the following Bessel function returns:

```
B := besselJ(53/2, PI):  
float(B)
```

```
-2854.225191
```

Plot the function  $J_x(\pi)$  for the values of  $x$  around  $53/2$ . The function plot shows that the floating-point approximation is incorrect:

```
plot(besselJ(x, PI), x = 26..27)
```



Sometimes, to see that the floating-point approximation is incorrect, you must zoom the particular parts of the function plot. For example, the numeric solver finds the unexpected zero of the Zeta function:

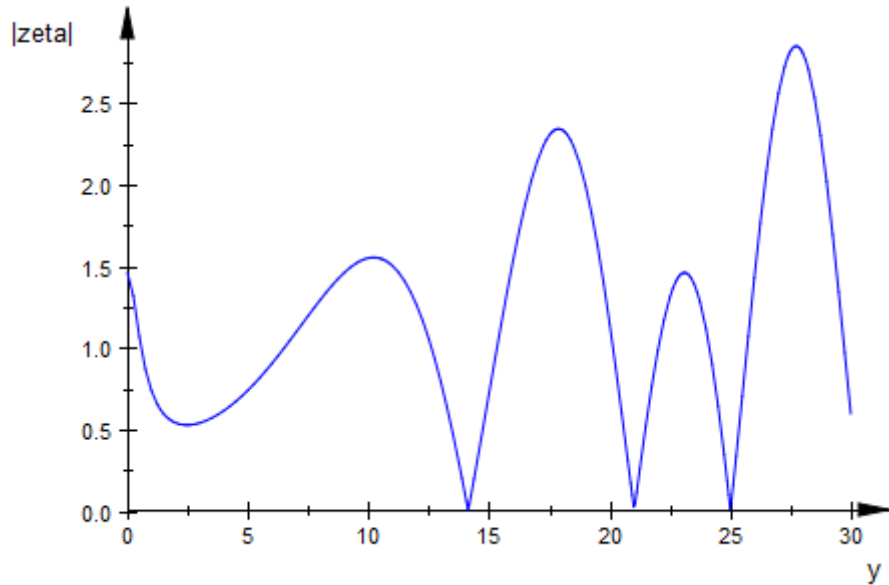
```
numeric::solve(zeta(1000000001/2000000000 + I*y), y = 14..15)
{14.13472514}
```

To investigate whether the Zeta function actually has a zero at that point or whether the result appears because of the roundoff error, plot the absolute

value of Zeta function  $\left| \zeta\left(\frac{1000000001}{2000000000} + iy\right) \right|$ .

```
plot(abs(zeta(1000000001/2000000000 + I*y)), y = 0..30,
```

```
AxesTitles = ["y", "|zeta|"])
```




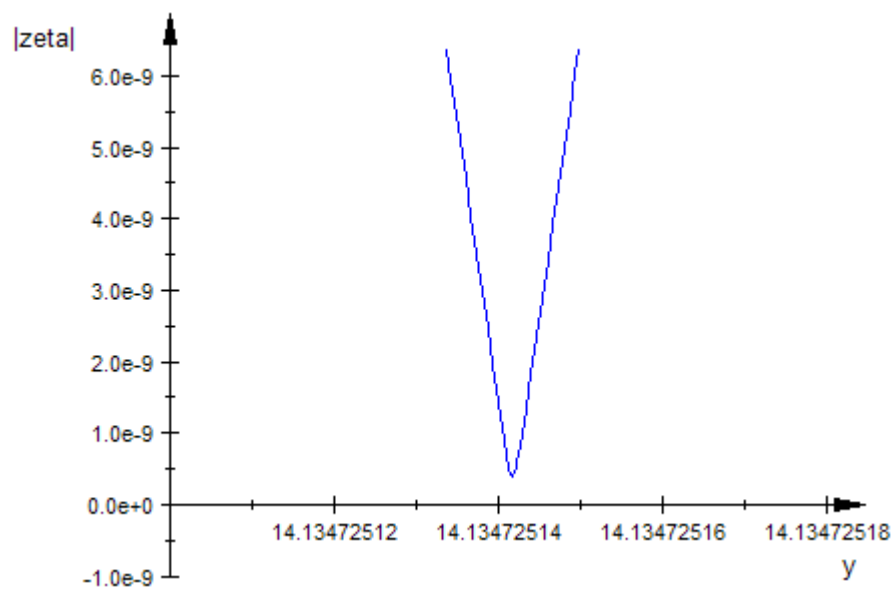
To see more details of the function plot near the possible zero, zoom the plot. To see that the numeric result is incorrect, enlarge that part of the function plot beyond the numeric working precision, and then reevaluate the plot. When you zoom and reevaluate, MuPAD recalculates the part of the function plot with the increased numeric precision.

---

**Note** When zooming, MuPAD does not automatically reevaluate the function plot.

---

To get accurate results after zooming the plot, use the Recalculate button . After zooming and reevaluating the plot, you can see that the function does not have a zero at that interval.



## Integral Transforms

### In this section...

“Fourier and Inverse Fourier Transforms” on page 3-251

“Laplace and Inverse Laplace Transforms” on page 3-254

### Fourier and Inverse Fourier Transforms

There are several commonly used conventions for defining Fourier transforms. MuPAD defines the Fourier transform (FT) as:

$$F(\omega) = c \int_{-\infty}^{\infty} f(t) e^{i s \omega t} dt$$

Here  $c$  and  $s$  are the parameters of the Fourier transform. By default,  $c = 1$  and  $s = -1$ . `Pref::fourierParameters` lets you specify other values for these parameters. For the inverse Fourier transform (IFT), MuPAD uses the following definition:

$$f(t) = \frac{|s|}{2 \pi c} \int_{-\infty}^{\infty} F(\omega) e^{-i s \omega t} d\omega$$

To compute the Fourier transform of an arithmetical expression, use the `fourier` function. For example, compute the Fourier transforms of the following exponential expression and the Dirac delta distribution:

```
fourier(exp(-t^2), t, w),
fourier(dirac(t), t, w)
```

$$\sqrt{\pi} e^{-\frac{w^2}{4}}, 1$$

If you know the Fourier transform of an expression, you can find the original expression or its mathematically equivalent form by computing the inverse

Fourier transform. To compute the inverse Fourier transform, use the `ifourier` function. For example, find the original exponential expression and the Dirac delta distribution:

```
ifourier(PI^(1/2)*exp(-w^2/4), w, t),
ifourier(1, w, t)
```

$$e^{-t^2}, \delta(t)$$

Suppose, you compute the Fourier transform of an expression, and then compute the inverse Fourier transform of the result. In this case, MuPAD can return an expression that is mathematically equivalent to the original one, but presented in a different form. For example, compute the Fourier transforms of the following trigonometric expressions:

```
Cosine := fourier(cos(t), t, w);
Sine := fourier(sin(t^2), t, w)
```

$$\pi (\delta(w - 1) + \delta(w + 1))$$

$$\frac{\sqrt{2} \sqrt{\pi} \left( \cos\left(\frac{w^2}{4}\right) - \sin\left(\frac{w^2}{4}\right) \right)}{2}$$

Now, compute the inverse Fourier transforms of the resulting expressions `Cosine` and `Sine`. The results differ from the original expressions:

```
invCosine := ifourier(Cosine, w, t);
invSine := ifourier(Sine, w, t)
```

$$\frac{e^{-ti}}{2} + \frac{e^{ti}}{2}$$

$$\frac{\sqrt{2} (\sqrt{2} \sqrt{\pi} (\cos(t^2) + \sin(t^2)) - \sqrt{2} \sqrt{\pi} (\cos(t^2) - \sin(t^2)))}{4 \sqrt{\pi}}$$

Simplifying the resulting expressions `invCosine` and `invSine` gives the original expressions:

```
simplify(invCosine), simplify(invSine)
```

```
cos(t), sin(t^2)
```

Besides arithmetical expressions, the `fourier` and `ifourier` functions also accept matrices of arithmetical expressions. For example, compute the Fourier transform of the following matrix:

```
A := matrix(2, 2, [exp(-t^2), t*exp(-t^2),
                  t^2*exp(-t^2), t^3*exp(-t^2)]):
fourier(A, t, w)
```

$$\begin{pmatrix} \sqrt{\pi} \sigma_1 & -\frac{\sqrt{\pi} w \sigma_1 i}{2} \\ \frac{\sqrt{\pi} \sigma_1}{2} - \frac{\sqrt{\pi} w^2 \sigma_1}{4} - \frac{3 \sqrt{\pi} w \sigma_1 i}{4} + \frac{\sqrt{\pi} w^3 \sigma_1 i}{8} & \end{pmatrix}$$

where

$$\sigma_1 = e^{-\frac{w^2}{4}}$$

The `fourier` and `ifourier` functions let you evaluate the transforms of an expression or a matrix at a particular point. For example, evaluate the Fourier transform of the matrix `A` for the values `w = 0` and `w = 2*x`:

```
fourier(A, t, 0);
fourier(A, t, 2*x)
```

$$\begin{pmatrix} \sqrt{\pi} & 0 \\ \frac{\sqrt{\pi}}{2} & 0 \end{pmatrix}$$

$$\left( \begin{array}{cc} \sqrt{\pi} e^{-x^2} & -\sqrt{\pi} x e^{-x^2} i \\ \frac{\sqrt{\pi} e^{-x^2}}{2} - \sqrt{\pi} x^2 e^{-x^2} & -\frac{3\sqrt{\pi} x e^{-x^2} i}{2} + \sqrt{\pi} x^3 e^{-x^2} i \end{array} \right)$$

If MuPAD cannot compute the Fourier transform of an expression, it returns an unresolved transform:

`fourier(f(t), t, w)`

`fourier(f(t), t, w)`

If MuPAD cannot compute the inverse Fourier transform of an expression, it returns the result in terms of an unresolved direct Fourier transform:

`ifourier(F(w), w, t)`

`$\frac{\text{fourier}(F(w), w, -t)}{2 \pi}$`

## Laplace and Inverse Laplace Transforms

The Laplace transform is defined as follows:

$$F(s) = \int_0^{\infty} f(t) e^{-s t} dt$$

The inverse Laplace transform is defined by a contour integral in the complex plane:

$$f(t) = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} F(s) e^{s t} ds$$



where  $c$  is a real value. To compute the Laplace transform of an arithmetical expression, use the `laplace` function. For example, compute the Laplace transform of the following expression:

```
tsine := laplace(t*sin(a*t), t, s)
```

$$\frac{2as}{(a^2 + s^2)^2}$$

To compute the original expression from its Laplace transform, perform the inverse Laplace transform. To compute the inverse Laplace transform, use the `ilaplace` function. For example, compute the inverse Laplace transform of the resulting expression `tsine`:

```
ilaplace(tsine, s, t)
```

$$t \sin(at)$$

Suppose, you compute the Laplace transform of an expression, and then compute the inverse Laplace transform of the result. In this case, MuPAD can return an expression that is mathematically equivalent to the original one, but presented in a different form. For example, compute the Laplace transforms of the following expression:

```
L := laplace(t*ln(t), t, s)
```

$$\frac{1}{s^2} - \frac{\text{EULER}}{s^2} - \frac{\ln(s)}{s^2}$$

Now, compute the inverse Laplace transform of the resulting expression `L`. The result differs from the original expression:

```
invL := ilaplace(L, s, t)
```

$$t - \text{EULER } t + t(\text{EULER} + \ln(t) - 1)$$

Simplifying the expression `invL` gives the original expression:

```
simplify(invL)
```

$$t \ln(t)$$

Besides arithmetical expressions, the `laplace` and `ilaplace` functions also accept matrices of arithmetical expressions. For example, compute the Laplace transform of the following matrix:

```
A := matrix(2, 2, [1, t, t^2, t^3]):
laplace(A, t, s)
```

$$\begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{6}{s^4} \end{pmatrix}$$

When computing a transform of an expression, you can use assumptions on mathematical properties of the arguments. For example, compute the Laplace transform of the Dirac delta distribution:

```
d := laplace(dirac(t - t_0), t, s) assuming t_0 >= 0
```

$$e^{-s t_0}$$

Restore the Dirac delta distribution from the resulting expression `d`:

```
ilaplace(d, s, t) assuming t_0 >= 0
```

$$\delta(t - t_0)$$

The `laplace` function provides the transforms for some special functions. For example, compute the Laplace transforms of the following Bessel functions:

```
laplace(besselJ(0, t), t, s);
laplace(besselJ(1, t), t, s);
laplace(besselJ(1/2, t), t, s)
```

$$\frac{1}{\sqrt{s^2 + 1}}$$

$$\frac{1}{\sqrt{s^2 + 1} (s + \sqrt{s^2 + 1})}$$

$$- \frac{\sqrt{2} \left( \frac{1}{\sqrt{s-i}} - \frac{1}{\sqrt{s+i}} \right) i}{2}$$

The laplace and ilaplace functions let you evaluate the transforms of an expression or a matrix at a particular point. For example, evaluate the Laplace transform of the following expression for the value  $s = 10$ :

```
laplace(t*exp(-t), t, 10)
```

$$\frac{1}{121}$$

Now, evaluate the inverse Laplace transform of the following expression for the value  $t = x + y$ :

```
ilaplace(1/(1 + s)^2, s, x + y)
```

$$e^{-x-y} (x+y)$$

If MuPAD cannot compute the Laplace transform or the inverse Laplace transform of an expression, it returns an unresolved transform:

```
laplace(f(t), t, s)
```

```
laplace(f(t), t, s)
```

```
ilaplace(F(s), s, t)
```

`ilaplace(F(s), s, t)`

## Z-Transforms

The Z-transform of the function  $F(z)$  is defined as follows:

$$F(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k}$$

If  $R$  is a positive number, such that the function  $F(Z)$  is analytic on and outside the circle  $|z| = R$ , then the inverse Z-transform is defined as follows:

$$f(k) = \frac{1}{2\pi i} \oint_{|z|=R} F(z)z^{k-1} dz, \quad k = 0, 1, 2, \dots$$

You can consider the Z-transform as a discrete equivalent of the Laplace transform.

To compute the Z-transform of an arithmetical expression, use the `ztrans` function. For example, compute the Z-transform of the following expression:

```
S := ztrans(sinh(n), n, z)
```

$$\frac{z \sinh(1)}{z^2 - 2 \cosh(1) z + 1}$$

If you know the Z-transform of an expression, you can find the original expression or a mathematically equivalent form by computing the inverse Z-transform. To compute the inverse Z-transform, use the `iztrans` function. For example, compute the inverse Z-transform of the expression `S`:

```
iztrans(S, z, n)
```

$$\sinh(n)$$

Suppose, you compute the Z-transform of an expression, and then compute the inverse Z-transform of the result. In this case, MuPAD can return

an expression that is mathematically equivalent to the original one, but presented in a different form. For example, compute the Z-transform of the following expression:

```
C := ztrans(exp(n), n, z)
```

$$\frac{z}{z-e}$$

Now, compute the inverse Z-transform of the resulting expression C. The result differs from the original expression:

```
invC := iztrans(C, z, n)
```

$$e \left( e^{-1} e^n - e^{-1} \delta_{n,0} \right) + \delta_{n,0}$$

Simplifying the resulting expression invC gives the original expression:

```
simplify(invC)
```

$$e^n$$

Besides arithmetical expressions, the ztrans and iztrans functions also accept matrices of arithmetical expressions. For example, compute the Z-transform of the following matrix:

```
A := matrix(2, 2, [1, n, n + 1, 2*n + 1]):
ZA := ztrans(A, n, z)
```

$$\begin{pmatrix} \frac{z}{z-1} & \frac{z}{(z-1)^2} \\ \frac{z}{z-1} + \frac{z}{(z-1)^2} & \frac{z}{z-1} + \frac{2z}{(z-1)^2} \end{pmatrix}$$

Computing the inverse Z-transform of ZA gives the original matrix A:

`iztrans(ZA, z, n)`

$$\begin{pmatrix} 1 & n \\ n+1 & 2n+1 \end{pmatrix}$$

The `ztrans` and `iztrans` functions let you evaluate the transforms of an expression or a matrix at a particular point. For example, evaluate the Z-transform of the following expression for the value  $z = 2$ :

`ztrans(1/n!, n, 2)`

$$\sqrt{e}$$

Evaluate the inverse Z-transform of the following expression for the value  $n = 10$ :

`iztrans(z/(z - exp(x)), z, 10)`

$$e^{10x}$$

If MuPAD cannot compute the Z-transform or the inverse Z-transform of an expression, it returns an unresolved transform:

`ztrans(f(n), n, z)`

$$ztrans(f(n), n, z)$$

`iztrans(F(z), z, n)`

$$iztrans(F(z), z, n)$$

## Discrete Fourier Transforms

The discrete Fourier transform (DFT) is an equivalent of the Fourier transform for discrete data. The one-dimensional discrete Fourier transform of  $N$  data elements  $L = [L_1, \dots, L_N]$  is defined as the list  $F = [F_1, \dots, F_N]$ , such that

$$F_k = \sum_{j=1}^N L_j e^{-2\pi i (j-1)(k-1)/N}, k = 1, \dots, N$$

The inverse discrete Fourier transform is defined as the list  $L$  of the following elements:

$$L_j = \frac{1}{N} \sum_{k=1}^N F_k e^{2\pi i (j-1)(k-1)/N}, j = 1, \dots, N$$

MuPAD uses a fast Fourier transform (FFT) algorithm to compute the discrete and the inverse discrete Fourier transforms. For any  $N$ , the computing costs are  $O(N \log_2(N))$ . To compute the discrete Fourier transforms, use the following functions:

- `numeric::fft` to compute the Fourier transform
- `numeric::invfft` to compute the inverse Fourier transform

These functions accept lists (domain type `DOM_LIST`), arrays (domain type `DOM_ARRAY`), hardware floating-point arrays (domain type `DOM_HFARRAY`), and matrices (category `Cat::Matrix`). The accepted data structures (except for lists) can be one- or multidimensional. You can use arbitrary arithmetical expressions as entries.

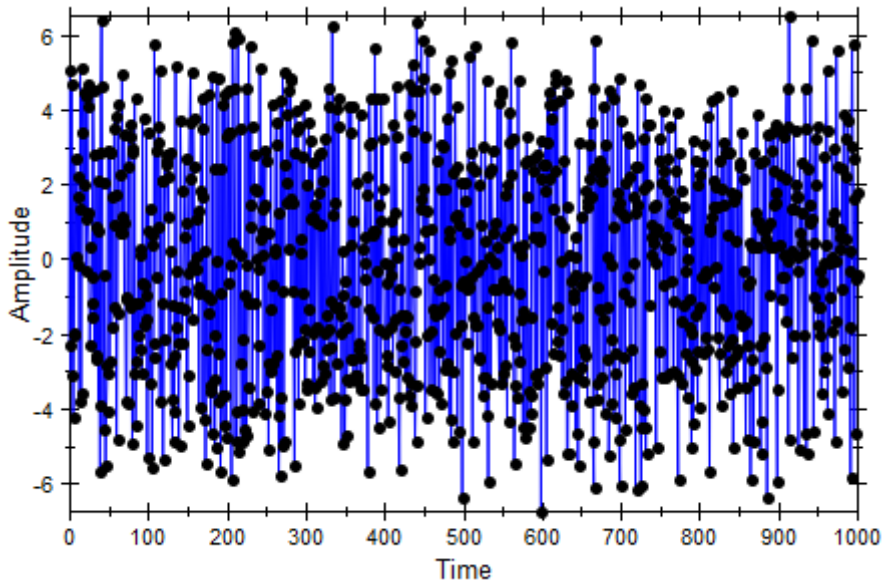
The discrete Fourier transform is often used in signal processing. It allows you to decompose a signal into a set of periodic signals with different frequencies and to analyze those frequencies. Suppose, you have a discrete set of values of a signal sampled at a fixed rate. The signal might be periodic, but the noise effectively hides the period. For example, the following data list represents such a signal:



```
f1 := 150: f2 := 300:
data := [sin(f1*2*PI*t/1000) + sin(f2*2*PI*t/1000)
         + 10*(frandom() - 1/2) $t = 0..1000]:
```

When you plot the data, the signal seems random. The noise effectively hides the two main frequencies of the signal:

```
plot(plot::Listplot(data, t = 0..1000),
     AxesTitles = ["Time", "Amplitude"],
     YAxisTitleOrientation = Vertical,
     XAxisTitleAlignment = Center, YAxisTitleAlignment = Center)
```

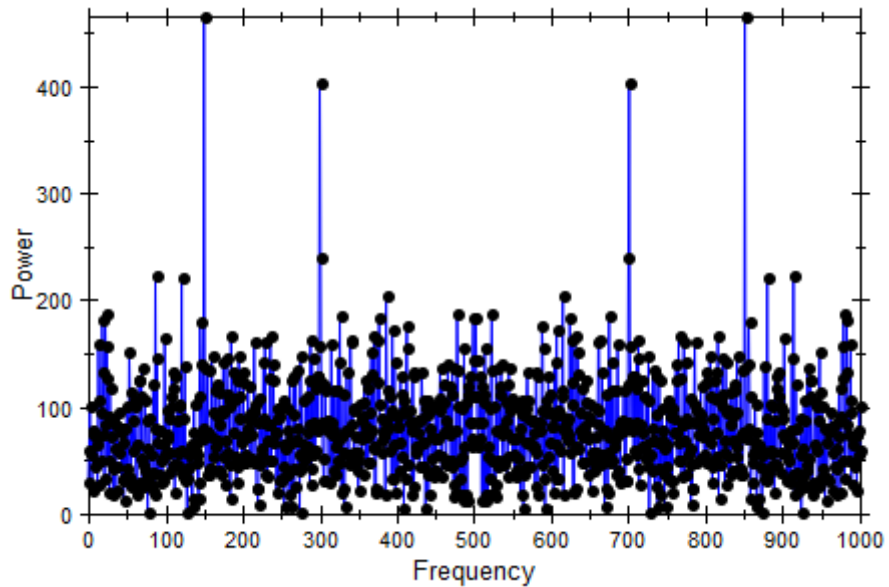


To prove that the signal has a strong periodic component and to find the main frequencies, compute the discrete Fourier transform of the signal:

```
fft := abs(numeric::fft(data)):
```

The plot of `fft` shows four peaks. The peaks correspond to the two main frequencies of the original signal ( $f_1 = 150$  and  $f_2 = 300$ ):

```
plot(plot::Listplot(fft, f = 0..1000),
      AxesTitles = ["Frequency", "Power"],
      YAxisTitleOrientation = Vertical,
      XAxisTitleAlignment = Center, YAxisTitleAlignment = Center)
```



The `numeric::fft` and `numeric::invfft` functions accept options. Use the `Symbolic` option to prevent the conversion of your data to floating-point numbers. For example, create a list of the following exact values:

```
exactData := [sin(1/3*2*PI*n/10) $n = 0..3]
```

$$\left[ 0, \sin\left(\frac{\pi}{15}\right), \sin\left(\frac{2\pi}{15}\right), \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4} \right]$$

Compute the discrete Fourier transform keeping the data in its exact symbolic form:

```
fft := numeric::fft(exactData, Symbolic)
```

$$\left[ \sin\left(\frac{\pi}{15}\right) + \sigma_1 + \sigma_3, -\sin\left(\frac{\pi}{15}\right) i - \sigma_1 + \sigma_2, \sigma_1 - \sin\left(\frac{\pi}{15}\right) - \sigma_3, \sin\left(\frac{\pi}{15}\right) i - \sigma_1 - \sigma_2 \right]$$

where

$$\sigma_1 = \sin\left(\frac{2\pi}{15}\right)$$

$$\sigma_2 = \frac{\sqrt{2} \sqrt{5 - \sqrt{5}} i}{4}$$

$$\sigma_3 = \frac{\sqrt{2} \sqrt{5 - \sqrt{5}}}{4}$$

Compute the inverse discrete Fourier transform of the resulting list. Use the `numeric::invfft` function with the `Symbolic` to prevent the data from the conversion to floating-point values:

```
numeric::invfft(fft, Symbolic)
```

$$\left[ 0, \sin\left(\frac{\pi}{15}\right), \sin\left(\frac{2\pi}{15}\right), \frac{\sqrt{2} \sqrt{5 - \sqrt{5}}}{4} \right]$$

Also, you can clean results by removing very small terms. To discard all entries of the result with absolute values smaller than  $10^{1/\text{DIGITS}}$  times the maximal absolute value of all operands of the result, use the `Clean` option. This option also removes tiny imaginary terms that often appear as a result of roundoff effects. For example, without the `Symbolic` option, the inverse Fourier transform from the previous example returns the following list of floating-point values:

```
numeric::invfft(fft)
```

$$\left[ 0.0, 0.2079116908 + 1.162988943 \cdot 10^{-17} i, 0.4067366431, 0.5877852523 - 1.162988943 \cdot 10^{-17} i \right]$$

When you use the `Clean` option, the `numeric::invfft` function discards small terms that appear in the second and fourth entries of the resulting list:

```
numeric::invfft(fft, Clean)
```

```
[0.0, 0.2079116908, 0.4067366431, 0.5877852523]
```

## Use Custom Patterns for Transforms

### In this section...

“Add New Patterns” on page 3-267

“Overwrite Existing Patterns” on page 3-269

### Add New Patterns

MuPAD provides common patterns for integral and Z-transform computations and the corresponding inverse transforms computations. Also, the system lets you add your own patterns for the transforms.

---

**Note** MuPAD does not save custom patterns permanently. To use a custom pattern, add it in the current MuPAD session.

---

The following example demonstrates how to add the pattern for the Fourier transform of the function  $f(t)$ . By default, there is no pattern for  $f(t)$ . The `fourier` function returns the unresolved transform:

```
fourier(f(t), t, s)
```

```
fourier(f(t), t, s)
```

Suppose, you want to add the pattern  $F(s)$  for the Fourier transform of the function  $f(t)$ . To add a new pattern for the Fourier transform, use the `fourier::addpattern` function:

```
fourier::addpattern(f(t), t, s, F(s)):
```

Now, when you compute the Fourier transform of  $f(t)$ , MuPAD returns  $F(s)$ :

```
fourier(f(t), t, s)
```

```
F(s)
```

MuPAD can use the new pattern indirectly:

```
fourier(sin(t^2) + f(10*t + 33), t, s)
```

$$\frac{F\left(\frac{s}{10}\right) e^{\frac{33 s i}{10}}}{10} + \frac{\sqrt{2} \sqrt{\pi} \left(\cos\left(\frac{s^2}{4}\right) - \sin\left(\frac{s^2}{4}\right)\right)}{2}$$

When you add a pattern for the Fourier transform, MuPAD does not automatically add the pattern for the inverse Fourier transform:

```
ifourier(F(s), s, t)
```

$$\frac{\text{fourier}(F(s), s, -t)}{2 \pi}$$

To add the corresponding pattern for the inverse Fourier transform, use the `ifourier::addpattern` function. For example, add the pattern  $f(t)$  for the inverse Fourier transform of  $F(s)$ :

```
ifourier::addpattern(F(s), s, t, f(t)):
ifourier(F(s), s, t)
```

$f(t)$

Using the same method, you can add your own patterns for the Laplace transform, inverse Laplace transform, Z-transform, and inverse Z-transform. Use the following functions to add patterns for computing these transforms:

- `laplace::addpattern` to add a pattern for computing the Laplace transform
- `ilaplace::addpattern` to add a pattern for computing the inverse Laplace transform
- `ztrans::addpattern` to add a pattern for computing the Z-transform
- `iztrans::addpattern` to add a pattern for computing the inverse Z-transform

## Overwrite Existing Patterns

You can introduce a new transform pattern for the expression for which MuPAD already has a pattern. In this case, the system replaces the standard existing pattern with the new one. For example, the Laplace transform of the hyperbolic sine function has a standard pattern implemented in MuPAD:

```
laplace(sinh(t), t, s)
```

$$\frac{1}{s^2 - 1}$$

Suppose, you want to change this pattern. Use the `laplace::addpattern` function to replace the existing standard pattern with your custom pattern:

```
laplace::addpattern(sinh(t), t, s, 1/2*(1/(s - 1) - 1/(s + 1))):  
laplace(sinh(t), t, s)
```

$$\frac{1}{2(s-1)} - \frac{1}{2(s+1)}$$

This change is temporary, it only affects the current MuPAD session.

## Supported Distributions

MuPAD supports standard continuous and discrete distributions. The system associates the following routines with each implemented distribution:

- A probability density function (PDF) for continuous distributions or probability function (PF) for discrete distributions
- A cumulative distribution function (CDF)
- An inverse cumulative distribution function (quantile function)
- A random number generator

The following continuous distributions are available in MuPAD.

Name	PDF	CDF	Quantile	Random Generator
$\beta$ -distribution	stats::betaPDF	stats::betaCDF	stats::betaQuantile	stats::betaRandom
Cauchy distribution	stats::cauchyPDF	stats::cauchyCDF	stats::cauchyQuantile	stats::cauchyRandom
$\chi^2$ -distribution	stats::chisquarePDF	stats::chisquareCDF	stats::chisquareQuantile	stats::chisquareRandom
Erlang distribution	stats::erlangPDF	stats::erlangCDF	stats::erlangQuantile	stats::erlangRandom
Exponential distribution	stats::exponentialPDF	stats::exponentialCDF	stats::exponentialQuantile	stats::exponentialRandom
F-distribution	stats::fPDF	stats::fCDF	stats::fQuantile	stats::fRandom
$\gamma$ -distribution	stats::gammaPDF	stats::gammaCDF	stats::gammaQuantile	stats::gammaRandom
Logistic distribution	stats::logisticPDF	stats::logisticCDF	stats::logisticQuantile	stats::logisticRandom
Lognormal distribution	stats::lognormalPDF	stats::lognormalCDF	stats::lognormalQuantile	stats::lognormalRandom
Normal distribution	stats::normalPDF	stats::normalCDF	stats::normalQuantile	stats::normalRandom
Student's t-distribution	stats::tPDF	stats::tCDF	stats::tQuantile	stats::tRandom



<b>Name</b>	<b>PDF</b>	<b>CDF</b>	<b>Quantile</b>	<b>Random Generator</b>
Uniform distribution	stats::uniformPDF	stats::uniformCDF	stats::uniformQuantile	stats::uniformRandom
Weibull distribution	stats::weibullPDF	stats::weibullCDF	stats::weibullQuantile	stats::weibullRandom

The following discrete distributions are available in MuPAD.

<b>Name</b>	<b>PF</b>	<b>CDF</b>	<b>Quantile</b>	<b>Random Generator</b>
Binomial distribution	stats::binomialPDF	stats::binomialCDF	stats::binomialQuantile	stats::binomialRandom
Empirical distribution	stats::empiricalPDF	stats::empiricalCDF	stats::empiricalQuantile	stats::empiricalRandom
Distribution of a Finite Sample Space	stats::finitePDF	stats::finiteCDF	stats::finiteQuantile	stats::finiteRandom
Geometric Distribution	stats::geometricPDF	stats::geometricCDF	stats::geometricQuantile	stats::geometricRandom
Hypergeometric Distribution	stats::hypergeometricPDF	stats::hypergeometricCDF	stats::hypergeometricQuantile	stats::hypergeometricRandom
Poisson Distribution	stats::poissonPDF	stats::poissonCDF	stats::poissonQuantile	stats::poissonRandom

## Import Data

If you have an external data set and want to analyze it in MuPAD, import the data to the MuPAD session. To import an ASCII data file to the MuPAD session, use the `import::readdata` function. Suppose, you want to analyze the world population growth and compare it to the US population growth between 1970 and 2000. The text file "WorldPopulation" contains the required data. To be able to work with the data in MuPAD, import the contents of the file line-by-line by using the `import::readdata` function. The function returns the following nested list:

```
data := import::readdata("WorldPopulation")
```

```
[[year, world(thousands), US(thousands), AnnualRateWorld, AnnualRateUS],
 [1970, 3711962, 205052, 2.07, 1.26], [1971, 3789539, 207661, 1.99, 1.07],
 [1972, 3865804, 209896, 1.94, 0.95], [1973, 3941551, 211909, 1.87, 0.91],
 [1974, 4016056, 213854, 1.79, 0.99], [1975, 4088612, 215973, 1.73, 0.95],
 [1976, 4159763, 218035, 1.71, 1.01], [1977, 4231510, 220239, 1.68, 1.1],
 [1978, 4303134, 222585, 1.71, 1.18], [1979, 4377497, 225055, 1.7, 0.98],
 [1980, 4452548, 227726, 1.7, 0.96], [1981, 4528882, 229966, 1.75, 0.91],
 [1982, 4608682, 232188, 1.75, 0.87], [1983, 4690278, 234307, 1.7, 0.89],
 [1984, 4770468, 236348, 1.7, 0.91], [1985, 4852052, 238466, 1.71, 0.89],
 [1986, 4935874, 240651, 1.73, 0.91], [1987, 5022023, 242804, 1.71, 0.94],
 [1988, 5108860, 245021, 1.69, 1.12], [1989, 5195713, 247342, 1.68, 1.33],
 [1990, 5283687, 250132, 1.57, 1.33], [1991, 5367185, 253493, 1.56, 1.3],
 [1992, 5451672, 256894, 1.5, 1.21], [1993, 5534138, 260255, 1.46, 1.18],
 [1994, 5615311, 263436, 1.44, 1.16], [1995, 5696677, 266557, 1.4, 1.2],
 [1996, 5776857, 269667, 1.35, 1.17], [1997, 5855087, 272912, 1.31, 1.15],
 [1998, 5932091, 276115, 1.28, 1.02], [1999, 6008255, 279295, 1.25, 1.01],
 [2000, 6083550, 282172, 1.24, 0.94]]
```

You can convert the resulting nested list to other data structures, For example, represent the imported data as a *sample*. A sample is a collection of statistical data in the form of a matrix. To convert the nested list of imported data to a sample, use the `stats::sample` function:

```
s := stats::sample(data)
```

year	world(thousands)	US(thousands)	AnnualRateWorld	AnnualRateUS
1970	3711962	205052	2.07	1.26
1971	3789539	207661	1.99	1.07
1972	3865804	209896	1.94	0.95
1973	3941551	211909	1.87	0.91
1974	4016056	213854	1.79	0.99
1975	4088612	215973	1.73	0.95
1976	4159763	218035	1.71	1.01
1977	4231510	220239	1.68	1.1
1978	4303134	222585	1.71	1.18
1979	4377497	225055	1.7	0.98
1980	4452548	227726	1.7	0.96
1981	4528882	229966	1.75	0.91
1982	4608682	232188	1.75	0.87
1983	4690278	234307	1.7	0.89
1984	4770468	236348	1.7	0.91
1985	4852052	238466	1.71	0.89
1986	4935874	240651	1.73	0.91
1987	5022023	242804	1.71	0.94
1988	5108860	245021	1.69	1.12
1989	5195713	247342	1.68	1.33
1990	5283687	250132	1.57	1.33
1991	5367185	253493	1.56	1.3
1992	5451672	256894	1.5	1.21
1993	5534138	260255	1.46	1.18
1994	5615311	263436	1.44	1.16
1995	5696677	266557	1.4	1.2
1996	5776857	269667	1.35	1.17
1997	5855087	272912	1.31	1.15
1998	5932091	276115	1.28	1.02
1999	6008255	279295	1.25	1.01
2000	6083550	282172	1.24	0.94

The first row in that sample contains text. The statistical functions cannot work with the text. Before you start analyzing the data, delete the first row:

```
s := stats::sample::delRow(s, 1)
```

1970	3711962	205052	2.07	1.26
1971	3789539	207661	1.99	1.07
1972	3865804	209896	1.94	0.95
1973	3941551	211909	1.87	0.91
1974	4016056	213854	1.79	0.99
1975	4088612	215973	1.73	0.95
1976	4159763	218035	1.71	1.01
1977	4231510	220239	1.68	1.1
1978	4303134	222585	1.71	1.18
1979	4377497	225055	1.7	0.98
1980	4452548	227726	1.7	0.96
1981	4528882	229966	1.75	0.91
1982	4608682	232188	1.75	0.87
1983	4690278	234307	1.7	0.89
1984	4770468	236348	1.7	0.91
1985	4852052	238466	1.71	0.89
1986	4935874	240651	1.73	0.91
1987	5022023	242804	1.71	0.94
1988	5108860	245021	1.69	1.12
1989	5195713	247342	1.68	1.33
1990	5283687	250132	1.57	1.33
1991	5367185	253493	1.56	1.3
1992	5451672	256894	1.5	1.21
1993	5534138	260255	1.46	1.18
1994	5615311	263436	1.44	1.16
1995	5696677	266557	1.4	1.2
1996	5776857	269667	1.35	1.17
1997	5855087	272912	1.31	1.15
1998	5932091	276115	1.28	1.02
1999	6008255	279295	1.25	1.01
2000	6083550	282172	1.24	0.94

The MuPAD statistical functions accept the resulting sample because it contains only numeric data. Now, you can analyze the sample. For example, compute the correlation between the US population and total world population stored in the second and third columns of the sample. Use the `float` function to approximate the result:

```
float(stats::correlation(s, 2, 3))
```

0.9972426119

The correlation coefficient is close to 1. Therefore, the world population data and the US population data are linearly related. Now, compute the correlation coefficient for the population growth rates stored in the fourth and fifth columns of the sample. In this case, you can omit the `float` function. MuPAD returns a floating-point result because the input data contains floating-point numbers:

```
stats::correlation(s, 4, 5)
```

-0.2127844699

The correlation coefficient indicates that the data for the world population growth rates and the data for the US population growth rates are not linearly related.

## Store Statistical Data

MuPAD offers various data containers, such as lists, arrays, tables, and so on, to store and organize data. For details about the MuPAD data structures, see *Working with Data Structures*. Although, you can use any of these data containers to store statistical data, the following containers serve best. The reason is that many functions of the “Statistics” library accept these data containers as input parameters:

- Lists
- Statistical samples (`stats::sample`). This data structure is specifically designed for statistical data.

Using sets, tables, arrays, vectors, and matrices is less convenient. You cannot use these data containers as input parameters for most functions of the “Statistics” library. To use these functions on data stored in sets, tables, arrays, vectors, or matrices, transfer the data to lists or statistical samples.

## Compute Measures of Central Tendency

Measures of central tendency locate a distribution of data along an appropriate scale. There are several standard measures of central tendency. Knowing the properties of a particular data sample (such as the origin of the data sample and possible outliers and their values) can help you choose the most useful measure of central tendency for that data sample. MuPAD provides the following functions for calculating the measures of central tendency:

- The `stats::modal` function returns the most frequent value of a data sample and the number of occurrences of that value.
- The `stats::mean` function calculates the arithmetic mean

$$\frac{1}{n} \left( \sum_{i=1}^n x_i \right) \text{ of a data sample } x_1, x_2, \dots, x_n.$$

- The `stats::quadraticMean` function calculates the quadratic mean

$$\sqrt{\frac{1}{n} \left( \sum_{i=1}^n x_i^2 \right)} \text{ of a data sample } x_1, x_2, \dots, x_n.$$

- The `stats::median` function returns the element  $x_n$  of a sorted data sample  $x_1, x_2, \dots, x_{2n}$ .

- The `stats::geometricMean` function calculates the geometric mean

$$\left( x_1 x_2 \dots x_n \right)^{1/n} \text{ of a data sample } x_1, x_2, \dots, x_n.$$

- The `stats::harmonicMean` function calculates the harmonic mean

$$\frac{1}{\frac{1}{n} \left( \sum_{i=1}^n \frac{1}{x_i} \right)} \text{ of a data sample } x_1, x_2, \dots, x_n.$$

The arithmetic average is a simple and popular measure of central tendency. It serves best for data samples that do not have significant outliers. Unfortunately, outliers (for example, data-entry errors or glitches) exist in almost all real data. The arithmetic average and quadratic mean are sensitive to these problems. One bad data value can move the average away from the center of the rest of the data by an arbitrarily large distance. For example, create the following two lists of entries that contain only one outlier. The outlier is equal to 100 in the first list and to 1 in the second list:

```
L := [1, 1, 1, 1, 1, 100.0]:  
S := [100, 100, 100, 100, 100, 1.0]:
```

The `stats::modal` function shows that the most frequent entry of the first list is 1. The most frequent entry of the second list is 100. A most frequent entry appears in each list five times:

```
modalL = stats::modal(L);  
modalS = stats::modal(S)
```

```
modalL = ([1], 5)
```

```
modalS = ([100], 5)
```

If the value of the outlier is large, the outlier can significantly move the mean and the quadratic mean away from the center:

```
meanL = stats::mean(L);  
quadraticMeanL = stats::quadraticMean(L)
```

```
meanL = 17.5
```

```
quadraticMeanL = 40.83503398
```

Large outliers affect the geometric mean and the harmonic mean less than they affect the simple arithmetic average. Nevertheless, both geometric and harmonic means are also not completely resistant to outliers:

```
geometricMeanL = stats::geometricMean(L);  
harmonicMeanL = stats::harmonicMean(L)
```

```
geometricMeanL = 2.15443469
```

```
harmonicMeanL = 1.19760479
```



If the value of the outlier is small, the impact on the mean of a data set is less noticeable. Quadratic mean can effectively mitigate the impact of a few small outliers:

```
meanS = stats::mean(S);  
quadraticMeanS = stats::quadraticMean(S)
```

```
meanS = 83.5
```

```
quadraticMeanS = 91.28800578
```

The small outlier significantly affects the geometric and harmonic means computed for the list S:

```
geometricMeanS = stats::geometricMean(S);  
harmonicMeanS = stats::harmonicMean(S)
```

```
geometricMeanS = 46.41588834
```

```
harmonicMeanS = 5.714285714
```

The median is usually resistant to both large and small outliers:

```
medianL = stats::median(L);  
medianS = stats::median(S)
```

```
medianL = 1
```

```
medianS = 100
```

For data samples that contain an even number of elements, MuPAD can use two definitions of the median. By default, `stats::median` returns the  $n/2$ -th element of a sorted data sample:

```
z := [1, 1, 1, 100, 100, 100]:  
medianZ = stats::median(z)
```

```
medianZ = 1
```

When you use the `Averaged` option, `stats::median` returns the arithmetic average of the two central elements of a sorted data sample:

```
z := [1, 1, 1, 100, 100, 100]:  
medianZ = stats::median(z, Averaged)
```

```
medianZ =  $\frac{101}{2}$ 
```

Nevertheless, the median is not always the best choice for measuring central tendency of a data sample. For example, the following data sample distribution has a step in the middle:

```
z := [1, 1, 1, 2, 100, 100, 100]:  
medianZ = stats::median(z)
```

```
medianZ = 2
```

## Compute Measures of Dispersion

The measures of dispersion summarize how spread out (or scattered) the data values are on the number line. MuPAD provides the following functions for calculating the measures of dispersion. These functions describe the deviation from the arithmetic average (mean) of a data sample:

- The `stats::variance` function calculates the variance

$$\frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^2 \right), \text{ where } \bar{x} \text{ is the arithmetic mean of the data sample } x_1, x_2, \dots, x_n.$$

- The `stats::stdev` function calculates the standard deviation

$$\sqrt{\frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^2 \right)}, \text{ where } \bar{x} \text{ is the arithmetic average of the data sample } x_1, x_2, \dots, x_n.$$

- The `stats::meandev` function calculates the mean deviation

$$\frac{1}{n} \left( \sum_{i=1}^n |x_i - \bar{x}| \right), \text{ where } \bar{x} \text{ is the arithmetic average of the data sample } x_1, x_2, \dots, x_n.$$

The standard deviation and the variance are popular measures of dispersion. The standard deviation is the square root of the variance and has the desirable property of being in the same units as the data. For example, if the data is in meters, the standard deviation is also in meters. Both the standard deviation and the variance are sensitive to outliers. A data value that is separate from the body of the data can increase the value of the statistics by an arbitrarily large amount. For example, compute the variance and the standard deviation of the list `x` that contains one outlier:

```
L := [1, 1, 1, 1, 1, 1, 1, 1, 1, 100.0]:
variance = stats::variance(L);
stdev = stats::stdev(L)
```

```
variance = 1089.0
```

```
stdev = 33.0
```

The mean deviation is also sensitive to outliers. Nevertheless, the large outlier in the list  $x$  affects the mean deviation less than it affects the variance and the standard deviation:

```
meandev = stats::meandev(L)
```

```
meandev = 19.55555556
```

Now, compute the variance, the standard deviation, and the mean deviation of the list  $y$  that contains one small outlier. Again, the mean deviation is less sensitive to the outlier than the other two measures:

```
S := [100, 100, 100, 100, 100, 100, 100, 100, 1.0]:  
variance = stats::variance(S);  
stdev = stats::stdev(S);  
meandev = stats::meandev(S)
```

```
variance = 1089.0
```

```
stdev = 33.0
```

```
meandev = 19.55555556
```

## Compute Measures of Shape

The measures of shape indicate the symmetry and flatness of the distribution of a data sample. MuPAD provides the following functions for calculating the measures of shape:

- The `stats::moment` function that calculates the k-th moment

$$\frac{1}{n} \left( \sum_{i=1}^n (x_i - X)^k \right)$$

- The `stats::obliquity` function that calculates the obliquity (skewness)

$$\frac{\frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^3 \right)}{\left( \frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^2 \right) \right)^{3/2}},$$

where  $\bar{x}$  is the arithmetic average (mean) of the data sample  $x_1, x_2, \dots, x_n$ .

- The `stats::kurtosis` function that calculates the kurtosis (excess)

$$\frac{\frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^4 \right)}{\left( \frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})^2 \right) \right)^2} - 3,$$

where  $\bar{x}$  is the arithmetic average (mean) of the data sample  $x_1, x_2, \dots, x_n$ .

The `stats::moment` function enables you to compute the kth moment of a data sample centered around an arbitrary value  $X$ . One of the popular measures in descriptive statistics is a central moment. The kth central moment of a data sample is the kth moment centered around the arithmetic average (mean) of that data sample. The following statements are valid for any data sample:

- The zero central moment is always 1.
- The first central moment is always 0.
- The second central moment is equal to the variance computed by using a divisor  $n$ , rather than  $n - 1$  (available via the `Population` option of `stats::variance`).

For example, create the lists `L` and `S` as follows:

```
L := [1, 1, 1, 1, 1, 1, 1, 1, 1, 100.0]:  
S := [100, 100, 100, 100, 100, 100, 100, 100, 1.0]:
```

Calculate the arithmetic average of each list by using the `stats::mean` function:

```
meanL := stats::mean(L);  
meanS := stats::mean(S)
```

```
12.0
```

```
89.0
```

Calculate the first four central moments of the list L:

```
stats::moment(0, meanL, L),  
stats::moment(1, meanL, L),  
stats::moment(2, meanL, L),  
stats::moment(3, meanL, L)
```

```
1.0, 0.0, 968.0, 74536.0
```

The zero and first central moments are the same for any data sample. The second central moment is the variance computed with the divisor n:

```
stats::variance(L, Population)
```

```
968.0
```

Now, calculate the first four central moments of the list S:

```
stats::moment(0, meanS, S),  
stats::moment(1, meanS, S),  
stats::moment(2, meanS, S),  
stats::moment(3, meanS, S)
```

```
1.0, 0.0, 968.0, -74536.0
```

Again, the zero central moment is 1, the first central moment is 0, and the second central moment is the variance computed with the divisor  $n$ :

```
stats::variance(S, Population)
```

```
968.0
```

The obliquity (skewness) is a measure of the symmetry of a distribution. If the distribution is close to symmetrical around its mean, the value of obliquity is close to zero. Positive values of obliquity indicate that the distribution function has a longer tail to the right of the mean. Negative values indicate that the distribution function has a longer tail to the left of the mean. For example, calculate the obliquity of the lists L and S:

```
stats::obliquity(L);  
stats::obliquity(S)
```

```
2.474873734
```

```
-2.474873734
```

The kurtosis measures the flatness of a distribution. For normally distributed data, the kurtosis is zero. Negative kurtosis indicates that the distribution function has a flatter top than the normal distribution. Positive kurtosis indicates that the peak of the distribution function is sharper than it is for the normal distribution:

```
stats::kurtosis(-2, -1, -0.5, 0, 0.5, 1, 2),  
stats::kurtosis(-1, 0.5, 0, 0, 0, 0, 0.5, 1)
```

```
-0.8333333333, 0.1606648199
```

## Compute Covariance and Correlation

If you have two or more data samples with an equal number of elements, you can estimate how similar these data samples are. The most common measures of similarity of two data samples are the covariance and the correlation. MuPAD provides the following functions for computing the covariance and the correlation of two data samples:

- The `stats::covariance` function calculates the covariance

$\frac{1}{n} \left( \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right)$ . Here  $\bar{x}$  is the arithmetic average of the data sample  $x_1, x_2, \dots, x_n$ , and  $\bar{y}$  is the arithmetic average of the data sample  $y_1, y_2, \dots, y_n$ .

- The `stats::correlation` function calculates the linear (Bravais-Pearson) correlation coefficient

$\frac{\sum_i (x_i(y_i - \bar{y}) - \bar{x}(y_i - \bar{y}))}{\sqrt{\left( \sum_i (x_i - \bar{x})^2 \right) \left( \sum_i (y_i - \bar{y})^2 \right)}}$ . Here  $\bar{x}$  is the arithmetic average of the data sample  $x_1, x_2, \dots, x_n$ , and  $\bar{y}$  is the arithmetic average of the data sample  $y_1, y_2, \dots, y_n$ .

Create the lists `x` and `y`:

```
x := [1, 1, 0.1]:
y := [1, 2, 0.1]:
```

To estimate the similarity of these lists, compute their covariance. For completely uncorrelated (nonsimilar) data, the covariance is a small value. A positive covariance indicates that the data change in the same direction (increases or decreases together). A negative covariance indicates the data change in opposite directions. There are two common definitions of the covariance. By default, the `stats::covariance` function uses the definition with the divisor  $n - 1$ . To switch to the alternative definition, use the `Population` option:

```
stats::covariance(x, y),
stats::covariance(x, y, Population)
```



0.42, 0.28

The covariance of a data sample with itself is the variance of that data sample:

```
stats::covariance(x, x) = stats::variance(x)
```

0.27 = 0.27

The correlation of data samples indicates the degree of similarity of these data samples. For completely uncorrelated data, the value of the correlation (as well as the covariance) tends to 0. For correlated data that change in the same direction, the correlation tends to 1. For correlated data that change in the opposite directions, the correlation tends to -1. Compute the correlation of x and y:

```
stats::correlation(x, y),  
stats::correlation(x, x),  
stats::correlation(x, -x)
```

0.8504394349, 1.0, -1.0

## Handle Outliers

The outliers are data points located far outside the range of the majority of the data. Glitches, data-entry errors, and inaccurate measurements can produce outliers in real data samples. The outliers can significantly affect the analysis of data samples. If you suspect that the data that you want to analyze contains outliers, you can discard the outliers or replace them with the values typical for that data sample.

Before you discard or replace the outliers, try to verify that they are actual errors. The outliers can be a part of the correct data sample, and discarding them can lead you to incorrect conclusions. If you cannot determine whether the outliers are correct data or errors, the recommended strategy is to analyze the data with the outliers and without them.

To discard outliers, use the `stats::cutoff` function. For example, discard the outliers with the values smaller than the 1/10 quantile (10th percentile) and larger than 9/10 quantile of the list `x`:

```
x := [1/100, 1, 2, 3, 4, 5, 6, 7, 8, 9, 100]:  
stats::cutoff(x, 1/10)
```

```
[1, 2, 3, 4, 5, 6, 7, 8, 9]
```

To replace the outliers with the value of a *k*-th quantile, use the `stats::winsorize` function. In the list `x`, replace the values of the outliers with the 10th and 90th percentiles:

```
stats::winsorize(x, 1/10)
```

```
[1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 9]
```

## Bin Data

The `stats::frequency` function categorizes the numerical data into a number of bins given by semiopen intervals  $(a_i, b_i]$ . This function returns a table with the entries (rows) corresponding to the bins. Each entry shows the following information:

- The number of the bin
- The interval defining the bin
- The number of data in the bin
- The data contained in the bin

The `stats::frequency` function enables you to specify the number of bins. By default, `stats::frequency` categorizes the data into 10 bins. For example, categorize the following data into 10 bins:

```
x := [-10.1, -1, 1.1, 3.5, 13, 0, -5.5, 0.5,
      7.9, 15, 0.15, 6.7, 2, 9]:
stats::frequency(x)
```

1	$[[-\infty, -7.59], 1, [-10.1]]$
2	$[[ -7.59, -5.08], 1, [-5.5]]$
3	$[[ -5.08, -2.57], 0, []]$
4	$[[ -2.57, -0.06], 1, [-1]]$
5	$[[ -0.06, 2.45], 5, [0, 0.15, 0.5, 1.1, 2]]$
6	$[[2.45, 4.96], 1, [3.5]]$
7	$[[4.96, 7.47], 1, [6.7]]$
8	$[[7.47, 9.98], 2, [7.9, 9]]$
9	$[[9.98, 12.49], 0, []]$
10	$[[12.49, 15.0], 2, [13, 15]]$

Now, categorize the same data into 5 bins:

```
stats::frequency(x, 5)
```

```
1 | [[-∞, -5.08], 2, [-10.1, -5.5]]
2 | [[-5.08, -0.06], 1, [-1]]
3 | [[-0.06, 4.96], 6, [0, 0.15, 0.5, 1.1, 2, 3.5]]
4 | [[4.96, 9.98], 3, [6.7, 7.9, 9]]
5 | [[9.98, 15.0], 2, [13, 15]]
```

When creating the bins, you can specify the intervals. For example, divide the data into two bins: one bin contains the numbers that are less or equal to zero, and the other bin contains the numbers that are greater than zero:

```
stats::frequency(x, [[-infinity, 0], [0, infinity]])
```

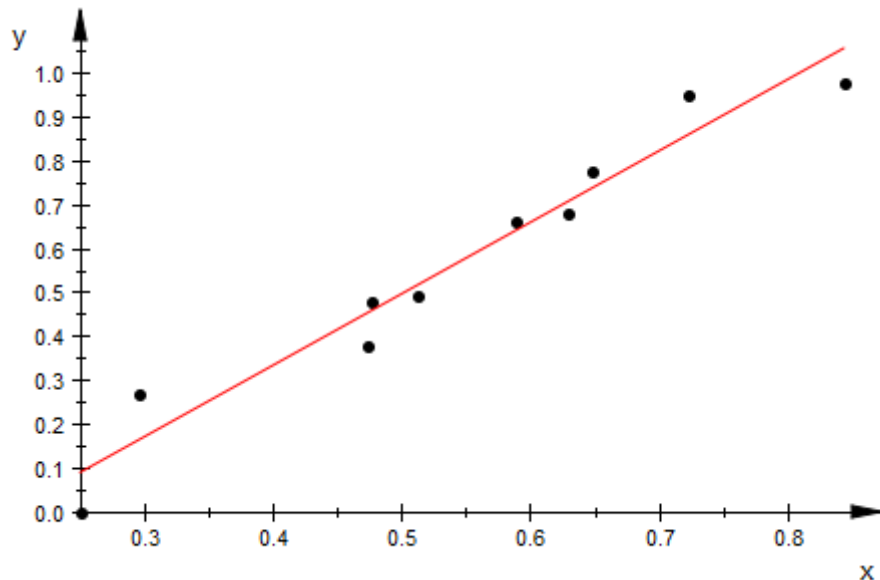
```
1 | [[-∞, 0], 4, [-10.1, -5.5, -1, 0]]
2 | [[0, ∞], 10, [0.15, 0.5, 1.1, 2, 3.5, 6.7, 7.9, 9, 13, 15]]
```

For graphical interpretation of the data binning, see “Create Bar Charts, Histograms, and Pie Charts” on page 3-295.

## Create Scatter and List Plots

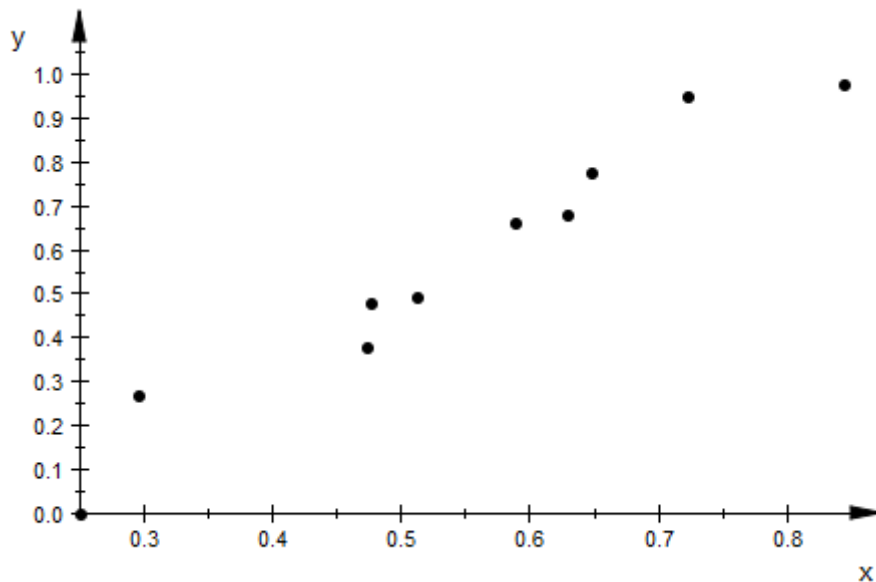
Scatter plots can help you identify the relationship between two data samples. A scatter plot is a simple plot of one variable against another. For two discrete data samples  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$ , a scatter plot is a collection of points with coordinates  $[x_1, y_1], [x_2, y_2], \dots, [x_n, y_n]$ . To create a scatter plot in MuPAD, use the `plot::Scatterplot` function. For example, create the scatter plot for the following data samples  $x$  and  $y$ :

```
x := [0.25, 0.295, 0.473, 0.476, 0.512,
      0.588, 0.629, 0.648, 0.722, 0.844]:
y := [0.00102, 0.271, 0.378, 0.478, 0.495,
      0.663, 0.68, 0.778, 0.948, 0.975]:
plot(plot::Scatterplot(x, y))
```



By default, the `plot::Scatterplot` function also displays a regression line. This line shows the linear dependency that best fits the two data samples. To hide the regression line, use the `LinesVisible` option:

```
plot(plot::Scatterplot(x, y, LinesVisible = FALSE))
```

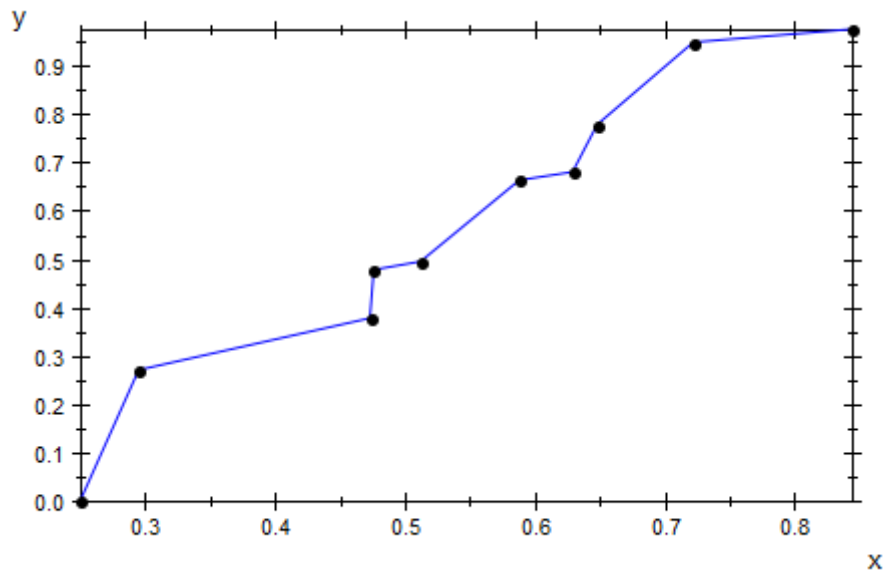


Another plot that can help you identify the relationship between two discrete data samples is a list plot. List plots are convenient for plotting one data sample with equidistant x-values. They are also convenient for plotting combined data samples, such as  $[[x_1, y_1], [x_2, y_2], \dots, [x_n, y_n]]$ . If you have two separate data samples, you can combine the data of these samples pairwise:

```
xy := [[x[i], y[i]] $ i = 1..10]:
```

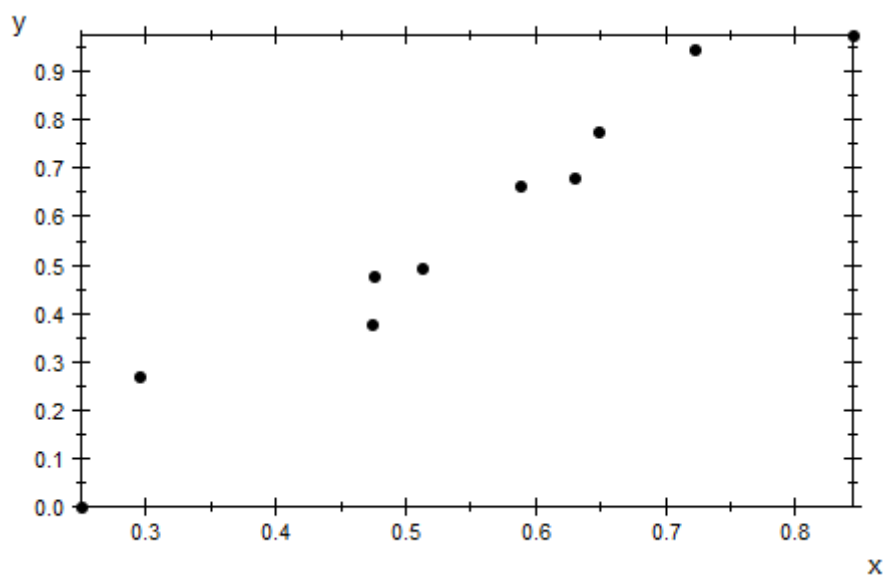
To create a list plot, use the `plot::Listplot` function:

```
plot(plot::Listplot(xy), AxesTitles = ["x", "y"])
```



By default, the `plot::Listplot` function connects adjacent points on the plot by straight lines. To hide these connections, use the `LinesVisible` option:

```
plot(plot::Listplot(xy),  
      AxesTitles = ["x", "y"],  
      LinesVisible = FALSE)
```





## Create Bar Charts, Histograms, and Pie Charts

### In this section...

“Bar Charts” on page 3-295

“Histograms” on page 3-297

“Pie Charts” on page 3-298

Bar charts, histograms, and pie charts help you compare different data samples, categorize data, and see the distribution of data values across a sample. These types of plots are very useful for communicating results of data analysis. Bar charts, histograms, and pie charts can help your audience understand your ideas, results, and conclusions quickly and clearly.

### Bar Charts

To compare different data samples or to show how individual elements contribute to an aggregate amount, use bar charts. A bar chart represents each element of a data sample as one bar. Bars are distributed along the horizontal or vertical axis, with each data element at a different location. To compare data samples, create a bar chart for two or more data samples. In this case, MuPAD accesses elements with the same index and plots the bars for these elements next to each other. For example, create three lists of random numbers:

```
x := [frandom() $ i = 1..10];
y := [frandom() $ i = 1..10];
z := [frandom() $ i = 1..10]
```

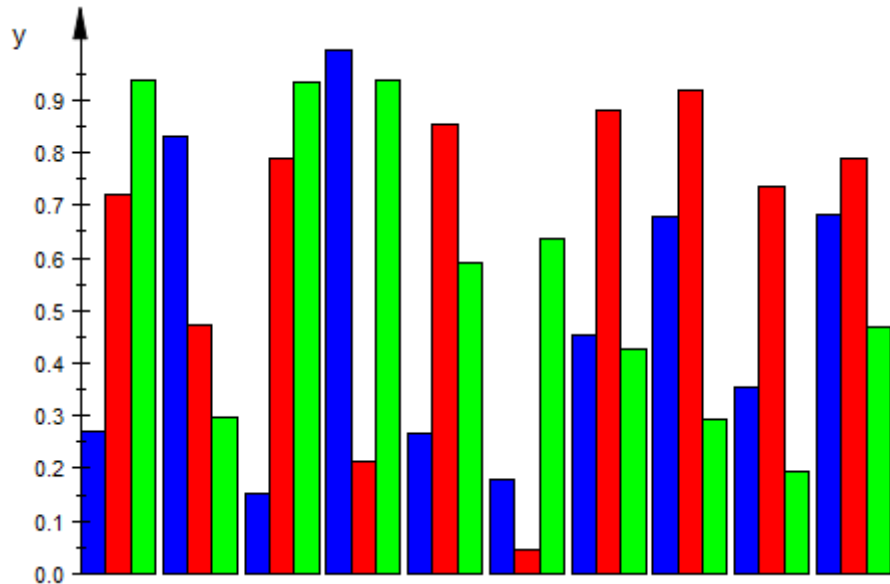
```
[0.2703581656, 0.8310371787, 0.153156516, 0.9948127808, 0.2662729021, 0.1801642277,
0.452083055, 0.6787819563, 0.3549849261, 0.6818588132]
```

```
[0.7219186551, 0.4738297742, 0.7889814922, 0.2115258358, 0.8556871754, 0.04489739417,
0.8791601269, 0.9193848479, 0.7350574234, 0.7875450269]
```

```
[0.9371484273, 0.2953238727, 0.9334772314, 0.9362730734, 0.5910800883, 0.6358075032,
0.4285065377, 0.2939293408, 0.1940618534, 0.4678382754]
```

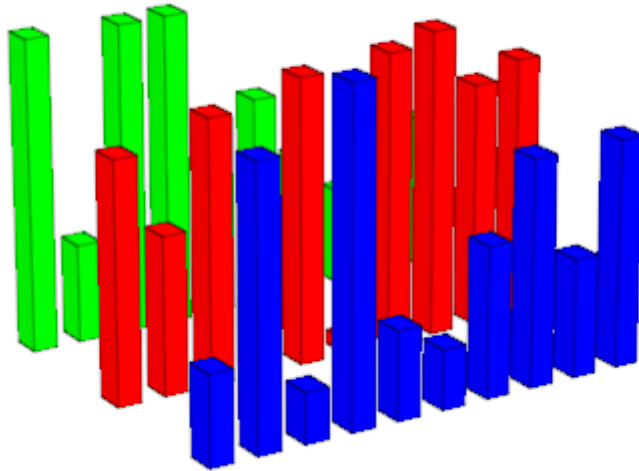
To create a 2-D bar chart, use the `plot::Bars2d` function. The chart displays data from the data samples `x`, `y`, and `z`. The resulting plot shows the elements with the same index clustered together. Small gaps separate each group of elements from the previous and the next group:

```
plot(plot::Bars2d(x, y, z))
```



To create a 3-D bar chart, use the `plot::Bars3d` function. This function accepts matrices and arrays. The function also accepts nested lists with flat inner lists. The `plot::Bars3d` function draws each element as a separate 3-D block. The elements of each row of an array or a matrix (or the elements of each flat list) appear along one horizontal axis. Bars that represent elements in the first column of an array or a matrix appear along the other horizontal axis. If you use a nested list, the elements of the inner lists with the same indices appear along the other horizontal axis. By default, the `plot::Bars3d` function does not display gaps between the groups of elements. Use the `Gap` option to create gaps and specify their size:

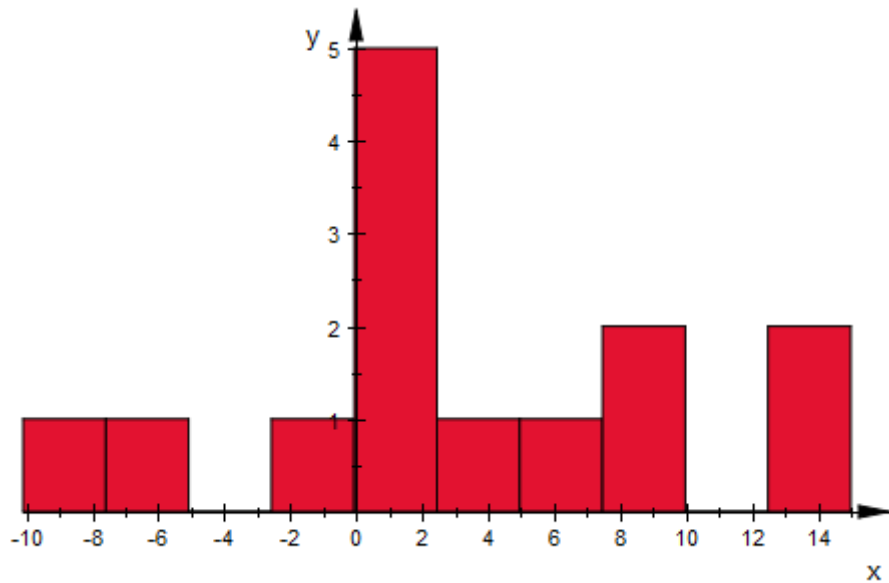
```
plot(plot::Bars3d([x, y, z], Gap = [0.5, 0.8]))
```



## Histograms

Histograms show the distribution of data values across a data range. They divide the data range into a certain number of intervals (bins), tabulate the number of values that fall into each bin, and plot these numbers using bars of varying height. To create a histogram, use the `plot::Histogram2d` function. By default, this function divides the data range into seven bins. To specify the number of bins, use the `Cells` option. For example, create the histogram of the following data sample categorizing the data into 10 bins:

```
data := [-10.1, -1, 1.1, 3.5, 13, 0, -5.5, 0.5,  
         7.9, 15, 0.15, 6.7, 2, 9]:  
plot(plot::Histogram2d(data, Cells = 10))
```



## Pie Charts

Pie charts can help you effectively communicate a portion (or percentage) that each element of a data sample contributes to the total number of all elements. To create a 2-D pie chart, use the `plot::Piechart2d` function. To create a 3-D pie chart, use the `plot::Piechart3d` function. A 3-D pie chart does not show any additional information. The 3-D view simply adds depth to the presentation by plotting the chart on top of a cylindrical base and lets you rotate the plot.

Suppose, you need to analyze the following list of numbers:

```
data := [-10.1, -1, 1.1, 3.5, 13, 0, -5.5, 0.5, 7.9,
         15, 0.15, 6.7, 2, 9]:
```

First, use the `stats::frequency` function to categorize the data into bins. (See Data Binning for more details.)

```
T := stats::frequency(data)
```

```

1 | [[-∞, -7.59], 1, [-10.1]]
2 | [[-7.59, -5.08], 1, [-5.5]]
3 | [[-5.08, -2.57], 0, []]
4 | [[-2.57, -0.06], 1, [-1]]
5 | [[-0.06, 2.45], 5, [0, 0.15, 0.5, 1.1, 2]]
6 | [[2.45, 4.96], 1, [3.5]]
7 | [[4.96, 7.47], 1, [6.7]]
8 | [[7.47, 9.98], 2, [7.9, 9]]
9 | [[9.98, 12.49], 0, []]
10| [[12.49, 15.0], 2, [13, 15]]

```

The result is a table that shows the intervals (bins), number of elements in those bins, and the data elements in each bin. The `plot::Piechart2d` and `plot::Piechart3d` functions do not accept tables as arguments. They accept lists, vectors, and arrays with one row or one column. Before creating a pie chart, extract the bins and the number of elements in them into two separate tables:

```

Counts := map(T, op, 2);
Bins := map(T, op, 1)

```

```

1 | 1
2 | 1
3 | 0
4 | 1
5 | 5
6 | 1
7 | 1
8 | 2
9 | 0
10| 2

```

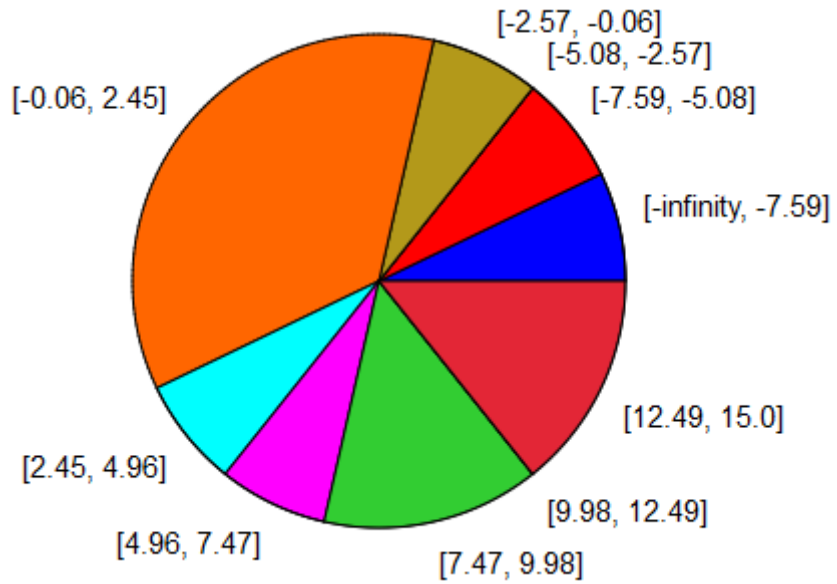
1	$[-\infty, -7.59]$
2	$[-7.59, -5.08]$
3	$[-5.08, -2.57]$
4	$[-2.57, -0.06]$
5	$[-0.06, 2.45]$
6	$[2.45, 4.96]$
7	$[4.96, 7.47]$
8	$[7.47, 9.98]$
9	$[9.98, 12.49]$
10	$[12.49, 15.0]$

Now, extract the entries from the Bins and Counts tables and create the lists containing these entries:

```
slices := [Counts[i] $ i = 1..10]:  
titles := [expr2text(Bins[i]) $ i = 1..10]:
```

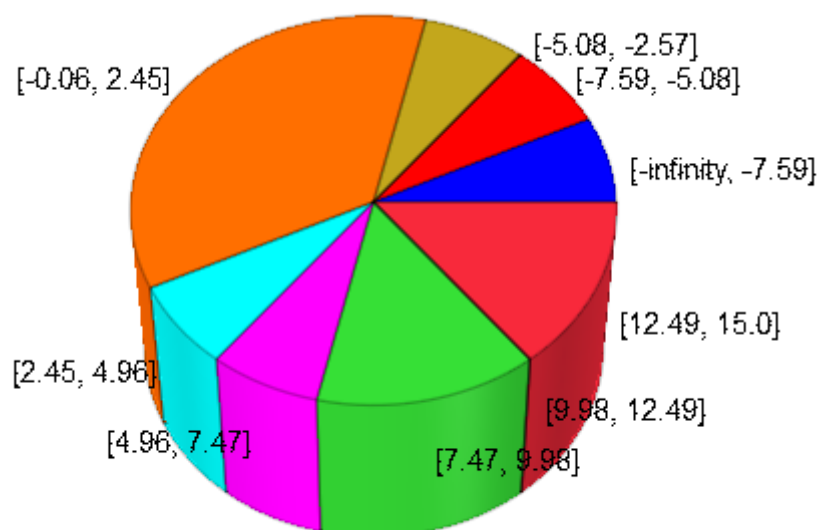
Create a 2-D pie chart by using the `plot::Piechart2d` function. The `slices` list specifies the portions that each bin contributes to the total number of all elements of the data sample. The `titles` list specifies the titles for each piece on the pie chart:

```
plot(plot::Piechart2d(slices, Titles = titles))
```



Create a 3-D pie chart from the same data by using the `plot::Piechart3d` function. To rotate the resulting 3-D chart, click any place on the chart, hold the mouse button and move the cursor:

```
plot(plot::Piechart3d(slices, Titles = titles, Radius = 0.3))
```





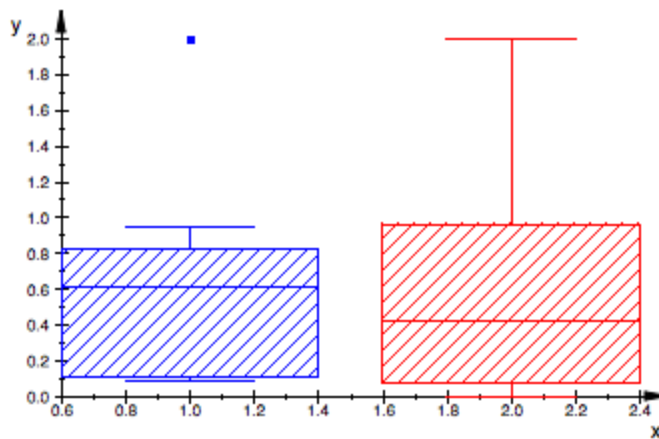
## Create Box Plots

Box plots reduce data samples to a number of descriptive parameters. Box plots are very useful for a quick overview and comparison of discrete data samples. To create a box plot, use the `plot::Boxplot` function. For example, create a box plot for the data samples `data1` and `data2` that contain random floating-point numbers from the interval  $[0.0, 1.0)$  and the value 2 (the outlier):

```
data1 := [frandom() $ i = 1..10]:
data1 := append(data1, 2);
data2 := [frandom() $ i = 1..10]:
data2 := append(data2, 2);
p := plot::Boxplot(data1, data2):
plot(p)
```

```
[0.9505346935, 0.7671696761, 0.3879781556, 0.6800823524, 0.09165554125, 0.6105917364,
0.6165219703, 0.08539849696, 0.8255921764, 0.1102019865, 2]
```

```
[0.9646295764, 0.9636463557, 0.6686584004, 0.07436099118, 0.4239603781, 0.3171767607,
0.1045730269, 0.004271391317, 0.04522333971, 0.8062151715, 2]
```

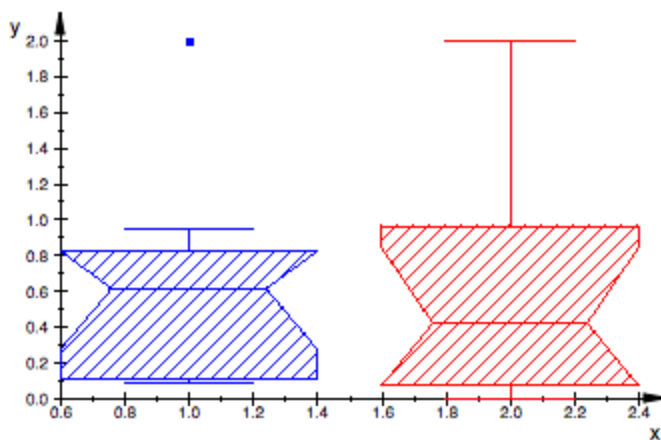


This plot demonstrate the following features:

- The tops and bottoms of each *box* are the 25th and 75th percentiles of the data samples, respectively. The distances between the tops and bottoms are the interquartile ranges.
- The line in the middle of each box is the sample median. A median is not always in the center of the box. The median shows the sample obliquity (skewness) of the sample distribution.
- The lines extending above and below each box are the whiskers. Whiskers extend from the ends of the interquartile ranges to the furthest observations within the maximum whisker length. The maximum whisker length is  $3/2$  of the height of the central box measured from the top or bottom of the box.
- The data points that lay beyond the whisker lengths are the outliers. Outliers are values that are more than  $3/2$  times the interquartile range away from the top or bottom of the box.

The Notched option enables you to create a box plot with notches. Notches display the variability of the median between samples:

```
p := plot::Boxplot(data1, data2, Notched = TRUE):  
plot(p)
```

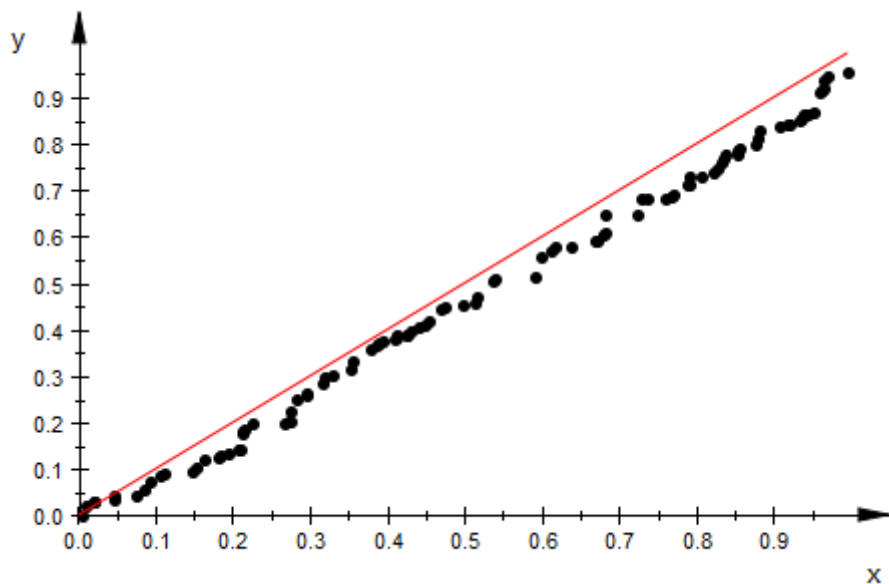


## Create Quantile-Quantile Plots

Quantile-quantile plots help you determine whether two samples come from the same distribution family. Quantile-quantile plots are scatter plots of quantiles computed from each sample together with a reference line along the diagonal of the plot. If the data forms the line, it is reasonable to assume that the two samples come from the same distribution family. If the data falls near the reference line, you also can assume that the two samples have the same mean and the same variance.

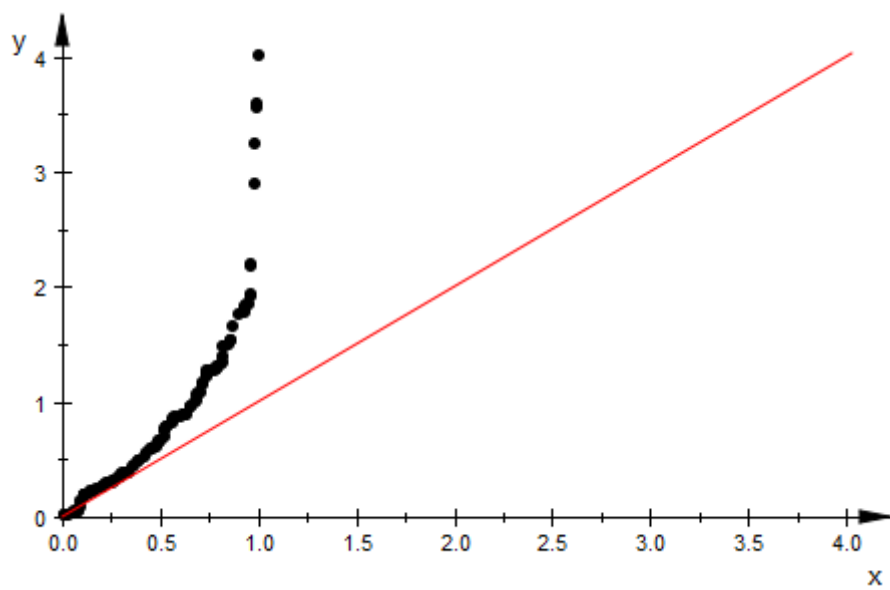
To create a quantile-quantile plot, use the `plot::QQplot` function. For example, create the data samples `data1` and `data2` that contain random floating-point numbers from the interval  $[0.0, 1.0)$ . Use the `frandom` function to create the `data1` sample. Use the `stats::uniformRandom` function to create the `data2` sample. Both functions produce uniformly distributed numbers. The quantile-quantile plot of these two data samples confirms that the samples come from the same distribution family. The plot is close to the line with a slope of 1:

```
data1 := [frandom() $ i = 1..100]:  
data2 := [stats::uniformRandom(0, 1)() $ k = 1..100]:  
p := plot::QQplot(data1, data2):  
plot(p)
```



The following quantile-quantile plot clearly shows that these two data samples come from different distribution families:

```
data1 := [stats::uniformRandom(0, 1)() $ k = 1..100]:  
data2 := [stats::exponentialRandom(0, 1)() $ k = 1..100]:  
p := plot::QQplot(data1, data2):  
plot(p)
```



## Univariate Linear Regression

Regression is the process of fitting models to data. Linear regression assumes that the relationship between the dependent variable  $y_i$  and the independent variable  $x_i$  is linear:  $y_i = a + bx_i$ . Here  $a$  is the offset and  $b$  is the slope of the linear relationship.

For linear regression of a data sample with one independent variable, MuPAD provides the `stats::linReg` function. This function uses the least-squares approach for computing the linear regression. `stats::linReg` chooses the parameters  $a$  and  $b$  by minimizing the quadratic error:

$$\chi^2 = \sum_i |y_i - a - bx_i|^2$$

The function also can perform weighted least-squares linear regression that minimizes

$$\chi^2 = \sum_i w_i |y_i - a - bx_i|^2$$

with the positive weight  $w_i$ . By default, the weights are equal to 1.

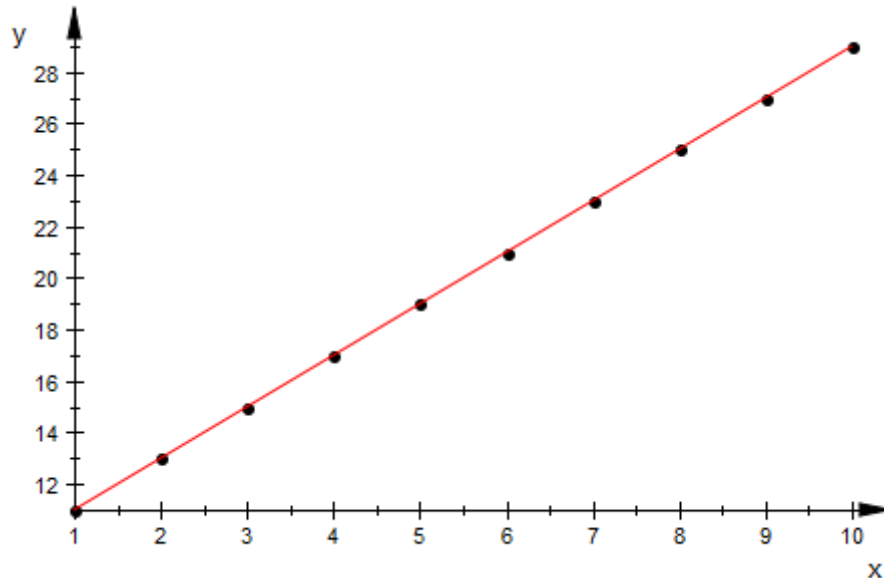
Besides the slope  $a$  and the offset  $b$  of a fitted linear model, `stats::linReg` also returns the value of the quadratic deviation  $\chi^2$ . For example, fit the linear model to the following data:

```
x := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]:
y := [11, 13, 15, 17, 19, 21, 23, 25, 27, 29]:
stats::linReg(x, y)
```

```
[[9, 2], 0]
```

The linear model  $y_i = 9 + 2x_i$  fits this data perfectly. The quadratic error for this model is zero. To visualize the data and the resulting model, plot the data by using the `plot::Scatterplot` function. The plot shows the regression line  $y_i = 9 + 2x_i$  computed by `stats::linReg`:

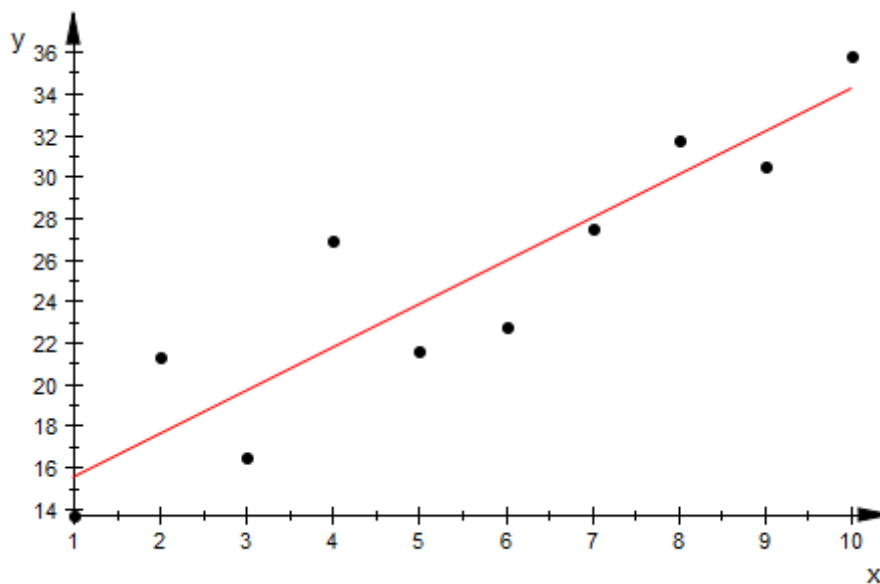
```
plot(plot::Scatterplot(x, y))
```



When you work with experimental data samples, the data almost never completely fits any linear model. The value of the quadratic error indicates how far the actual data deviate from the fitted model. For example, modify the data from the previous example by adding small random floating-point values to the entries of the list `y`. Then, perform linear regression for the entries of the lists `x` and `y1` and plot the data:

```
y1 := y + [10*frandom() $ i = 1..10]:
stats::linReg(x, y1);
plot(plot::Scatterplot(x, y1))
```

```
[[13.43518739, 2.077876934], 76.58789632]
```

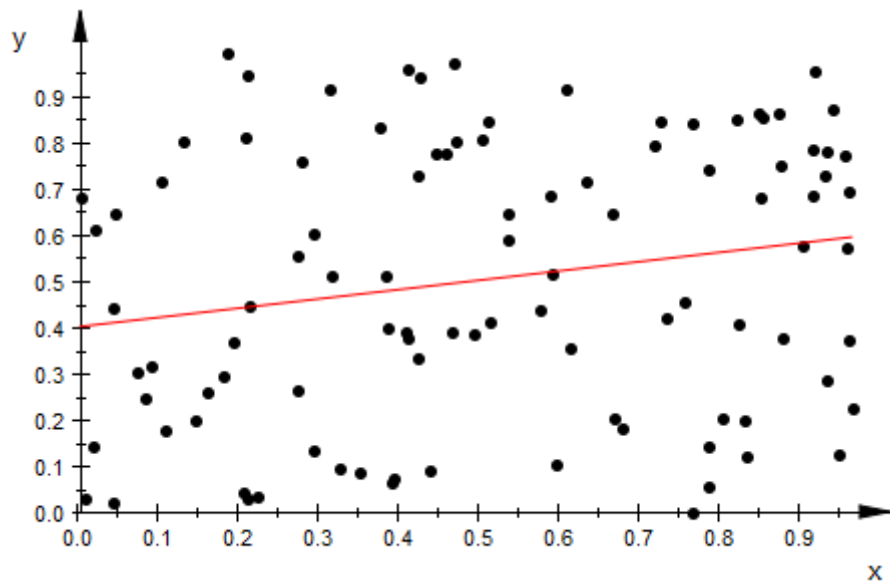


The fact that `stats::linReg` finds a linear model to fit your data does not guarantee that the linear model is a good fit. For example, you can find a linear model to fit the following uniformly distributed random data points:

```
x := [frandom() $ i = 1..100]:  
y := [frandom() $ i = 1..100]:  
stats::linReg(x, y);  
plot(plot::Scatterplot(x, y))
```

```
[[0.4005452605, 0.2002242944], 8.272613418]
```





The large value of the quadratic error indicates that the linear model is a poor fit for these data.

delete x, y, y1

## Univariate Nonlinear Regression

Nonlinear regression can assume any type of relationship between the dependent variable  $y$  and independent variables  $x_j$ . For nonlinear regression, MuPAD provides the `stats::reg` function. This function uses the least-squares approach for computing the regression. `stats::reg` chooses the parameters  $p_1, \dots, p_n$  by trying to minimize the quadratic error:

$$\chi^2(p_1, \dots, p_n) = \sum_{i=1}^k |y_i - f(x_{i1}, \dots, x_{im}, p_1, \dots, p_n)|^2$$

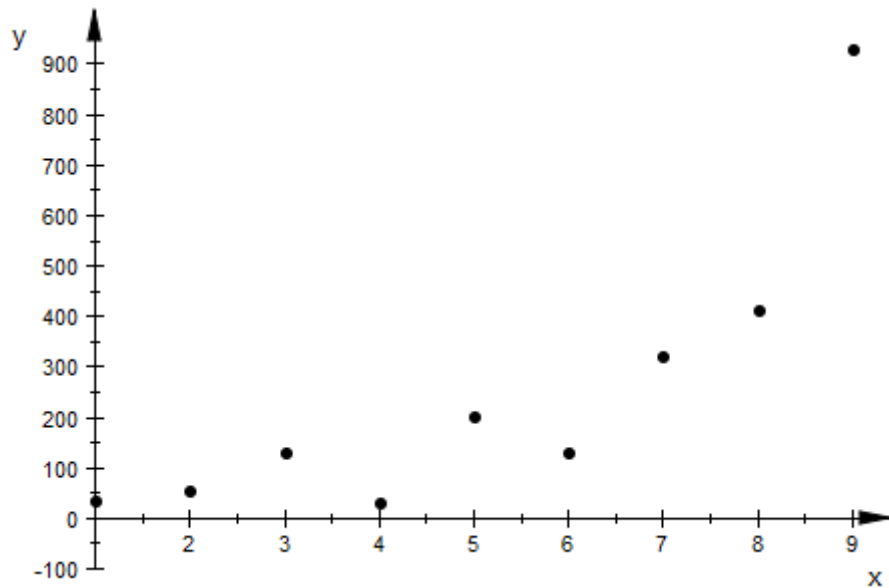
Here  $x_{ij}$  is the  $i$ th measurement of the independent variable  $x_j$ . The `stats::reg` function also can perform weighted least-squares nonlinear regression. By default, weights are equal to 1.

`stats::reg` returns a list of optimized parameters  $[p_1, \dots, p_n]$  and the minimized value of the quadratic error for the specified model. Suppose, you want to find a model for the following data:

```
sampleX := [1, 2, 3, 4, 5, 6, 7, 8, 9]:
sampleY := [36.97666099, 54.14911101, 131.3852077,
            30.43939553, 202.2004454, 129.5801972,
            321.0663718, 411.3959961, 929.597986]:
```

Plotting the data can help you choose the model:

```
plot(
    plot::Scatterplot(sampleX, sampleY,
                      LinesVisible = FALSE)
):
```



The scatter plot clearly shows that linear models do not fit the data. The dependency looks similar to exponential. Therefore you can try to fit the data to different models involving exponential functions. Suppose, you want to try

fitting the data to the expression  $a + \frac{b^2 e^{x1}}{x1}$ :

```
fit := stats::reg(sampleX, sampleY, p1 + p2^2*exp(x1)/x1,
                  [x1], [p1, p2])
```

```
[[87.35956975, 0.9696856368], 26612.78899]
```

The `stats::reg` function returns the parameters of the model and the quadratic error as a nested list. To access the parameters separately, use the following commands:

```
a := fit[1][1];
b := fit[1][2];
chi2 := fit[2]
```

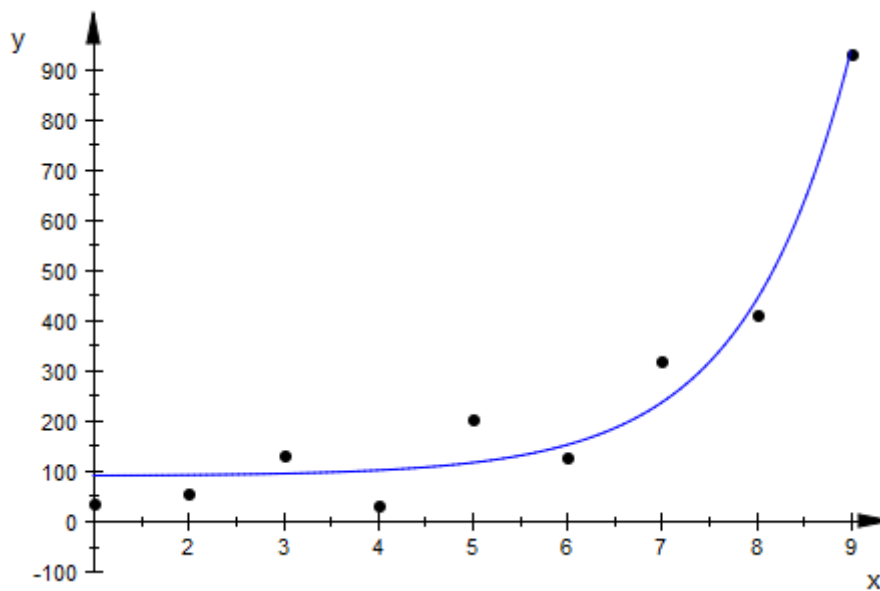
87.35956975

0.9696856368

26612.78899

Now, plot the data and expression that you used to fit the data on the same plot. This plot shows how the model expression fits the data:

```
plot2 := plot::Function2d(a + b^2*exp(x)/x, x = 1..9):  
plot(plot1, plot2)
```



## Multivariate Regression

The `stats::reg` function also performs linear and nonlinear regressions with two or more independent variables. The following example demonstrates how to perform regression and fit the data to a function of two variables. Suppose, you have three data lists that contain coordinates  $x$ ,  $y$ , and  $z$  of discrete points:

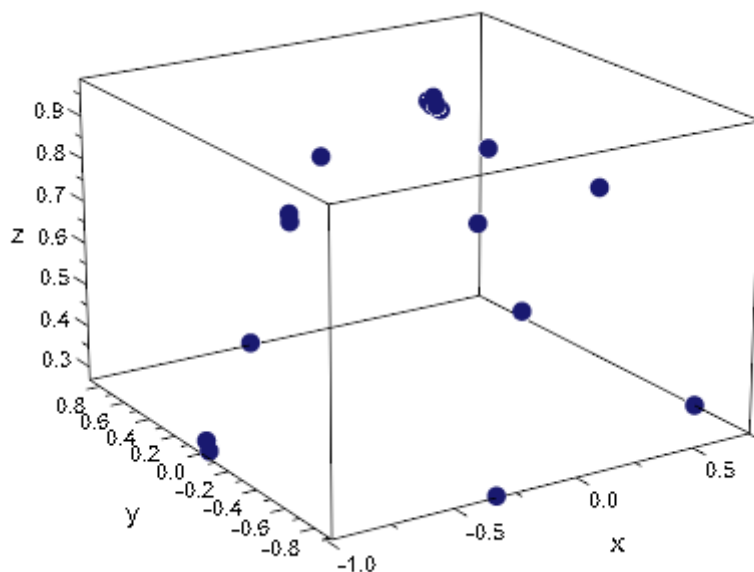
```
sampleX := [
  -0.9612553839, -0.329576986, 0.7544749248, 0.7339191669,
  -0.294101483, -0.9809519422, -0.6251624775, -0.1885706545,
  0.4729466504, 0.4402179092, -0.1883574567, -0.6260246367,
  -0.0274947885, -0.01843922645, -0.02687538212, -0.03682895886,
  -0.009212115975, -0.04956242636]:
sampleY := [
  -0.02185415496, -0.9146217269, -0.5792023459, 0.5440822742,
  0.8848317212, -0.03925037966, -0.02360776024, -0.5657632266,
  -0.3461422332, 0.3429495709, 0.5113552882, -0.02089004809,
  -0.03700165982, -0.0226531849, -0.004897297126, -0.03063832565,
  -0.03469956571, -0.01391540741]:
sampleZ := [
  0.2755344332, 0.272077192, 0.2682296712, 0.2915713541,
  0.2737466882, 0.3060314064, 0.7624231851, 0.8013891042,
  0.7755723041, 0.7631156115, 0.7816602999, 0.7807856826,
  0.9679031724, 0.9661527172, 0.9632260164, 0.986479402,
  0.9554368723, 0.9768285979]:
```

Suppose, you want to find the surface that fits these points. Start with plotting the data. The `plot::PointList3d` function, which plots a finite number of discrete points, requires the coordinates of each point to be grouped together. The following commands create separate lists for the coordinates of each point:  $[x_i, y_i, z_i]$  and put these lists into one nested list:

```
points := [[sampleX[i], sampleY[i], sampleZ[i]]
           $ i = 1..nops(sampleX)]:
```

Now, use the `plot::PointList3d` function to visualize the data:

```
plot1 := plot::PointList3d(points, PointSize = 2.5):
plot(plot1)
```



Rotating the plot can help you guess which surface can possibly fit the data. This plot shows that the data can belong to the upper half of a sphere. Thus, try to fit the data to a sphere with an unknown radius  $r$ . The `stats::reg` function searches for the best fitting value for the parameter  $r$ :

```
fit := stats::reg(sampleX, sampleY, sampleZ,
  sqrt(r^2 - x1^2 - y1^2), [x1, y1], [r])
```

```
[[0.9977031578], 0.0519038156]
```

The `stats::reg` function also accepts the nested list points:

```
fit := stats::reg(points, sqrt(r^2 - x1^2 - y1^2),
  [x1, y1], [r])
```

```
[[0.9977031578], 0.0519038156]
```

The `stats::reg` function returns the parameters of the model and the quadratic error in a form of a nested list. To access the parameters separately, use the following commands:

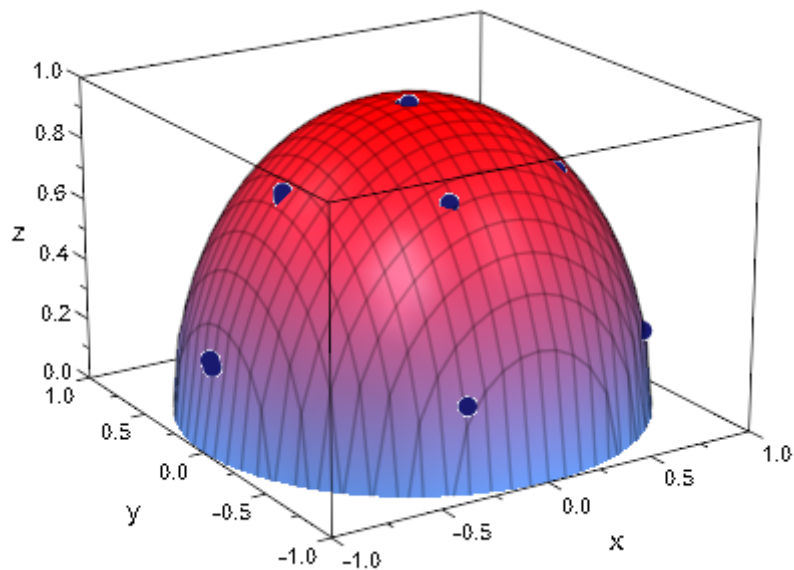
```
R := op(fit)[1][1];  
chi2 := op(fit)[2]
```

```
0.9977031578
```

```
0.0519038156
```

Now, plot the data and sphere that you used to fit the data on the same plot. Rotate the plot to see how the sphere fits the data:

```
plot2 := plot::Function3d(sqrt(R^2 - x^2 - y^2),  
                           x = -1..1, y = -1..1):  
plot(plot1, plot2)
```



## Principles of Hypothesis Testing

Hypothesis (goodness-of-fit) testing is a common method that uses statistical evidence from a sample to draw a conclusion about a population. In hypothesis testing, you assert a particular statement (a null hypothesis) and try to find evidence to support or reject that statement. A null hypothesis is an assumption about a population that you would like to test. It is “null” in the sense that it often represents a status-quo belief, such as the absence of a characteristic or the lack of an effect. You can formalize it by asserting that a population parameter, or a combination of population parameters, has a certain value. MuPAD enables you to test the following null hypotheses:

- The data has the distribution function  $f$ . If  $f$  is a cumulative distribution function (CDF), you can use the classical chi-square goodness-of-fit test or the Kolmogorov-Smirnov test. If  $f$  is probability density function (PDF), a discrete probability function (PF), or an arbitrary distribution function, use the classical chi-square goodness-of-fit test.
- The data has a normal distribution function with a particular mean and a particular variance. For cumulative distribution functions, use the classical chi-square goodness-of-fit test or the Kolmogorov-Smirnov test. For other distribution functions, use the classical chi-square goodness-of-fit test.
- The data has a normal distribution function (with unknown mean and variance). Use the Shapiro-Wilk goodness-of-fit test for this hypothesis.
- The mean of the data is larger than some particular value. Use the t-Test for this hypothesis.

The main result returned by the hypothesis tests is the p-value (PValue). The p-value of a test indicates the probability, under the null hypothesis, of obtaining a value of the test statistic as extreme or more extreme than the value computed from the sample. If the p-value is larger than the significance level (stated and agreed upon before the test), the null hypothesis passes the test. A typical value of a significance level is 0.05. P-values below a significance level provide strong evidence for rejecting the null hypothesis.



## Perform chi-square Test

For the classical chi-square goodness-of-fit test, MuPAD provides the `stats::csGOFT` function. This function enables you to test the data against an arbitrary function `f`. For example, you can define `f` by using any of the cumulative distribution functions, probability density functions, and discrete probability functions available in the MuPAD “Statistics” library. You also can define `f` by using your own distribution function. For example, create the data sequence `x` that contains a thousand random entries:

```
reset()

f := stats::normalRandom(0, 1/2):
x := f() $ k = 1..1000:
```

Suppose, you want to test whether the entries of that sequence are normally distributed with the mean equal to 0 and the variance equal to 1/2. The classical chi-square test uses the following three-step approach:

- 1 Divide the line of real values into several intervals (also called bins or cells).
- 2 Compute the number of data elements in each interval.
- 3 Compare those numbers with the numbers expected for the specified distribution.

When you use the `stats::csGOFT` function, specify the cell boundaries as an argument. You must specify at least three cells. The recommended minimum number of cells for a sample of  $n$  data elements is  $2n^{2/5}$ . The recommended method for defining the cells is to use the `stats::equiprobableCells` function. This function creates equiprobable cells when the underlying distribution is continuous:

```
q := stats::normalQuantile(0, 1/2):
cells := stats::equiprobableCells(40, q):
```

Now, call the `stats::csGOFT` function to test the data sequence `x`. For example, compare `x` with the cumulative normal distribution function with the same mean and variance. The `stats::csGOFT` returns a large p-value for this test. Therefore, the null hypothesis (`x` is normally distributed with

the mean equal to 0 and the variance equal to 1/2) passes this test. Besides the p-value, `stats::csGOF`T returns the observed value of the chi-square statistics and the minimum of the expected cell frequencies:

```
stats::csGOF(x, cells, CDF = stats::normalCDF(0, 1/2))
```

```
[PValue = 0.7728609833, StatValue = 32.16, MinimalExpectedCellFrequency = 25.0]
```

The `stats::csGOF`T enables you to test the data against any distribution function. For example, testing the sequence `x` against the probability density function gives the same result:

```
stats::csGOF(x, cells, PDF = stats::normalPDF(0, 1/2))
```

```
[PValue = 0.7728609833, StatValue = 32.16, MinimalExpectedCellFrequency = 25.0]
```

If you test the same data sequence `x` against the normal distribution function with different values of the mean and the variance, `stats::csGOF`T returns the p-value that is below the typical significance level 0.05. The null hypothesis does not pass the test:

```
stats::csGOF(x, cells, CDF = stats::normalCDF(0, 1))
```

```
[PValue = 9.164721121 10-21, StatValue = 184.3794698,
```

```
MinimalExpectedCellFrequency = 17.68346079]
```

## Perform Kolmogorov-Smirnov Test

For the Kolmogorov-Smirnov goodness-of-fit test, MuPAD provides the `stats::ksGOFT` function. This function enables you to test the data against any cumulative distribution available in the MuPAD “Statistics” library. The Kolmogorov-Smirnov test returns two p-values. The null hypothesis passes the test only if both values are larger than the significance level. For example, create the following data sequence `x` which contains a thousand entries:

```
f := stats::normalRandom(1, 1/3):
x := f() $ k = 1..1000:
```

Use the function `stats::ksGOFT` to test whether the sequence `x` has a normal distribution with the mean 1 and the variance 1/3. Suppose, you apply the typical significance level 0.05. Since both p-values are larger than the significance level, the sequence passes the test:

```
stats::ksGOFT(x, CDF = stats::normalCDF(1, 1/3))
```

```
[PValue1 = 0.06518463285, StatValue1 = 1.163141302, PValue2 = 0.8735153294,
  StatValue2 = 0.2548375105]
```

Test the same sequence, but this time compare it to the normal distribution with the variance 1. Both p-values are much smaller than the significance level. The null hypothesis states that the sequence `x` has a normal distribution with the mean 1 and the variance 1. This hypothesis must be rejected:

```
stats::ksGOFT(x, CDF = stats::normalCDF(1, 1))
```

```
[PValue1 = 1.7315647 10-17, StatValue1 = 4.377042911, PValue2 = 2.761005529 10-14,
  StatValue2 = 3.938439155]
```

## Perform Shapiro-Wilk Test

The Shapiro-Wilk goodness-of-fit test asserts the hypothesis that the data has a normal distribution. For the Shapiro-Wilk goodness-of-fit test, MuPAD provides the `stats::swGOFT` function. For example, create the normally distributed data sequence `x` by using the `stats::normalRandom` function:

```
fx := stats::normalRandom(0, 1/2):  
x := fx() $ k = 1..1000:
```

Also, create the data sequence `y` by using the `stats::poissonRandom` function. This function generates random numbers according to the Poisson distribution:

```
fy := stats::poissonRandom(10):  
y := fy() $ k = 1..1000:
```

Now, use the `stats::swGOFT` function to test whether these two sequences are normally distributed. Suppose, you use the typical significance level 0.05. For the first sequence (`x`), the resulting p-value is above the significance level. The entries of the sequence `x` can have a normal distribution. For the second sequence (`y`), the resulting p-value is below the significance level. Therefore, reject the hypothesis that the entries of this sequence are normally distributed:

```
stats::swGOFT(x);  
stats::swGOFT(y)
```

```
[PValue = 0.4388845871, StatValue = 0.9983128403]
```

```
[PValue = 0.0000000004748026729, StatValue = 0.9812478768]
```

## Perform t-Test

The t-Test compares the actual mean value of a data sample with the specified value  $m$ . The null hypothesis for this test states that the actual mean value is larger than  $m$ . For the t-Test, MuPAD provides the `stats::tTest` function. For example, create the normally distributed data sequence of 1000 entries by using the `stats::normalRandom` function:

```
f := stats::normalRandom(1, 2):  
x := f() $ k = 1..1000:
```

Now, use the `stats::tTest` function to test whether the actual mean value of  $x$  is larger than 1. Use the significance level 0.05. The returned p-value indicates that the hypothesis passes the t-Test:

```
stats::tTest(x, 1)
```

```
[PValue = 0.1353274218, StatValue = -1.102166281]
```

With the same significance level 0.05, the hypothesis that the actual mean value is larger than 2 does not pass the t-Test:

```
stats::tTest(x, 2)
```

```
[PValue = 6.722832021 10-98, StatValue = -23.50521869]
```

## Divisors

### In this section...

“Compute Divisors and Number of Divisors” on page 3-324

“Compute Greatest Common Divisors” on page 3-325

“Compute Least Common Multiples” on page 3-326

### Compute Divisors and Number of Divisors

Studying divisibility of integers by other integers is a common task in number theory. The MuPAD numlib library contains the functions that support this task. These functions return all divisors, the sum of all divisors, the number of divisors, and the number of prime divisors. For example, to find all positive integer divisors of an integer number, use the `numlib::divisors` function:

```
numlib::divisors(12345)
```

```
[1, 3, 5, 15, 823, 2469, 4115, 12345]
```

To find only prime divisors of an integer, use the `numlib::primedivisors` function:

```
numlib::primedivisors(12345)
```

```
[3, 5, 823]
```

To compute the number of all divisors of an integer, use the `numlib::numdivisors` function. To compute the number of prime divisors, use the `numlib::numprimedivisors` function. For example, the number 123456789987654321 has 192 divisors. Only seven of these divisors are prime numbers:

```
numlib::numdivisors(123456789987654321),  
numlib::numprimedivisors(123456789987654321)
```

```
192, 7
```

The `numlib::numprimedivisors` function does not take into account multiplicities of prime divisors. This function counts a prime divisor with multiplicity as one prime divisor. To compute the sum of multiplicities of prime divisors, use the `numlib::Omega` function. For example, the number 27648 has 44 divisors, and 2 of them are prime numbers. The prime divisors of 27648 have multiplicities; the total sum of these multiplicities is 13:

```
numlib::numdivisors(27648),  
numlib::numprimedivisors(27648),  
numlib::Omega(27648)
```

```
44, 2, 13
```

You can factor the number 27648 into prime numbers to reveal the multiplicities. To factor an integer into primes, use the `ifactor` function:

```
ifactor(27648)
```

```
210 33
```

To compute the sum of all positive integer divisors of an integer number, use the `numlib::sumdivisors` function. For example, compute the sum of positive divisors of the number 12345:

```
numlib::sumdivisors(12345)
```

```
19776
```

## Compute Greatest Common Divisors

The largest nonnegative integer that divides all the integers of a sequence exactly (without remainders) is called the greatest common divisor of a sequence. To compute the greatest common divisor of a sequence of integers, use the `igcd` function. For example, compute the greatest common divisor of the following numbers:

```
igcd(12345, 23451, 34512, 45123, 51234)
```

3

The `icontent` function computes the greatest common divisor of the coefficients of a polynomial. All coefficients must be integers:

```
icontent(12*x^2 + 16*x + 24)
```

4

### Compute Least Common Multiples

The smallest integer that is exactly divisible (without remainders) by all integers of a sequence is called the least common multiple of a sequence. To compute the least common multiple of a sequence of integers, use the `ilcm` function. For example, compute the least common multiple of the following numbers:

```
ilcm(12, 5, 2, 21)
```

420



## Primes and Factorizations

In this section...
“Operate on Primes” on page 3-327
“Factorizations” on page 3-329
“Prove Primality” on page 3-330

### Operate on Primes

Prime numbers are positive integers larger than 1 that have only two positive integer divisors: 1 and the number itself. In MuPAD, you can check whether the number is prime by using the `isprime` function. For example, 809 is a prime number while 888 and 1 are not primes:

```
isprime(809), isprime(888), isprime(1)
```

```
TRUE, FALSE, FALSE
```

In rare cases, the `isprime` function can return a false positive result. The function performs the Miller-Rabin primality test and uses 10 independent random bases. For a more accurate (and also slower) method, see Proving Primality.

The sequence of prime numbers is infinite. It starts with 2, 3, 5, 7, 11, 13, 17, and goes on. The `ithprime` function lets you quickly find and display any entry of this sequence. For example, to find the 100th prime number, enter `ithprime(100)`:

```
ithprime(100)
```

```
541
```

To find the prime number that appears in the sequence before a particular value, use the `prevprime` function. To find the prime number that appears after a particular value, use the `nextprime` function. For example, find the prime numbers that precede and follow the number 1000:

```
prevprime(1000), nextprime(1000)
```

```
997, 1009
```

---

**Note** `prevprime` and `nextprime` use the probabilistic primality test (the Miller-Rabin test). In rare cases, these functions can return nonprime numbers.

---

MuPAD stores a precalculated table of the prime numbers up to a certain value. The `ifactor` function with the `PrimeLimit` option returns that value:

```
ifactor(PrimeLimit)
```

```
1000000
```

The `ithprime` function with the `PrimeLimit` option returns the number of primes in that table:

```
ithprime(PrimeLimit)
```

```
78498
```

The `ithprime` function extracts the prime number from this table. To compute larger prime numbers (which MuPAD does not store in the table), `ithprime` chooses some number as a starting point, and then recursively calls the `nextprime` function. Although the internal algorithm tries to reduce the number of computation steps, computing huge prime numbers can be very slow:

```
ithprime(100000000)
```

```
2038074743
```

Suppose, you want to display a sequence of prime numbers. For the numbers that MuPAD stores in the table, call the `ithprime` function to find each number in the sequence:

```
ithprime(i) $ i = 1000..1010
```

```
7919, 7927, 7933, 7937, 7949, 7951, 7963, 7993, 8009, 8011, 8017
```

If the numbers exceed the value returned by `ifactor(PrimeLimit)`, MuPAD does not store them. In this case, calling `ithprime` for each number can be very slow. More efficiently, use `ithprime` to find the first number in the sequence, and then use `nextprime` to find all following numbers:

```
(n := ithprime(3*10^7)), (n := nextprime(n + 1)) $i = 1..10
```

```
573259391, 573259433, 573259439, 573259483, 573259523, 573259529, 573259573, 573259627,  
573259637, 573259651, 573259679
```

To find how many prime numbers do not exceed a particular value, use the `numlib::pi` function:

```
numlib::pi(2), numlib::pi(3), numlib::pi(20),  
numlib::pi(1000), numlib::pi(15.789)
```

```
1, 2, 8, 168, 6
```

## Factorizations

You can represent any integer number as a product of primes. This process is called factoring an integer. To factor an integer, use the `ifactor` function. For example, factor the number 362880:

```
ifactor(362880)
```

```
27 34 5 7
```

## Prove Primality

The function `numlib::proveprime` implements the Atkin-Goldwasser-Kilian-Morain algorithm for proving primality. For information about primality proving and this particular algorithm, see the following papers:

- Atkin, A. O., and F. Morain. “Elliptic curves and primality proving.” *Mathematics of Computation*. Vol. 61, Number 203, 1993.
- Goldwasser, S., and J. Kilian. “Almost all primes can be quickly certified”. *Proceedings of the 18th annual ACM symposium on theory of computing*. Berkeley, CA, US, 1986, pp. 316-329.

For relatively small prime numbers, `numlib::proveprime` returns the value `TRUE`. For composite numbers, the function returns `FALSE`:

```
numlib::proveprime(541), numlib::proveprime(243)
```

```
TRUE, FALSE
```

For larger prime numbers, `numlib::proveprime` returns a certificate of primality:

```
certificate := numlib::proveprime(1299709)
```

```
[1299709, 15, [2, 2, 2, 2, 2, 107, 379], 700619, 67686, 0, 796444, [107, 379]]
```

Generated primality certificates provide all data that you need for proving primality of a number by the Atkin-Goldwasser-Kilian-Morain algorithm. You can substitute the numbers into the algorithm and verify the primality of a number. The `numlib::checkPrimalityCertificate` function can verify the certificate for you:

```
numlib::checkPrimalityCertificate(certificate)
```

```
TRUE
```

## Modular Arithmetic

In this section...
“Quotients and Remainders” on page 3-331
“Common Modular Arithmetic Operations” on page 3-333
“Residue Class Rings and Fields” on page 3-334

### Quotients and Remainders

Computing the quotient and the remainder of the division of two integers is a common operation in number theory. The MuPAD standard library provides the `div` function for computing the quotient of a division. The standard library also provides the modulo operator `mod` for computing the remainder of a division:

`17 div 3, 17 mod 3`

`5, 2`

MuPAD supports two definitions of the remainder of a division. The first definition states that the remainder is always a nonnegative number. To compute the remainder by this definition, use the positive modulo function `modp` or the default modulo operator `mod`:

`123 mod 5, modp(123, 5)`

`3, 3`

The symmetric modulo function `mods` implements the alternative definition of the remainder of a division. This definition allows the remainder to be negative. The symmetric modulo function compares the absolute values of the negative and the positive remainders, and returns the remainder that has the least absolute value:

`mods(122, 5), mods(123, 5)`

2, -2

You can redefine the modulo operator `mod` by assigning `mods` or `modp` to `mod`:

```
123 mod 5;
_mod := mods:
123 mod 5
```

3

-2

Redefining the modulo operator `mod` does not affect the `div` function that computes the quotient. The `div` function always computes the quotient by assuming that the remainder must be nonnegative:

```
123 div 5, 123 mod 5
```

24, -2

For further computations, restore the default behavior of the modulo operator:

```
_mod := modp:
```

Now, suppose that the number is represented as  $a^b$ , where  $a$  and  $b$  are integers. Suppose, you want to compute the modular power of  $a^b$  over  $c$ , which is the remainder of the division of  $a^b$  by an integer  $c$ . You can compute the modular power by using the modulo operator `mod`:

```
987^124 mod 12
```

9

When you use the `mod` operator to compute the modular power, MuPAD performs computations in two steps. First, the system computes the power of an integer, and then it divides the result and finds the remainder. This approach works if both the number and its power are small. However, if the

numbers are large, the first step (computing the power of an integer) can be extremely slow. To avoid this step, use the `powermod` function. This function computes the modular power more efficiently, especially for large numbers:

```
powermod(987, 124, 12),
powermod(987, 123456789, 12)
```

9, 3

## Common Modular Arithmetic Operations

MuPAD implements a large number of algorithms that enable you to perform various modular arithmetic operations. For example, you can quickly compute primitive roots, orders of residue classes, and the Euler's totient function.

If  $g$  is a primitive root modulo  $n$ , then for any integer  $a$  there exists an integer  $k$  such that  $g^k \equiv a \pmod{n}$ . This congruence has solutions (primitive roots exist) only if  $\text{igcd}(a, n) = 1$ . To compute the least primitive root modulo  $n$ , use the `numlib::primroot` function. For example, compute the primitive roots modulo 19, 23, 191, and 311:

```
numlib::primroot(19),
numlib::primroot(23),
numlib::primroot(191),
numlib::primroot(311)
```

2, 5, 19, 17

The `numlib::order` function computes the order of a residue class. For an integer  $a$  and a positive integer  $n$ , this function returns the least number  $k$ , such that  $a^k \equiv 1 \pmod{n}$ . For example, compute the order of the residue class of 3 in the unit group modulo 7:

```
numlib::order(3, 7)
```

6

The Euler's totient function of an integer  $n$  counts all positive integers that satisfy the following two conditions:

- The integer is coprime to  $n$ .
- The integer is smaller or equal to  $n$ .

The Euler's totient function returns the number of such integers. To compute the Euler's totient function in MuPAD, use the `numlib::phi` function:

```
numlib::phi(i) $ i = 1..20
```

```
1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8
```

For other modular arithmetic functions available in MuPAD, see the `numlib` library.

## Residue Class Rings and Fields

For the two integers  $a$  and  $m$ , all integers  $b$ , such that  $a \equiv b \pmod{m}$ , construct a residue class. The MuPAD domain `Dom::IntegerMod` lets you create a ring that consists of all residue classes modulo some integer  $m$ . For example, create a residue class ring of integers modulo 11:

```
Z11:= Dom::IntegerMod(11)
```

```
Dom::IntegerMod(11)
```

Now, create the elements  $a$  and  $b$  of this ring. Then, compute the sum of  $a$  and  $b$ :

```
a:= Z11(1); b:= Z11(6); a + b
```

```
1 mod 11
```

```
6 mod 11
```



## 7 mod 11

You can use `Dom::IntegerMod` to specify polynomials over a coefficient ring. When computing the coefficients of a polynomial, `Dom::IntegerMod(7)` uses the positive modulo function `modp`:

```
poly([[9, 3], [13, 1]], [x], Dom::IntegerMod(11))
```

```
poly(9 x3 + 2 x, [x], Dom::IntegerMod(11))
```

For specifying polynomials over a residue class ring  $\mathbb{Z}_n$ , the `poly` function also provides the `IntMod` option. This option enables you to create a polynomial with the coefficients that belong to the `IntMod(n)` ring. Here  $n$  is an integer greater than 1. Mathematically, this ring coincides with `Dom::IntegerMod(n)`. However, MuPAD operates differently on the polynomials created over `IntMod(n)` and the polynomials created over `Dom::IntegerMod(n)`. In particular, MuPAD performs arithmetic operations for polynomials over the `IntMod` ring faster. Also, if you use `IntMod(n)`, MuPAD uses the symmetric modulo function `mods`:

```
poly([[9, 3], [13, 1]], [x], IntMod(11))
```

```
poly(-2 x3 + 2 x, [x], IntMod(11))
```

The domain `Dom::GaloisField` enables you to create the residue class field  $\mathbb{Z}_p[X]/\langle f \rangle$ , which is a finite field with  $p^n$  elements. If you do not specify `f`, MuPAD randomly chooses `f` from all irreducible polynomials of the degree  $n$ . For more information, see the `Dom::GaloisField` help page.

## Congruences

### In this section...

“Linear Congruences” on page 3-336

“Systems of Linear Congruences” on page 3-337

“Modular Square Roots” on page 3-338

“General Solver for Congruences” on page 3-342

### Linear Congruences

If  $a$ ,  $b$ , and  $m$  are integers, and  $(a - b)/m$  is also an integer, then the numbers  $a$  and  $b$  are congruent modulo  $m$ . The remainder of the division  $a/m$  is equal to the remainder of the division of  $b/m$ . For example,  $11 \equiv 5 \pmod{3}$ :

$$5 \pmod{3} = 11 \pmod{3}$$

$$2 = 2$$

For known integers  $a$  and  $m$ , all integers  $b$ , such that  $a \equiv b \pmod{m}$ , form a residue class. Thus, the numbers 5 and 11 belong to the same residue class modulo 3. The numbers  $5 + 3n$ , where  $n$  is an integer, also belong to this residue class.

Suppose, you want to solve an equation  $ax \equiv b \pmod{m}$ , where  $a$ ,  $b$ , and  $m$  are integers and  $x$  is an unknown integer. Such equations are called linear congruence equations. To solve a linear congruence equation, use the `numlib::lincongruence` function. This function returns only the solutions  $x < m$ . For example, solve the linear congruence equation  $56x \equiv 77 \pmod{49}$ :

```
numlib::lincongruence(56, 77, 49)
```

```
[4, 11, 18, 25, 32, 39, 46]
```

A linear congruence equation  $ax \equiv b \pmod{m}$  has at least one solution if and only if the parameters  $a$ ,  $b$ , and  $m$  satisfy the following condition:  $b \equiv 0 \pmod{\gcd(a, m)}$ . If the parameters of a linear congruence equation do not

satisfy this condition, the equation does not have a solution. In this case, `numlib::lincongruence` returns FAIL:

```
numlib::lincongruence(56, 77, 48)
```

FAIL

You can divide a rational number  $u/v$  modulo an integer  $m$ , and compute the remainder of such a division. Here  $u$  and  $v$  are nonzero coprime integers. The modulo operator computes an integral solution  $r$  of the linear congruence  $vr \equiv u \pmod{m}$ . For example, compute the remainder of the division of the rational number  $3/7$  modulo 1231:

```
r := 3/7 mod 1231
```

528

To reconstruct a linear congruence from the modulus  $m$  and the remainder  $r$ , use the `numlib::reconstructRational(r, m)` function call. For  $m = 1231$  and  $r = 528$ , the `numlib::reconstructRational` function returns the coefficients of the linear congruence  $7r \equiv 3 \pmod{1231}$ :

```
numlib::reconstructRational(528, 1231)
```

3, 7

## Systems of Linear Congruences

The Chinese remainder theorem states that if the integers  $m_i$  ( $i = 1, \dots, n$ ) are pairwise coprime, the system of  $n$  linear congruences  $x \equiv a_i \pmod{m_i}$  has a solution. The numbers  $m_i$  ( $i = 1, \dots, n$ ) are pairwise coprime if the greatest common divisor of any pair of numbers  $m_i, m_j$  ( $i \neq j$ ) is 1. The solution is unique up to multiples of the least common multiple (lcm) of  $m_1, m_2, \dots, m_n$ . To solve a system of linear congruence equations, use the `numlib::ichrem` function:

```
numlib::ichrem([3, 1, 10], [6, 5, 13])
```

231

The Chinese remainder theorem does not state that the system of linear congruences is solvable only if numbers  $m_1, \dots, m_n$  are pairwise coprime. If these numbers are not pairwise coprime, the system still can have a solution. Even if the numbers are not pairwise coprime, the solution is still unique up to multiples of the least common multiple (lcm) of  $m_1, m_2, \dots, m_n$ :

```
numlib::ichrem([5, 7, 9, 6], [10, 11, 12, 13])
```

3945

If the numbers are not pairwise coprime, a system of linear congruences does not always have a solution. For unsolvable systems, `numlib::ichrem` returns FAIL:

```
numlib::ichrem([5, 1, 9, 6], [10, 15, 12, 13])
```

FAIL

## Modular Square Roots

### Compute Modular Square Roots

To compute modular square roots  $x < m$  of the equation  $x^2 \equiv a \pmod{m}$ , use the `numlib::msqrts` function. Here the integers  $a$  and  $m$  must be coprime. For example, solve the congruence equation  $x^2 \equiv 13 \pmod{17}$ :

```
numlib::msqrts(13, 17)
```

[8, 9]

If the congruence does not have any solutions, `numlib::msqrts` returns an empty set:

```
numlib::msqrts(10, 17)
```

□

If  $a$  and  $m$  are not coprime, `numlib::msqrts` errors:

```
numlib::msqrts(17, 17)
```

```
Error: Arguments must be relative prime. [numlib::msqrts]
```

If `numlib::msqrts` cannot solve a congruence, try using the `numlib::mroots` function. For more information, see [General Solver for Congruences](#).

### Use Solvability Tests: Legendre and Jacobi Symbols

The Legendre symbol determines the solvability of the congruence  $x^2 \equiv a \pmod{m}$ , where  $m$  is a prime. You can compute the Legendre symbol only if the modulus is a prime number. The following table demonstrates the dependency between the value of the Legendre symbol and solvability of the congruence:

If the Legendre number is...	The congruence...
1	Has one or more solutions
0	Cannot be solved by <code>numlib::msqrts</code> . Try <code>numlib::mroots</code> .
-1	Has no solution

MuPAD implements the Legendre symbol as the `numlib::legendre` function. If, and only if, the congruence  $x^2 \equiv a \pmod{m}$  is solvable, the Legendre symbol is equal to 1:

```
numlib::legendre(12, 13)
```

```
1
```

```
numlib::msqrts(12, 13)
```

```
[5, 8]
```

If, and only if, the congruence  $x^2 \equiv a \pmod{m}$  does not have any solutions, the Legendre symbol is equal to -1:

```
numlib::legendre(11, 13)
```

```
-1
```

```
numlib::msqrts(11, 13)
```

```
[]
```

If  $a$  and  $m$  are not coprime, the Legendre symbol is equal to 0. In this case, `numlib::legendre` function returns 0, and `numlib::msqrts` errors:

```
numlib::legendre(13, 13)
```

```
0
```

```
numlib::msqrts(13, 13)
```

```
Error: Arguments must be relative prime. [numlib::msqrts]
```

You can compute the Legendre symbol only if the modulus is a prime number. If a congruence has a nonprime odd modulus, you can compute the Jacobi symbol. The Jacobi symbol determines the unsolvable congruences  $x^2 \equiv a \pmod{m}$ . You cannot compute the Jacobi symbol if the modulus is an even number. The following table demonstrates the dependency between the value of the Jacobi symbol and the solvability of the congruence:

If the Jacobi number is...	The congruence...
1	Might have solutions
0	Cannot be solved by numlib::msqrts. Try numlib::mroots.
-1	Has no solutions

MuPAD implements the Jacobi symbol as the `numlib::jacobi` function. If the Jacobi symbol is equal to -1, the congruence does not have a solution:

```
numlib::jacobi(19, 21)
```

```
-1
```

```
numlib::msqrts(19, 21)
```

```
[]
```

If Jacobi symbol is equal to 1, the congruence might have solutions:

```
numlib::jacobi(16, 21)
```

```
1
```

```
numlib::msqrts(16, 21)
```

```
[4, 10, 11, 17]
```

However, the value 1 of the Jacobi symbol does not guarantee that the congruence has solutions. For example, the following congruence does not have any solutions:

```
numlib::jacobi(20, 21)
```

1

```
numlib::msqrts(20, 21)
```

[]

If  $a$  and  $m$  are not coprime, the Jacobi symbol is equal to 0. In this case, `numlib::jacobi` function returns 0, and `numlib::msqrts` errors:

```
numlib::jacobi(18, 21)
```

0

```
numlib::msqrts(18, 21)
```

```
Error: Arguments must be relative prime. [numlib::msqrts]
```

## General Solver for Congruences

Besides solving a linear congruence or computing modular square roots, MuPAD also enables you to solve congruences of a more general type of  $P(x) \equiv 0 \pmod{m}$ . Here  $P(x)$  is a univariate or multivariate polynomial. To solve such congruences, use the `numlib::mroots` function. For example, solve the congruence  $x^3 + x^2 + x + 1 \equiv 0 \pmod{3}$ . First, define the left side of the congruence as a polynomial by using the `poly` function:

```
p := poly(x^3 + x^2 + x + 1)
```

```
poly(x^3 + x^2 + x + 1, [x])
```

Now, use the `numlib::mroots` function to solve the congruence:

```
numlib::mroots(p, 299)
```

```
[229, 252, 298]
```



Using the `numlib::mroots` function, you also can solve the congruence for a multivariate polynomial. For a multivariate polynomial  $P(x_1, \dots, x_n)$ , `numlib::mroots` returns a nested list as a result. Each inner list contains one solution  $x_1, \dots, x_n$ . For example, find modular roots of the following multivariate polynomial:

```
p := poly(x^3*y^2 + x^2*y + x + y + 1):  
numlib::mroots(p, 11)
```

```
[[0, 10], [3, 2], [3, 7], [7, 3], [7, 5], [8, 5], [8, 8], [9, 5], [9, 8], [10, 0], [10, 2]]
```

## Sequences of Numbers

### In this section...

“Fibonacci Numbers” on page 3-344

“Mersenne Primes” on page 3-344

“Continued Fractions” on page 3-345

### Fibonacci Numbers

The Fibonacci numbers are a sequence of integers. The following recursion formula defines the  $n$ th Fibonacci number:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}.$$

To compute the Fibonacci numbers, use the `numlib::fibonacci` function. For example, the first 10 Fibonacci numbers are:

```
numlib::fibonacci(n) $ n = 0..9
```

```
0, 1, 1, 2, 3, 5, 8, 13, 21, 34
```

### Mersenne Primes

The Mersenne numbers are the prime numbers  $2^p - 1$ . Here  $p$  is also a prime. The `numlib::mersenne` function returns the list that contains the following currently known Mersenne numbers:

```
numlib::mersenne()
```

```
[2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689,
9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433,
1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951,
30402457, 32582657, 37156667, 42643801, 43112609, 57885161]
```

## Continued Fractions

The continued fraction approximation of a real number  $r$  is an expansion of the following form:

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots + a_{k-1} + \frac{1}{a_k + \dots}}}}$$

Here  $a_1$  is the integer floor( $r$ ), and  $a_2, a_3, \dots$  are positive integers.

To create a continued fraction approximation of a real number, use the `numlib::contfrac` function. For example, approximate the number  $123456/123456789$  by a continued fraction:

```
numlib::contfrac(123456/123456789)
```

$$1000 + \frac{1}{156 + \frac{1}{2 + \frac{1}{8 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \dots}}}}}}}$$

Alternatively, you can use the more general `contfrac` function. This function belongs to the standard library. While `numlib::contfrac` accept only real numbers as parameters, `contfrac` also accepts symbolic expressions. When working with real numbers, `contfrac` internally calls `numlib::contfrac`, and returns the result of the domain type `numlib::contfrac`:

```
a := contfrac(123456/123456789);
domtype(a)
```

$$\begin{array}{r} \frac{1}{1000 + \frac{1}{156 + \frac{1}{2 + \frac{1}{8 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \dots}}}}} \end{array}$$

`numlib::contfrac`

Since `contfrac` internally calls `numlib::contfrac`, calling the `numlib::contfrac` directly can speed up your computations.

# Programming Fundamentals

---

- “Data Type Definition” on page 4-3
- “Choose Appropriate Data Structures” on page 4-6
- “Convert Data Types” on page 4-8
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## Data Type Definition

In this section...
“Domain Types” on page 4-3
“Expression Types” on page 4-3

### Domain Types

MuPAD stores all objects as elements of particular domains. There are two types of domains in MuPAD: basic domains and library domains. The system includes the basic domains written in C++. The names of the basic domains start with `DOM_`. For example, the domain of integers `DOM_INT`, the domain of rational numbers `DOM_RAT`, and the domain of identifiers `DOM_IDENT` are basic domains. Most of the basic domains available in MuPAD are listed in “Basic Domains”.

The system also includes library domains, such as the domain `Dom::Matrix()` of matrices, the domain `Dom::ArithmeticalExpression` of arithmetical expressions, the domain `Dom::Interval` of floating-point intervals, and the domain `stats::sample` of statistical samples. Many library domains are listed in “Library Domains”. Other library domains are listed in the corresponding libraries. For example, you can find the library domain `solvelib::BasicSet` of the basic infinite sets under the “Utilities for the Solver” category. The library domains are written in the MuPAD programming language. You also can create your own library domains.

Overloading works differently for the basic domains and the library domains. The system can overload any method of a library domain. For basic domains, the system overloads only some methods.

### Expression Types

The basic domain `DOM_EXPR` includes MuPAD expressions, such as expressions created by arithmetical or indexed operators, statements, and function calls. MuPAD classifies the elements of the domain `DOM_EXPR` further by defining *expression types*. Expression types provide more detailed information about a particular expression. For example, you might want to know whether the expression is an arithmetical expression, such as a sum or a product, a

Boolean expression, a function call, or some other type of expression. For these expressions, the `domtype` function returns their domain type `DOM_EXPR`:

```
domtype(a + b), domtype(a*b),  
domtype(a and b), domtype(sin(a)),  
domtype(a = b)
```

```
DOM_EXPR, DOM_EXPR, DOM_EXPR, DOM_EXPR, DOM_EXPR
```

To find the expression types of these expressions, use the `type` function:

```
type(a + b), type(a*b),  
type(a and b), type(sin(a)),  
type(a = b)
```

```
"_plus", "_mult", "_and", "sin", "_equal"
```

If an operator or a function has a “type” slot, the `type` function returns the string value stored in that slot. For example, the “type” slot of the addition operator contains the string value “\_plus”, the “type” slot of the multiplication operator contains “\_mult”, the “type” slot of the sine function contains “sin”, and so on.

An expression can include more than one operator. Typically, MuPAD associates the expression type of such expressions with the lowest precedence operator. If you visualize an expression as an expression tree, the lowest precedence operator appears at the root of that tree. See *Visualizing Expression Trees* for more information. For example, consider the expression  $a + b*c$ . When evaluating this expression, the system performs multiplication, and then performs addition. Therefore, the addition operator is the lowest precedence operator in the expression. This operator determines the expression type:

```
type(a + b*c)
```

```
"_plus"
```



If the lowest precedence operator in the expression does not have a “type” slot, the `type` function returns the string “function”:

```
type(f(a^2 + a + 2))
```

```
"function"
```

The `domtype` and `type` functions return the same results for elements of most MuPAD domains:

```
domtype(5/7), type(5/7);  
domtype(1.2345), type(1.2345);  
domtype(a), type(a);
```

```
DOM_RAT, DOM_RAT
```

```
DOM_FLOAT, DOM_FLOAT
```

```
DOM_IDENT, DOM_IDENT
```

If your code relies on the assumption that an object belongs to a particular domain type or expression type, verify that assumption before executing the code. To test whether a MuPAD object belongs to a particular type, use the `testtype` function. Use this function to test both domain types and expression types:

```
testtype(a + b, DOM_EXPR), testtype(a + b, "_plus")
```

```
TRUE, TRUE
```

## Choose Appropriate Data Structures

When you create a new MuPAD object, you choose the domain type of that object either explicitly or implicitly. Typically, MuPAD does not require you to declare domain types of simple objects, such as numbers, identifiers, or expressions. The system associates an object with a particular domain during run time.

When you create more complicated data structures, such as sets, lists, arrays, matrices, procedures, and so on, the syntax that you use to create these structures is a shortcut to the domain constructors. For example, `matrix` is a shortcut for the domain constructor `Dom::Matrix` with the default ring `Dom::ExpressionField()`.

Also, when defining your own procedure, you can specify the types of arguments accepted by that procedure. In this case, for every call to that procedure, MuPAD automatically checks the types of provided arguments. For more information, see [Checking Types of Arguments](#).

The `domtype` function returns the name of a domain to which an object belongs:

```
domtype([a, b, c])
```

```
DOM_LIST
```

When choosing a data structure for a new object, try to answer these questions:

- **Which features are essential to the new object?** For example, some structures keep the initial order of elements, while other structures can change the order. Another example is that you can create multidimensional arrays, but you cannot create MuPAD matrices with more than two dimensions.
- **Which functions do you want to use on that object?** Each MuPAD function accepts only objects of particular domain types. If an object that you pass to a function is not one of the acceptable domain types for that function, the function issues an error. To determine the domain types

acceptable for a particular function, see the help page for that function. For example, you cannot use standard mathematical operations for MuPAD arrays.

If you already created an object, and then realized that it must belong to another domain type, try to convert the domain type of the object. See [Converting Data Types](#).

## Convert Data Types

In this section...
“Use the coerce Function” on page 4-9
“Use the expr Function” on page 4-10
“Use Constructors” on page 4-13

When creating new objects in MuPAD, the best practice is to consider which domain the object must belong to. Choosing the correct data type from the beginning helps you avoid unnecessary conversions. Nevertheless, some computation tasks require data types conversions. For such tasks, MuPAD enables you to convert the elements of some domains to elements of other domains.

To convert an object to a different domain type, use one of the following alternatives. Some of the alternatives do not work for conversions between particular domains.

- “Use the coerce Function” on page 4-9. Call the `coerce` function to convert an object to an element of a specified domain. The `coerce` function can call the `convert`, `convert_to`, and `coerce` domain methods. If the conversion is not possible or if none of these methods is implemented for the domains participating in conversion, the `coerce` function returns `FAIL`.
- “Use the expr Function” on page 4-10. Call the `expr` function to convert an object to an element of a basic domain. The `expr` function calls the `expr` method. If the conversion is not possible or if the `expr` method is not implemented for the specified domains, the `expr` function returns `FAIL`.
- “Use Constructors” on page 4-13. Call the domain constructor of the domain to which you want to convert an object. This approach typically works, but it can fail in some cases.
- Calling the conversion methods of a domain directly. To use this approach, you must know which conversion methods are implemented for the participating domains. This alternative is not recommended. Use the `coerce` or `expr` function instead.

---

**Note** If you implement a new domain, consider implementing the conversion methods for that domain.

---

## Use the coerce Function

To convert a MuPAD object to an element of a specified domain, use the `coerce` function. For example, convert the following element of the domain `DOM_LIST` to an element of the domain `DOM_SET`:

```
L := [1, 2, 3, 4, 5, 6];
S := coerce(L, DOM_SET)
```

```
[1, 2, 3, 4, 5, 6]
```

```
{1, 2, 3, 4, 5, 6}
```

```
domtype(L), domtype(S)
```

```
DOM_LIST, DOM_SET
```

Results of the conversion can depend on the original domain type. For example, create an array and a matrix with the same number of elements and equal dimensions:

```
M := matrix(2, 3, [1, 2, 3, 4, 5, 6]);
A := array(1..2, 1..3, [1, 2, 3, 4, 5, 6])
```

```
( 1 2 3 )
( 4 5 6 )
```

```
( 1 2 3 )
( 4 5 6 )
```

Verify that matrix `M` belongs to the domain `Dom::Matrix()`, and array `A` belongs to the domain `DOM_ARRAY`:

```
domtype(M), domtype(A)
```

```
Dom::Matrix(), DOM_ARRAY
```

Use the `coerce` function to convert both objects to elements of the domain `DOM_LIST`. The `coerce` function converts matrix `M` to a nested list, where the inner lists represent the rows of the original matrix. At the same time, `coerce` converts the array `A` to a flat list:

```
LM := coerce(M, DOM_LIST);  
LA := coerce(A, DOM_LIST)
```

```
[[1, 2, 3], [4, 5, 6]]
```

```
[1, 2, 3, 4, 5, 6]
```

Both new objects belong to the basic domain of lists `DOM_LIST`:

```
domtype(LM), domtype(LA)
```

```
DOM_LIST, DOM_LIST
```

For further computations, delete the identifiers `L`, `S`, `M`, `A`, `LM`, and `LA`:

```
delete L, S, M, A, LM, LA
```

### Use the `expr` Function

To convert an element of a library domain to an element of a basic kernel domain, use the `expr` function. This function does not allow you to select the basic domain to which it converts an object. If the conversion to basic domains is not possible, the function returns `FAIL`.

The `expr` function tries to find the simplest basic domain to which it can convert a given object. The function also can convert an element of a more complicated basic domain to the simpler basic domain. For example, it converts the polynomial represented by a single variable `x` to the identifier `x`:

```
y := poly(x);
expr(y)
```

```
poly(x, [x])
```

```
x
```

The original object `y` belongs to the domain of polynomials. The result of conversion belongs to the domain of identifiers `DOM_IDENT`:

```
domtype(y), domtype(expr(y))
```

```
DOM_POLY, DOM_IDENT
```

If you call the `expr` function for a more complicated polynomial, the function converts that polynomial to the expression:

```
p := poly(x^2 + x + 2);
expr(p)
```

```
poly(x^2 + x + 2, [x])
```

```
x^2 + x + 2
```

Again, the original polynomial belongs to the domain of polynomials. This time, the result of the conversion belongs to the domain of expressions:

```
domtype(p), domtype(expr(p))
```

```
DOM_POLY, DOM_EXPR
```

MuPAD can apply the `expr` function to an object recursively. If an object contains terms that belong to library domains or complicated basic domains, `expr` also tries to convert those terms elements of simpler basic domains. For example, if the polynomial `p` is an element of a list, applying the `expr` function to the list converts the polynomial `p` to the expression:

```
matrix(2, 2, [p, 0, 0, 1]);
expr(matrix(2, 2, [p, 0, 0, 1]))
```

$$\begin{pmatrix} \text{poly}(x^2+x+2, [x]) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x^2+x+2 & 0 \\ 0 & 1 \end{pmatrix}$$

The `expr` function converts a matrix of the library domain `Dom::Matrix()` to an array of the basic kernel domain `DOM_ARRAY`:

```
M := matrix(2, 3, [1, 2, 3, 4, 5, 6])
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

```
domtype(M), domtype(expr(M))
```

`Dom::Matrix()`, `DOM_ARRAY`

The `expr` function converts an element of the series domain to the expression of a basic domain `DOM_EXPR`. Typically, the order of terms in the resulting expression changes:

```
s := series(exp(x), x);
expr(s)
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)$$

$$\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1$$

```
domtype(s), domtype(expr(s))
```



`Series::Puiseux, DOM_EXPR`

For further computations, delete the identifiers `y`, `p`, `M`, and `s`:

```
delete y, p, M, s
```

## Use Constructors

Instead of calling the conversion functions, you can use the constructor of a domain to which you want to convert an object. This approach works for most conversions, but it can fail in some cases.

Calling the constructor of the required domain is often the simplest way to convert an object to the element of that domain. Suppose you want to convert the following list `L` to a matrix:

```
L := [1, 2, 3, 4, 5, 6]
```

`[1, 2, 3, 4, 5, 6]`

To convert a list to a matrix, call the matrix constructor `matrix`. Use the arguments of the matrix constructor to specify the dimensions of the required matrix and the list `L` of the elements of a matrix. The resulting object is the  $2 \times 3$  matrix of the domain `Dom::Matrix()`:

```
matrix(2, 3, L);
domtype(%)
```

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

`Dom::Matrix()`

Alternatively, you can convert a list to a matrix by using `Dom::Matrix()`:

```
Dom::Matrix()(2, 3, L);
domtype(%)
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

`Dom::Matrix()`

For further computations, delete the identifier L:

`delete L`

## Define Your Own Data Types

MuPAD provides many predefined domains for a wide variety of symbolic and numeric computations. The system is also extensible. If the predefined MuPAD domains do not satisfy your needs, you can create your own domains and define operations on their elements. Also, MuPAD allows you to change the existing library domains, although this approach is not recommended. To create a new domain, use the `newDomain` function. For example, create the domain `TempF` that represents the Fahrenheit temperature scale:

```
TempF := newDomain("TempF");
```

Currently the new domain does not have any elements. To create an element of a domain, use the `new` function. For example, create the new element `x` of the domain `TempF`:

```
x := new(TempF, 15)
```

```
new(TempF, 15)
```

To verify that the object `x` belongs to the domain `TempF`, use the `testtype` function. You can also use the `domtype` or `type` function:

```
testtype(x, TempF), domtype(x), type(x)
```

```
TRUE, TempF, TempF
```

To check the internal representation of `x`, use the `extop` function:

```
extop(x)
```

```
15
```

Suppose you want the new method of the domain `TempF` to check whether an input argument is a valid temperature measurement. In this case, redefine the new method for your domain `TempF`:

```
TempF::new := t -> if testtype(t, Type::Real) then
```

```
if t >= -459.67 then
  new(dom, t);
else
  error("Temperature is below absolute zero.");
end_if;
else
  error("Expecting a real number above absolute zero.");
end_if;
```

The redefined method checks that an input argument is a number. It also checks that the provided temperature is not below absolute zero. Absolute zero is zero degrees on the Kelvin temperature scale, which is equivalent to -459.67 degrees on the Fahrenheit scale. The redefined method creates new elements in degrees Fahrenheit. Call the method `TempF::new` directly or use the shorter syntax `TempF(newElement)`:

```
TempF::new(0), TempF(32)
```

```
new(TempF, 0), new(TempF, 32)
```

The `TempF::new` method also ensures that all new elements represent valid temperatures:

```
TempF(-500)
```

```
Error: Temperature is below absolute zero. [TempF::new]
```

The `TempF::new` method requires new elements to be real numbers:

```
TempF(x + 2)
```

```
Error: Expecting a real number above absolute zero. [TempF::new]
```

```
TempF(1 + 2*I)
```

Error: Expecting a real number above absolute zero. [TempF::new]

As a next step, improve the output format for the elements of your new domain. To change the format that MuPAD uses for displaying the elements of a domain, redefine the `print` method for that domain. For example, when displaying elements of the domain `TempF::new`, show temperature measurements followed by the units of measurements (degrees Fahrenheit):

```
TempF::print := proc(TempF)
begin
  expr2text(extop(TempF, 1)).Symbol::deg."F"
end_proc:
```

Now, MuPAD displays the elements of the domain `TempF::new` as follows:

```
TempF(32), TempF(72), TempF(90), TempF(104.8)
```

```
32°F, 72°F, 90°F, 104.8°F
```

Suppose you want to perform additions for the elements of the new domain. By default, arithmetical operations treat the elements of a new domain as identifiers:

```
TempF(75) + TempF(10)
```

```
10°F + 75°F
```

You can implement arithmetical operations for your domain. For example, define addition for the domain `TempF`. The result of addition must also belong to the domain `TempF`:

```
TempF::_plus := proc()
local elements;
begin
  elements:= map([args()], op, 1);
  new(TempF, _plus(op(elements)))
end_proc:
```

The system add the elements of your new domain the same way as it adds numbers. The system also displays the degrees Fahrenheit unit for the resulting sum:

```
TempF(75) + TempF(10)
```

```
85°F
```

For further computations, delete the identifier x:

```
delete x
```

## Access Arguments of a Procedure

The terms *parameters* and *arguments* are often used interchangeably. Strictly speaking, the term *parameters* means object names that you specify while defining a procedure. The term *arguments* means objects that you use while calling a procedure. This documentation also uses the term *formal parameters* for parameters and the term *actual parameters* for arguments.

Many programming languages require the number of parameters to be equal to the number of arguments. MuPAD does not have this requirement. Therefore, you can call a procedure with the number of arguments different from the number of specified parameters. If you call a procedure using fewer arguments than parameters, the system treats additional parameters as additional local variables without specified values. If you call a procedure using more arguments than parameters, the system does not disregard these additional parameters, but enables you to access them.

The `args` function lets you access the arguments of a current procedure call. The call `args(i)`, where `i` is a positive integer, returns the `i`th argument of the current procedure call. The call `args(0)` returns the total number of arguments in the current procedure call. For example, the following procedure computes the sum of its arguments in each procedure call:

```
f := proc() begin
  _plus(args(i) $ i = 1..args(0)):
end_proc:
```

The procedure works for any number of arguments:

```
f(), f(100), f(1, 2, 3, 4, 5),
f(a, b, c), f(x $ x = 1..1000)
```

```
0, 100, 15, a + b + c, 500500
```

Also, you can access the whole sequence of arguments or any subsequence of that sequence. For example, the following procedure prints all arguments used in the current call. If the current call uses three or more arguments, the procedure also prints its first three arguments:

```
g := proc() begin
  print(Unquoted, "all arguments" = args());
  if args(0) > 2 then
    print(Unquoted, "first three arguments" = args(1..3));
  else
    print(Unquoted, "not enough arguments");
  end_if
end_proc:
```

Call the procedure `g` with five arguments:

```
g(10, 20, 30, 40, 50)
```

```
all arguments = (10, 20, 30, 40, 50)
```

```
first three arguments = (10, 20, 30)
```

When you pass arguments to a procedure, MuPAD evaluates these arguments. Then the system creates a local variable for each formal parameter specified in the procedure definition. The system assigns evaluated arguments to these local variables. Parameters outside the procedure do not change. For example, assign a new value to a formal parameter inside the procedure:

```
h := proc(a) begin
  a := b;
  print(args());
end_proc:
```

Assigning a new value to a formal parameter inside a procedure does not affect the parameter itself. This assignment affects the result returned by `args`. For example, if you pass any argument to the procedure `h`, that argument changes to a variable `b` inside the procedure:

```
h(100)
```



*b*

The formal parameter *a* does not change its value outside the procedure:

*a*

*a*

For further computations, delete the procedures *f*, *g*, and *h*:

`delete f, g, h`

## Test Arguments

**In this section...**

“Check Types of Arguments” on page 4-22

“Check Arguments of Individual Procedures” on page 4-24

### Check Types of Arguments

When writing a MuPAD procedure, you can specify the type of arguments accepted by the procedure. To define a procedure that accepts only particular types of arguments, specify the expected types when defining the formal parameters of a procedure. For example, the formal parameter of the following procedure accepts only those arguments that belong to the domain `DOM_INT`:

```
f:= proc(k:DOM_INT)
begin
  sin(PI*k/4)
end_proc:
```

Therefore, only an integer number is a valid first argument for this procedure:

`f(1)`

$$\frac{\sqrt{2}}{2}$$

The system compares the type of the formal parameter `k` and the type of an argument passed to the procedure. If the first argument that you pass to the procedure `f` is not an integer, MuPAD issues an error:

`f(2/3)`

```
Error: The type of argument number 1 must be 'DOM_INT'. The object '2
Evaluating: f
```

During a typical procedure call, for example a call to the `solve` or `int` function, MuPAD internally calls many other procedures. Testing argument types for each internal procedure call is computationally expensive. To provide better performance, MuPAD reduces the amount of type checks in the running code. By default, the system checks the types of arguments only in those procedures that you call interactively. If one procedure internally calls another procedure, MuPAD does not check types of the arguments of the internally called procedure. For example, create the following procedure `g` as a wrapper for the procedure `f`:

```
g := proc(n::Type::Numeric) begin f(n) end_proc:
```

MuPAD performs type checks only for the arguments of the procedure `g`, which you call interactively. It does not perform type checks for the arguments of the procedure `f`:

```
g(2/3)
```

$$\frac{1}{2}$$

The `Pref::typeCheck` function enables you to control type checking of procedure arguments. This function affects all procedures. It does not allow you to control type checking for individual procedures. The default setting of `Pref::typeCheck` is `Interactive`:

```
Pref::typeCheck()
```

`Interactive`

To perform type checks in all procedure calls, including internal calls, set the value of `Pref::typeCheck` to `Always`:

```
Pref::typeCheck(Always):
```

Now, the system realizes that `2/3` is not a valid argument of the internally called procedure `f`:

```
g(2/3)
```

```
Error: The type of argument number 1 must be 'DOM_INT'. The object '2  
Evaluating: f
```

To disable type checks in all procedure calls, including interactive calls, set the value of `Pref::typeCheck` to `None`:

```
Pref::typeCheck(None):
```

Now, the system does not check argument types in any procedure calls:

```
g(2/3), f(2/3)
```

```
 $\frac{1}{2}$ ,  $\frac{1}{2}$ 
```

To restore the default setting of `Pref::typeCheck`, use the `NIL` option:

```
Pref::typeCheck(NIL):
```

## Check Arguments of Individual Procedures

When writing a procedure in MuPAD, you can include your own tests for the procedure arguments inside the procedure itself. You can test types, values, or any other properties of the arguments of a procedure. For example, create the procedure that calculates the arcsine function of a real number. Suppose, you want to limit possible results of this procedure to real numbers. In this case, an input argument of the procedure must belong to the interval  $[-1, 1]$ . To ensure that the procedure accepts only the values from this interval, test the value of an input argument inside the procedure:

```
p := proc(x:Dom::Real)  
begin  
  if abs(x) > 1 then  
    error("invalid number. Choose a value from the interval [-1,1].")  
  end_if;  
  arcsin(x)  
end_proc:
```

Typically, when you call one MuPAD procedure, that procedure internally calls other MuPAD procedures. Some of these internal calls are multiple calls to the same procedure with different sets of arguments. Testing arguments for each internal procedure call can become computationally expensive. By default, the system uses the following general principle for testing arguments of a typical MuPAD procedure:

- If you call a procedure interactively, the procedure performs all argument checks.
- If one procedure internally calls another procedure, the second procedure skips argument checks.

Currently, the procedure `p` always checks whether the value of its argument belongs to the interval  $[-1, 1]$ . To follow the general principle for testing arguments, the procedure must be able to recognize internal and interactive calls, and skip argument checking when a call is internal. For this task, MuPAD provides the `testargs` function. When you call `testargs` inside an interactive procedure call, `testargs` returns the value `TRUE`. For internal procedure calls, `testargs` returns the value `FALSE` by default. For example, rewrite your procedure `p` as follows:

```
p := proc(x)
begin
  if testargs() then
    if abs(x) > 1 then
      error("invalid number. Choose a value from the interval [-1,1].");
    end_if;
  end_if;
  arcsin(x)
end_proc;
```

When you call the procedure `p`, it checks whether the input argument belongs to the specified interval:

```
p(1/2), p(1), p(0)
```

```
 $\frac{\pi}{6}$ ,  $\frac{\pi}{2}$ , 0
```

```
p(10)
```

```
Error: invalid number. Choose a value from the interval [-1,1]. [p]
```

Now, write the simple wrapper procedure `f` that calls the procedure `p`:

```
f := proc(x) begin p(x) end_proc:
```

When the wrapper procedure `f` calls `p`, the procedure `p` does not check its arguments because `testargs` returns the value `FALSE`:

```
f(10)
```

```
arcsin(10)
```

The `testargs` function also allows you to switch to the *argument checking mode*. In this mode, MuPAD checks arguments of all procedures, regardless of how a procedure is called. This mode can slow down your computations. Use this mode only for debugging your code. To switch to the *argument checking mode*, set the value of `testargs` to `TRUE`:

```
testargs(TRUE):
```

In the *argument checking mode*, the procedure `p` checks its argument during interactive and internal calls:

```
p(10)
```

```
Error: invalid number. Choose a value from the interval [-1,1]. [p]
```

```
f(10)
```

```
Error: invalid number. Choose a value from the interval [-1,1]. [p]
```

Always restore `testargs` to its default value `FALSE` after you finish debugging:

`testargs(FALSE)` :

## Verify Options

For many standard MuPAD procedures, you can use different options. If a MuPAD procedure accepts options, it has an embedded mechanism for collecting and verifying these options. For example, the `solve` function accepts the `Real` option. The option indicates that the solver must return only real solutions and accepts the values `TRUE` and `FALSE`. If an option accepts only `TRUE` and `FALSE` values, you can provide the option name without specifying its value:

```
solve(x^4 - 1 = 0, x, Real)
```

```
{-1, 1}
```

If you do not specify the `Real` option, the solver uses the default option value `FALSE`, and returns all complex solutions of the equation:

```
solve(x^4 - 1 = 0, x)
```

```
{-1, 1, -i, i}
```

You can explicitly specify the option-value pair, for example, `Real = TRUE` or `Real = FALSE`:

```
solve(x^4 - 1 = 0, x, Real = TRUE);  
solve(x^4 - 1 = 0, x, Real = FALSE)
```

```
{-1, 1}
```

```
{-1, 1, -i, i}
```

If you provide an unexpected option (for example, if you spell the option name incorrectly), MuPAD issues an error indicating the wrong option. If there are several wrong options, MuPAD indicates the first wrong option:

```
solve(x^4 - 1 = 0, x, Rea, Rel)
```



```
Error: The argument number 3 is invalid.
Evaluating: solvelib::getOptions
```

You can embed the same option checking mechanism in your own procedures. For this task, MuPAD provides the `prog::getOptions` function, which collects and verifies options used during a procedure call. When a user calls your procedure, `prog::getOptions` scans all arguments and returns a table that contains all expected options and their values. It also returns a list of all unexpected options.

When you pass arguments to `prog::getOptions`, always use the following order. The first argument of `prog::getOptions` is the number  $n + 1$ , where  $n$  is the number of required (non-optional) arguments of the procedure. Then you must provide the list of all actual arguments followed by the table of all acceptable options and their default values. To access a sequence of all arguments of a procedure call, including required arguments and options, use the `args` function.

The following example demonstrates the procedure that accepts the numeric coefficients  $a$ ,  $b$ , and  $c$  and solves the quadratic equation  $ax^2 + bx + c = 0$  using these coefficients. The procedure `solveQuadraticEqn` requires the user to provide three numeric values. Therefore, if you embed `prog::getOptions` into this procedure, the first parameter of `prog::getOptions` must be the number 4. The procedure also accepts the optional argument `PositiveOnly`. If the value of `PositiveOnly` is `TRUE`, the procedure returns only positive solutions of the quadratic equation. If the value is `FALSE`, the procedure returns all solutions. The following function call to `prog::getOptions` sets the default option value `PositiveOnly = FALSE`:

```
solveQuadraticEqn := proc(a::Numeric, b::Numeric,
                          c::Numeric)
local options, S;
begin
  options := prog::getOptions(4, [args()],
                              table(PositiveOnly = FALSE));
  S := solve(a*x^2 + b*x + c = 0, x);
  if options[1][PositiveOnly] = TRUE then
    S := select(S, testtype, Type::Positive)
```

```
end_if:  
return(S)  
end_proc:
```

If you call `solveQuadraticEqn` without the `PositiveOnly` option, the procedure returns all solutions of the quadratic equation:

```
solveQuadraticEqn(2, 3, -9)
```

$$\left\{-3, \frac{3}{2}\right\}$$

If you use the `PositiveOnly` option, the procedure returns only positive solutions:

```
solveQuadraticEqn(2, 3, -9, PositiveOnly)
```

$$\left\{\frac{3}{2}\right\}$$

By default, `prog::getOptions` does not error when it finds an unexpected option (an option that is not listed in the table of accepted options). Instead, it collects all unexpected options and returns them in a separate list. Thus, the procedure `solveQuadraticEqn` does not issue an error when you spell the option name incorrectly:

```
solveQuadraticEqn(2, 3, -9, Positive)
```

$$\left\{-3, \frac{3}{2}\right\}$$

The `prog::getOptions` function can issue an error when it finds an unexpected option. In a function call to `prog::getOptions`, the fourth argument indicates whether `prog::getOptions` must silently collect unexpected options or issue an error. This argument is optional. By default, it is set to `FALSE`. To issue an error instead of listing unexpected arguments, use `TRUE` as the fourth argument of `prog::getOptions`:

```
solveQuadraticEqn := proc(a:Type::Numeric, b:Type::Numeric,
```

```

                                                                    c::Type::Numeric)
local options, S;
begin
  options := prog::getOptions(4, [args()],
    table(PositiveOnly = FALSE), TRUE);
  S := solve(a*x^2 + b*x + c = 0, x);
  if options[1][PositiveOnly] = TRUE then
    S := select(S, testtype, Type::Positive)
  end_if;
  return(S)
end_proc;
```

Now, the procedure `solveQuadraticEqn` issues an error. The error message indicates the wrong option:

```
solveQuadraticEqn(2, 3, -9, Positive)
```

```
Error: The argument number 4 is invalid.
Evaluating: solveQuadraticEqn
```

## Debug MuPAD Code in the Tracing Mode

MuPAD provides two alternatives for debugging your code. First, you can use the MuPAD GUI debugger to debug your code interactively, observing execution of the code step by step. Another alternative is to generate a report showing the steps that were taken while executing your code. The mode in which MuPAD generates a report while executing a procedure, domain, method, or function environment is called the *tracing mode*. This mode is helpful for multistep procedures that require a long time to execute. Use this approach to debug a particular procedure, domain, method, or function environment, for which you want to print a report.

When debugging, you also can use the `print` and `fprint` functions for printing intermediate results produced by your code.

Suppose, you want to create a procedure that computes the Lucas numbers. The Lucas numbers are a sequence of integers. The recursion formula that defines the  $n$ th Lucas number is similar to the definition of the Fibonacci numbers:

$$L_0 = 1, L_1 = 3, L_{n+2} = L_n + L_{n+1}.$$

Although MuPAD does not provide a function that computes the Lucas numbers, writing your own procedure for this task is easy:

```
lucas := proc(n:Type::PosInt)
begin
  if n = 1 then
    1
  elif n = 2 then
    3
  else
    lucas(n - 1) + lucas(n - 2)
  end_if
end_proc:
```

The procedure call `lucas(n)` returns the  $n$ th Lucas number. For example, display the first 10 Lucas numbers:

```
lucas(n) $ n = 1..10
```

```
1, 3, 4, 7, 11, 18, 29, 47, 76, 123
```

Suppose you want to trace this procedure. To switch execution of a particular procedure, domain, method, or function environment to the tracing mode, use the `prog::trace` function. For example, to trace the `lucas` procedure, enter:

```
prog::trace(lucas):
```

Now, if you call the `lucas` procedure, the trace mechanism observes every step of the procedure call and generates the report for that call:

```
lucas(5)
```

```
enter lucas(5)
  enter lucas(4)
    enter lucas(3)
      enter lucas(2)
        computed 3
        enter lucas(1)
          computed 1
        computed 4
      enter lucas(2)
        computed 3
      computed 7
    enter lucas(3)
      enter lucas(2)
        computed 3
        enter lucas(1)
          computed 1
        computed 4
      computed 11
```

```
11
```

The `prog::traced()` function call returns the names of all currently traced procedures, domains, methods, and function environments:

```
prog::traced()
```

```
[lucas]
```

By using different options of the `prog::trace` function, you can customize generated reports. Most of these options are independent of a particular procedure call. They affect all reports generated after you use an option. See the `prog::trace` help page for more details.

If a procedure uses many nested procedure calls, the generated report for that procedure can be very long. To limit the number of nested procedure calls in a report, use the `Depth` option of `prog::trace`:

```
prog::trace(Depth = 2):
```

```
lucas(5)
```

```
enter lucas(5)
  enter lucas(4)
    computed 7
  enter lucas(3)
    computed 4
  computed 11
```

```
11
```

The `Depth` option affects all reports generated for further calls to procedures, domains, methods, and function environments. If you do not want to use this option for further calls, set its value to 0. The value 0 indicates that `prog::trace` must display all nested calls:

```
prog::trace(Depth = 0):
```

To display memory usage in each step of the procedure call, use the `Mem` option:

```
prog::trace(Mem):
```

```
lucas(5)
```

```
enter lucas(5) [mem: 5636064]
  enter lucas(4) [mem: 5636544]
    enter lucas(3) [mem: 5636944]
      enter lucas(2) [mem: 5637344]
        computed 3 [mem: 5637060]
        enter lucas(1) [mem: 5637424]
          computed 1 [mem: 5637140]
        computed 4 [mem: 5636756]
      enter lucas(2) [mem: 5637120]
        computed 3 [mem: 5636836]
      computed 7 [mem: 5636356]
    enter lucas(3) [mem: 5636720]
      enter lucas(2) [mem: 5637120]
        computed 3 [mem: 5636836]
      enter lucas(1) [mem: 5637200]
        computed 1 [mem: 5636916]
      computed 4 [mem: 5636532]
    computed 11 [mem: 5636052]
```

11

To stop using the Mem option for further calls, set its value to FALSE:

```
prog::trace(Mem = FALSE):
```

To stop tracing calls to the lucas procedure, use the prog::untrace function:

```
prog::untrace():
```

## Display Progress

By default, MuPAD procedures do not show progress information or comments on run time. For example, create the following procedure that returns the sign of an input number. (MuPAD provides the standard function `sign` for this task.)

```
S := proc(z:Type::Numeric)
begin
  if not(testtype(z, Dom::Real)) then
    z/abs(z)
  elif z > 0 then
    1
  elif z < 0 then
    -1
  else
    0
  end_if
end_proc:
```

When you execute this procedure, it returns only the final result:

```
S(10)
```

```
1
```

Typically, the final result is all that your users want to see. However, if executing a procedure takes a long time or if users can benefit from the comments on some procedure steps, you can extend the procedure to include additional information. To embed the progress information into your procedure, use the `print` function. For example, modify the procedure `S` so it reports its progress:

```
S := proc(z:Type::Numeric)
begin
  print(Unquoted, "Is ".expr2text(z)." a real number?");
  if not(testtype(z, Dom::Real)) then
    print(Unquoted, expr2text(z)." is a complex number. Computing the
    z/abs(z);
```



```
else
  print(Unquoted, expr2text(z)." is a real number");
  print(Unquoted, "Is ".expr2text(z)." a positive number?");
  if z > 0 then
    print(Unquoted, expr2text(z)." is a positive number");
    1
  else
    print(Unquoted, expr2text(z)." is not a positive number");
    print(Unquoted, "Is ".expr2text(z)." a negative number?");
    if z < 0 then
      print(Unquoted, expr2text(z)." is a negative number");
      -1
    else
      print(Unquoted, expr2text(z)." is not a negative number");
      print(Unquoted, expr2text(z)." is zero.");
      0
    end_if
  end_if
end_if
end_proc:
```

Now the procedure S displays the status messages:

```
S(0)
```

```
Is 0 a real number?
```

```
0 is a real number
```

```
Is 0 a positive number?
```

```
0 is not a positive number
```

Is 0 a negative number?

0 is not a negative number

0 is zero.

0

## Use Assertions

If you rely on correctness of particular statements for your code, then consider including these statements in the code. Such statements are called *assertions*. Assertions help you remember specific conditions under which you expected to execute your code. They can also help other developers who might need to review or update your code.

MuPAD lets you use *checked assertions*. If you switch to a special mode of executing your code, the system evaluates assertions during run time. If an assertion does not evaluate to TRUE, the system stops code execution and throws an error.

For example, this procedure solves the equation  $\sin(x) + \cos(x) = a^2$ . Suppose you get the parameter  $a$  as a result of some computations, and you expect the condition  $a^2 \leq \sqrt{2}$  to be always valid. Relying on this condition, you expect the solutions to be real. Specify this condition as an assertion by using `assert`:

```
f := proc(a)
begin
  assert(a^2 <= sqrt(2));
  s := solve(sin(x) + cos(x) = a^2)
end:
```

Assertions are checked only when you run your code in a special mode called the *argument checking mode*. Otherwise, the system ignores all assertions. For example, in this procedure call, MuPAD skips the assertion and returns the following complex result:

```
f(4/3)
```

$$(x) \in \left\{ \left( -\ln\left(\frac{8}{9} - \sigma_1 + \frac{8i}{9}\right) i + 2 \pi k \right) \mid k \in \mathbb{Z} \right\} \cup \left\{ \left( -\ln\left(\sigma_1 + \frac{8}{9} + \frac{8i}{9}\right) i + 2 \pi k \right) \mid k \in \mathbb{Z} \right\}$$

where

$$\sigma_1 = \frac{(-1)^{1/4} \sqrt{47}}{9}$$

To switch to the argument checking mode, set the value of `testargs` to `TRUE`:

`testargs(TRUE)`:

Now when you call the procedure `f`, MuPAD checks the assertion. For `a = 4/3`, the assertion evaluates to `FALSE`, and the procedure execution stops with the error:

`f(4/3)`

`Error: Assertion 'a^2 <= sqrt(2)' has failed. [f]`

For `a = 1`, the assertion evaluates to `TRUE`. The procedure call runs to completion, and returns the following set of real solutions:

`f(1)`

$$(x) \in \{(2 \pi k) \mid k \in \mathbb{Z}\} \cup \left\{ \left( \frac{\pi}{2} + 2 \pi k \right) \mid k \in \mathbb{Z} \right\}$$

The argument checking mode can slow down your computations. Use this mode only for debugging your code. Always restore `testargs` to its default value `FALSE` after you finish debugging:

`testargs(FALSE)`:

## Write Error and Warning Messages

When writing a procedure in MuPAD, you can include error and warning messages in your procedure. If the system encounters an error while executing a procedure, it terminates execution of the procedure and displays an error message. If the system encounters a warning, it prints the warning message and continues executing the procedure. You can specify your own error and warning messages.

The error function terminates execution of a current procedure with an error. For example, the following procedure converts the number of a month to the name of a month. The procedure checks the type of its argument and accepts only positive integer numbers. Since there are only 12 months in a year, the procedure must also ensure that the number does not exceed 12. The error function lets you terminate the procedure call with an appropriate error message if the number is greater than 12:

```
monthNumberToName := proc(n:Type::PosInt)
begin
  if n > 12 then
    error("Invalid number. The number must not exceed 12.")
  end_if;
  case n
  of 1 do return(January)
  of 2 do return(February)
  of 3 do return(March)
  of 4 do return(April)
  of 5 do return(May)
  of 6 do return(June)
  of 7 do return(July)
  of 8 do return(August)
  of 9 do return(September)
  of 10 do return(October)
  of 11 do return(November)
  of 12 do return(December)
  end_case;
end:
```

If you call `monthNumberToName` with any integer from 1 to 12, the procedure returns the name of a month:

```
monthNumberToName(12)
```

```
December
```

If you call `monthNumberToName` with an integer greater than 12, the procedure terminates with the specified error:

```
monthNumberToName(13)
```

```
Error: Invalid number. The number must not exceed 12. [monthNumberToName]
```

Warning messages help you inform your users about potential problems in the algorithm. These problems are typically minor and do not interfere with the execution of a procedure. For example, you can warn your users about limited functionality of a procedure or about implicit assumptions made by a procedure.

The following procedure uses the `simplify` function to simplify the fraction. The function implicitly assumes that the variable `x` is not equal to `a`. If your procedure uses the `simplify` function, you can add the following warning for your users:

```
simplifyW := proc(a)
begin
    warning("Assuming x <> ".expr2text(a));
    simplify((x^2 - a^2)/(x - a))
end:
```

Now, your procedure informs its users about the assumption:

```
simplifyW(10)
```

```
Warning: Assuming x <> 10 [simplifyW]
```

```
x + 10
```

## Handle Errors

Typically, when a MuPAD procedure encounters an error caused by evaluation of an object, the procedure terminates, the system returns to an interactive level and displays an error message. If you want to avoid terminating a procedure, use the `traperror` function to catch an error.

The `traperror` function returns the error number instead of the error itself. Later, you can reproduce the error by using the `lasterror` function. This function produces the last error that occurred in the current MuPAD session. If you call `lasterror` inside a statement or a procedure, it terminates that statement or procedure. If you want to display the error message without throwing the error itself, save the error number returned by `traperror`, and then retrieve the message by using the `getlasterror` function.

For example, the function `f` throws an error when  $\sin(\pi k)$  is equal to zero:

```
f := (k) -> 1/sin(PI*k)
```

$$k \rightarrow \frac{1}{\sin(\pi k)}$$

```
f(1)
```

```
Error: Division by zero. [_invert]
Evaluating: f
```

Suppose you want to compute `f` for the values  $-\pi \leq k \leq \pi$  increasing the value of `k` by  $\frac{\pi}{4}$  in each step. To call the function `f` for all required values, create the `for` loop. When you try to execute this statement, it terminates as soon as the function `f` encounters division by zero for the first time. To avoid terminating the `for` statement, use the `traperror` function to catch the error. To display the text of that error message without interrupting execution of the `for` statement, use `getlasterror()`:

```
for k from -1 to 1 step 1/4 do
  err := traperror(f(k));
```

```
if err = 0 then
    print(Unquoted, "k = ".expr2text(k), "f = ".expr2text(f(k)))
else
    print(Unquoted, "k = ".expr2text(k), getlasterror())
end_if
end_for
```

```
k = -1, [1025, Error: Division by zero. [_invert]]
    Evaluating: f
```

```
k = -3/4, f = -2^(1/2)
```

```
k = -1/2, f = -1
```

```
k = -1/4, f = -2^(1/2)
```

```
k = 0, [1025, Error: Division by zero. [_invert]]
    Evaluating: f
```

```
k = 1/4, f = 2^(1/2)
```

```
k = 1/2, f = 1
```

```
k = 3/4, f = 2^(1/2)
```



```
k = 1, [1025, Error: Division by zero. [_invert]]
      Evaluating: f
```

For errors created with the `error` function, including your custom error messages, `traperror` always returns the error code 1028. If an error message with the code 1028 is the last error message that occurred in a MuPAD session, you can retrieve that error message by using `lasterror` or `getlasterror`. For example, create the procedure `g` that computes the factorial of any number less than 10. If you pass a number greater than 10 to this procedure, the procedure reduces the number to 10, and then computes the factorial:

```
g := proc(n::Type::PosInt)
local test;
begin
  test := proc()
  begin
    if n > 10 then
      error("The number must not exceed 10.")
    end_if
  end_proc:
  if traperror(test(n)) <> 0 then
    g(n - 1)
  else
    n!
  end_if;
end_proc:
```

Call the procedure with the number 100. During run time, the procedure encounters the error, but does not terminate. It also does not display the error message. Instead, it returns the result:

```
g(100)

3628800
```

To retrieve and display the error message caught during execution of the procedure `g`, use the `lasterror` function:

```
lasterror()
```

```
Error: The number must not exceed 10. [test]
```

To display the error code and the text of that message, use the `getlasterror` function:

```
g(100):  
getlasterror()
```

```
[1028, "Error: The number must not exceed 10. [test]"]
```

## When to Analyze Performance

Symbolic computations can be very time consuming. For many tasks, MuPAD provides its own, optimized functions. When you use these functions, the system typically chooses algorithms that provide the best performance for your task.

However, in some cases you might want to measure and investigate your code performance. This task is also called *profiling*. For example, profiling is helpful when you:

- Feel that your code runs too slowly.
- Want to estimate the performance of a particular computation algorithm.
- Want to compare performances of different algorithms.
- Define and use your own data types and methods.

MuPAD provides tools to measure the running time of a particular code snippet or the whole MuPAD session. The system also provides the profiling tool to help you find and eliminate performance bottlenecks in your code.

## Measure Time

In this section...
“Calls to MuPAD Processes” on page 4-48
“Calls to External Processes” on page 4-51

### Calls to MuPAD Processes

The simplest tool you can use to estimate code performance is the `time` function. This function measures the running time of your code snippet. The `time` function does not count the time spent outside of MuPAD processes: external function calls or processes running in the background do not affect the results of `time`. The function returns results measured in milliseconds.

The following example demonstrates different algorithms implemented for the same task. The task is to check whether each integer from 1 to 1000 appears in a 1000×1000 matrix of random integers. To create a 1000×1000 matrix of random integers, use `linalg::randomMatrix`:

```
matrixSize := 1000:  
M := linalg::randomMatrix(matrixSize, matrixSize, Dom::Integer):
```

The direct approach is to write a procedure that checks every element of a matrix proceeding row by row and column by column:

```
f := proc(M, n, x)  
begin  
  for j from 1 to n do  
    for k from 1 to n do  
      if M[j, k] = x then  
        return(TRUE)  
      end_if  
    end_for  
  end_for;  
  return(FALSE)  
end_proc:
```

Call the procedure `f` 1000 times to check if each number from 1 to 1000 appears in that matrix:

```
g := proc()
begin
  f(M, matrixSize, i) $ i = 1..1000
end_proc:
```

This algorithm is very inefficient for the specified task. The function `f` performs  $10^4$  computation steps to find out that an integer does not occur in the matrix `M`. Since you call the function `f` 1000 times, executing this algorithm takes a long time:

```
time(g())

62435.902
```

In this example, the bottleneck of the chosen approach is obviously the algorithm that accesses each matrix element. To accelerate the computation, rewrite the procedure `f` using the bisection method. Before using this method, convert the matrix `M` to a list and sort the list. Then select the first and last elements of the sorted list as initial points. Each step in this algorithm divides the list of elements at the midpoint:

```
f := proc(M, n, x)
begin
  if (M[1] - x)*(M[n] - x) > 0 then
    return(FALSE)
  elif (M[1] - x)*(M[n] - x) = 0 then
    return(TRUE);
  else
    a := 1: b := n:
    while (b - a > 1) do
      if is(b - a, Type::Odd) then
        c := a + (b - a + 1)/2
      else
        c := a + (b - a)/2
      end_if;
      if M[c] - x = 0 then
        return(TRUE)
      end_if;
    end_while;
  end_if;
end_proc;
```

```
        elif (M[a] - x)*(M[c] - x) < 0 then
            b := c:
        else
            a := c:
        end_if;
    end_while;
end_if;
return(FALSE)
end_proc:
```

Use the `op` function to access all elements of the matrix `M`. This function returns a sequence of elements. Use brackets to convert this sequence to a list. Then use the `sort` function to sort the list in ascending order. Finally, call the procedure `f` for each integer from 1 to 1000:

```
g := proc()
local M1;
begin
    M1 := sort([op(M)]):
    f(M1, matrixSize^2, i) $ i = 1..1000
end_proc:
```

Using the bisection method instead of accessing each matrix element significantly improves the performance of the example:

```
time(g())

3724.233
```

Typically, the best approach is to use the appropriate MuPAD functions whenever possible. For example, to improve performance further, rewrite the code using the MuPAD function `has`. Also, converting a matrix to a set can reduce the number of elements. (MuPAD removes duplicate elements of a set.) In addition to speed up, this approach makes your code significantly shorter and easier to read:

```
g := proc()
local M1;
begin
    M1 := {op(M)}:
```

```
has(M1, i) $ i = 1..1000  
end_proc:
```

In this case, execution time is even shorter than for the code that implements the bisectional method:

```
time(g())  
  
1508.094
```

## Calls to External Processes

Results returned by the `time` function exclude the time spent on calls to external programs. If your code uses external programs, you can measure the total time spent by that code, including calls to external processes. To measure the total time, use the `rtime` function instead of `time`. For example, the function call `rtime()` returns the elapsed time of the current MuPAD session. This time includes idle time of the current session:

```
t := rtime():  
print(Unquoted, "This session runtime is ").stringlib::formatTime(t)  
  
This session runtime is 5 minutes, 25.579 seconds
```

When measuring code performance using `rtime`, avoid running other processes in the background. Also ensure that enough memory is available. The `rtime` function counts the total time, including idle time during which some other process uses the CPU or your computer swaps data between different types of memory.

## Profile Your Code

*Profiling* is a way to measure where a program spends time. MuPAD provides the `prog::profile` function to help you identify performance bottlenecks in your code. Use this function to analyze performance of complicated nested procedure calls. For simpler code snippets, measuring execution times is more convenient and can give you enough information to identify performance bottlenecks.

The `prog::profile` function evaluates the code that you want to profile, and returns the profiling report. The report contains:

- A table showing the time your code spent on each function (total and averaged per one call), number of calls to that function, children of that function, and so on. Information for the most heavily used functions appear on the top of the table. The first row in this table represents the total execution time.
- A dependence graph showing a function itself, the functions it calls (children), and the functions that call this function. The graph also shows the timing and number of calls for each function.

---

**Note** By default, `prog::profile` does not measure performance of single calls to kernel functions.

---

However, when `prog::profile` measures the performance of library functions, it also prints the accumulated time the system spends in kernel functions. To measure the performance of a single call to a kernel function, use `prog::trace` to trace that kernel function.

Suppose you want to write a procedure that checks whether each integer from 1 to 1000 appears in a  $1000 \times 1000$  matrix of random integers. The direct approach is to write a procedure that checks every element of a matrix proceeding row by row and column by column:

```
f := proc(M, n, x)
begin
  for j from 1 to n do
```



```

        for k from 1 to n do
            if M[j, k] = x then
                return(TRUE)
            end_if
        end_for
    end_for;
    return(FALSE)
end_proc:

```

Use the `linalg::randomMatrix` to create a 1000×1000 matrix of random integers:

```

matrixSize := 1000:
M := linalg::randomMatrix(matrixSize, matrixSize, Dom::Integer):

```

Then call the procedure `f` 1000 times to check if each number from 1 to 1000 appears in that matrix:

```

g := proc()
begin
    f(M, matrixSize, i) $ i = 1..1000
end_proc:

```

Measuring the time needed to run the procedure `g` shows that the procedure requires optimization. Although the performance bottleneck in this procedure is obvious, it is not always easy to identify performance bottlenecks in more complicated procedures. The `time` function does not indicate where the procedure spends most of its execution time:

```

time(g())

62023.876

```

To obtain the complete profiling report that shows timings for all inner function calls, use `prog::profile`:

```

prog::profile(g()):

percent usage of all

```

```

|         time self per single call
|         |         time self
call |         |         time children per single
|         |         |         time children
|         |         |         |         |         calls/normal exit
|         |         |         |         |         |         calls/remember exit
|         |         |         |         |         |         |         |         calls/e
|         |         |         |         |         |         |         |         |         [index] function name
-----
100.0   109982.9   109982.9   .
.       1 . . [0] procedure entry point
-----

45.9   .   50467.2   .
27149.6 3019825 . . [1] (Dom::Matrix(Dom::Integer))::_index_inte

19.7   .   21689.3
.       5460.3 3019825 . . [2] Dom::Integer::coerce

16.7
.       18373.1 .       77616.9 3019825 . . [3] (Dom::Matr

```

```

12.7      14.0      13984.8
   96.0      95990.0      1000 . . [4] f

5.0
.      5460.3      .      .      3019825 . . [5] Dom::Int

.
8.0      8.0      109974.9      109974.9      1 . . [6] g

```

```

-----

index
%time      self children      called [index] name

```

```

-----

[0]      100.0 109982.8      0
1 procedure entry point

```

```

      8.0 109974.8      1      [6] g

```

```

-----

      50467.24 27149.65      3019825
[3] (Dom::Matrix(Dom::Integer))::_index

[1]      45.9 50467.24 27149.65      3019825 (Dom::Matrix(Dom::Int

      21689.30 5460.346      3019825
[2] Dom::Integer::coerce

```

```

                21689.30  5460.346    3019825
[1] (Dom::Matrix(Dom::Integer))::_index_intern

[2]      19.7  21689.30  5460.346    3019825
Dom::Integer::coerce

                5460.346          0    3019825    [5] Dom::Integer::convert
-----

                18373.13  77616.89    3019825
[4] f

[3]      16.7  18373.13
77616.89    3019825 (Dom::Matrix(Dom::Integer))::_index

                50467.24  27149.65    3019825
[1] (Dom::Matrix(Dom::Integer))::_index_intern
-----

                13984.84  95990.02          1000
[6] g

[4]      12.7  13984.84
95990.02          1000  f

                18373.13  77616.89    3019825    [3] (Dom::Matrix(Dom::Int
-----

                5460.346          0    3019825
[2] Dom::Integer::coerce

[5]      5.0  5460.346          0    3019825  Dom::Integer::convert
-----

```

```

[6]      0.0      8.0 109974.8
1 g

          13984.84
95990.02 1000 [4] f

```

---

Time sum: 109982.873 ms

Top rows of the profiling report indicate that the procedure spends most of its time accessing each matrix element. To improve performance, rewrite the procedure so that it can access fewer elements of a matrix in each call to the procedure `f`. For example, use the algorithm based on the bisection method:

```

f := proc(M, n, x)
begin
  if (M[1] - x)*(M[n] - x) > 0 then
    return(FALSE)
  elif (M[1] - x)*(M[n] - x) = 0 then
    return(TRUE);
  else
    a := 1: b := n:
    while (b - a > 1) do
      if is(b - a, Type::Odd) then
        c := a + (b - a + 1)/2
      else
        c := a + (b - a)/2
      end_if;
      if M[c] - x = 0 then
        return(TRUE)
      elif (M[a] - x)*(M[c] - x) < 0 then
        b := c:
      else
        a := c:
      end_if;
    end_while;
  end_if;
  return(FALSE)
end_proc:

```





```

. 9981 . [8] is . .

```

---

The recommended approach for improving performance of your code is to use the MuPAD functions when possible. For example, MuPAD provides the `has` function for checking whether one MuPAD object contains another MuPAD object. Rewrite your code using the `has` function and combining the procedures `f` and `g`:

```

g := proc()
local M1;
begin
  M1 := {op(M)}:
  has(M1, i) $ i = 1..1000
end_proc:

```

This procedure also converts the matrix `M` to a set. Converting a matrix to a set can reduce the number of elements. (MuPAD removes duplicate elements of a set.)

The execution time for the procedure call `g()` is the shortest among the three implementations:

```

time(g())

1520.095

```

The profiling report shows that the procedure spends most of its execution time accessing the 1000×1000 matrix of random integers and converting it to a set. This example shows only the top of the profiling report:

```

prog::profile(g()):

percent usage of all
|   time self per single call

```



```

|          |          time self
call |          |          time children per single
|          |          |          time children
|          |          |          |          calls/normal exit
|          |          |          |          |          calls/remember exit
|          |          |          |          |          |          calls/errors
|          |          |          |          |          |          |          [index] function name

```

```

-----
100.0  1556.1  1556.1      .      .
  1 . . [0] procedure entry point

```

```

-----
78.9  1228.1  1228.1  188.0  188.0
  1 . . [1] (Dom::Matrix(Dom::Integer))::op

  9.0  140.0  140.0  1416.1  1416.1  1 .
. [2] g

  8.2   0.1  128.0      .      .  1000 .
. [3] `p -> [coeff(p, All)][2..-1]`

  3.9   0.1   60.0      .      .  1000 .
. [4] `l -> l.[Rzero $ r - nops(l)]`

. . . . . 2 .

```

. [5] Dom::Integer::hasProp

. [6] DomainConstructor::hasProp . 2 .

-----

## Techniques for Improving Performance

For most symbolic and numeric computation tasks, MuPAD implements the fastest and most reliable currently known algorithms. Among these algorithms, the system always tries to find the best algorithm for your particular computation task. Often, the system also allows you to choose an algorithm explicitly or implicitly. For example, you can create a sequence by using the sequence generator \$ or the for loop. Also, you can choose particular solvers, and simplification functions. Such choices can affect the performance of computations.

These techniques can accelerate your computations in MuPAD:

- Use built-in MuPAD data types and functions when possible. Typically, these functions are optimized to handle your computation tasks faster and smoother.
- Set assumptions on parameters when possible. Use the assumptions of variables sparingly or avoid them completely. For details about how assumptions affect performance, see *When to Use Assumptions*.
- Call special solvers directly instead of using general solvers. If you can determine the type of an equation or system that you want to solve, calling the special solver for that equation or system type is more efficient. See *Choosing a Solver*.
- Call numeric solvers directly if you know that a particular problem cannot be solved symbolically. This technique has a significant disadvantage: for nonpolynomial equations numeric solvers return only the first solution that they find.
- Try using options. Many MuPAD functions accept options that let the system reduce computation efforts. For information about the options of a particular MuPAD function, see the “Options” section of the function help page.
- Limit complexity of the expressions that you use.
- Use shorter data structures when possible. For example, converting a sequence with  $10^6$  entries to a list takes longer than converting 1000 sequences with 1000 entries each.

- Avoid creating large symbolic matrices and dense matrices when possible. For details about improving performance when working with matrices, see [Using Sparse and Dense Matrices](#).
- Avoid using `for` loops to create a sequence, a flat list, a string and similar data structures by appending new entries. Instead, use the sequence generator `$`.
- Use `for` loops as outer loops when creating deep nested structures. Use the sequence generator `$` for inner loops.
- Use the remember mechanism if you call a procedure with the same arguments more than once. The remember mechanism lets you avoid unnecessary reevaluations. See [Remember Mechanism](#). At the same time, avoid using the remember mechanism for nonrecurring procedure calls, especially if the arguments are numerical.
- Avoid storing lots of data in the history table. If you suspect that the history table uses a significant amount of memory, clear the history table or reset the engine. For information about the history table, see [History Mechanism](#).
- Avoid running large background processes, including additional MuPAD sessions, at the same time as you execute code in MuPAD.

## Display Memory Usage

### In this section...

“Use the Status Bar” on page 4-65

“Generate Memory Usage Reports Periodically” on page 4-66

“Generate Memory Usage Reports for Procedure Calls” on page 4-67

### Use the Status Bar

The amount of memory available on your computer can greatly affect your symbolic computations. First, some computations cannot run without enough memory. In this case, you get the “out of memory” error. Second, if the MuPAD engine uses virtual memory by swapping data on and off the storage device, computations can run much slower than they run if the system does not use virtual memory. Observing memory usage when executing your code can help you understand whether your code uses available memory efficiently.

The simplest tool for observing the memory usage is the status bar. You can find the status bar at the bottom of a MuPAD notebook. If you do not see the status bar, select **View > Status Bar**. The far left end of the status bar displays the current engine state, including memory and time used during the most recent computations. While a computation is still running, the status bar information keeps changing.



If the engine is not connected to your notebook, the status bar displays **Not Connected**. For more information about the status bar, see Viewing Status Information.

---

**Note** When you perform computations in several MuPAD notebooks, each notebook starts its own engine. In this case, watch for the total amount of memory used by all MuPAD engines (the mupkern.exe processes).

---

## Generate Memory Usage Reports Periodically

When a computation takes a long time to run, it can be helpful to display progress information. In this case, MuPAD can issue periodic messages showing active memory usage, reserved memory, and evaluation time. You can control the frequency with which MuPAD prints such messages.

To set the frequency of these periodic messages, use the `Pref::report` function. By default, the value of `Pref::report` is 0; MuPAD does not print periodic status messages. If you increase the value to 1, MuPAD prints status messages approximately every hour. (The exact frequency depends on your machine.) The maximum value accepted by `Pref::report` is 9.

Suppose you want to generate and sort the list of 10,000,000 random integer numbers. These operations take a long time because of the huge number of elements. If you set the value of `Pref::report` to 4, MuPAD displays a few status messages while executing these operations:

```
Pref::report(4):
```

```
sort([random() $ i = 1..10^7]):
```

```
[used=167852k, reserved=168579k, seconds=30]  
[used=294614k, reserved=295370k, seconds=60] [used=421376k, reserved=  
seconds=90]
```

If you increase the value of `Pref::report` to 6, MuPAD prints the status messages more frequently:

```
Pref::report(6):
```

```
sort([random() $ i = 1..10^7]):
```

```
[used=84035k, reserved=84661k, seconds=10]  
[used=126987k, reserved=127664k, seconds=21] [used=169940k, reserved=  
seconds=32] [used=212892k, reserved=213537k, seconds=43] [used=255844  
reserved=256540k, seconds=54] [used=298797k, reserved=299476k, second  
[used=341749k, reserved=342413k, seconds=76] [used=384701k, reserved=  
seconds=87] [used=427654k, reserved=428352k, seconds=98] [used=470606  
reserved=471355k, seconds=109]
```

Every time you execute this example, MuPAD adds a new list of  $10^7$  random numbers and stores that list in the history table. By default, the history table contains up to 20 elements. While this list remains in the history table, MuPAD cannot release the memory needed to store  $10^7$  integers. To release this memory, use one of these alternatives:

- Continue computations waiting until MuPAD writes 20 new elements to the history table. Performing computations with a reduced amount of available memory can be very slow.
- Terminate the MuPAD engine connected to the notebook by selecting **Notebook > Disconnect**. The new engine starts when you evaluate any command in the notebook.
- Clear the history table by setting the value of variable `HISTORY` to 0. This variable specifies the maximum number of elements in the history table. To restore the default value of `HISTORY`, enter `delete HISTORY`:

```
HISTORY := 0:
delete HISTORY:
HISTORY
```

```
20
```

For more information about the history mechanism in MuPAD, see [History Mechanism](#).

For further computations, also restore the default value of `Pref::report`:

```
Pref::report(NIL):
```

## Generate Memory Usage Reports for Procedure Calls

MuPAD can print memory usage information when you execute a procedure in the tracing mode. In this case, the system reports memory usage on each step of a procedure call.

For example, create the recursive procedure `juggler` that computes the Juggler number sequence for any initial positive integer `n`:

```
juggler := proc(n:Type::PosInt)
```

```
begin
  J := append(J, n);
  if n = 1 then
    return(J)
  end_if;
  if testtype(n, Type::Even) then
    juggler(floor(n^(1/2)))
  else
    juggler(floor(n^(3/2)))
  end_if
end_proc;
```

Suppose you want to see the memory usage report for every call to this procedure. First call the `prog::trace` function with the `Mem` option. Then switch execution of the `juggler` procedure to the tracing mode:

```
prog::trace(Mem);
prog::trace(juggler)
```

Now when you call the `juggler` procedure, the tracing report shows the memory usage for each call to `juggler`:

```
J := [];
juggler(7)
```

```
enter
juggler(7) [mem: 5338408]   enter juggler(18) [mem: 5373600]   ente
juggler(4) [mem: 5374080]   enter juggler(2) [mem: 5374584]
  enter juggler(1) [mem: 5375064]   computed [7, 18, 4,
2, 1] [mem: 5375032]   computed [7, 18, 4, 2, 1] [mem: 5374648]
  computed [7, 18, 4, 2, 1] [mem: 5374264]   computed [7, 18, 4,
2, 1] [mem: 5373880] computed [7, 18, 4, 2, 1] [mem: 5373524]
```

The `Mem` option is independent of the traced procedure. Now if you use `prog::trace` to trace any other procedure, `prog::trace` displays memory usage in every step of that procedure. Remove this global option for further computations:

```
prog::trace(Mem = FALSE)
```



To stop tracing the `juggler` procedure, use the `prog::untrace` function:

```
prog::untrace(juggler):
```

## Remember Mechanism

### In this section...

“Why Use the Remember Mechanism” on page 4-70

“Remember Results Without Context” on page 4-72

“Remember Results and Context” on page 4-73

“Clear Remember Tables” on page 4-74

“Potential Problems Related to the Remember Mechanism” on page 4-77

### Why Use the Remember Mechanism

If your code calls a procedure with the same arguments more than once, avoid unnecessary reevaluations, and thus, improve performance. Instead of multiple evaluations of a procedure call with the same arguments, MuPAD can store the results of the first procedure call in a special table. This table is called the *remember table*. The system stores the arguments of a procedure call as indices of the remember table entries, and the corresponding results as values of these entries. When you call a procedure using the same arguments as in previous calls, MuPAD accesses the remember table of that procedure. If the remember table contains the entry with the required arguments, MuPAD returns the value of that entry. Otherwise, MuPAD evaluates the procedure call, and writes the arguments and corresponding results to the remember table of the procedure.

Using the remember mechanism in MuPAD can significantly accelerate your computations, especially when you use recursive procedure calls. For example, create the procedure that computes the Lucas numbers. The Lucas numbers are a sequence of integers. The recursion formula that defines the  $n$ th Lucas number is similar to the definition of the Fibonacci numbers:

$$L_0 = 1, L_1 = 3, L_{n+2} = L_n + L_{n+1}.$$

The following recursive procedure returns any Lucas number:

```
lucas := proc(n:Type::PosInt)
begin
```

```

if n = 1 then
  1
elif n = 2 then
  3
else
  lucas(n - 1) + lucas(n - 2)
end_if
end_proc:

```

However, if the value  $n$  is large, computing the  $n$ th Lucas number can be very slow. The number of required procedure calls is exponential. Often, the procedure calls itself with the same arguments, and it reevaluates the result in every call:

```
time(lucas(35))
```

```
66156.25
```

Using the remember mechanism eliminates these reevaluations. To enable the remember mechanism for a particular procedure, use the `prog::remember` function. This function returns a modified copy of a procedure that stores results of previous calls in the remember table:

```
lucas := prog::remember(lucas):
```

When you call this procedure, MuPAD accesses the remember table. If the system finds the required entry in the remember table, it returns remembered results immediately. Now, MuPAD computes the 35th and even the 100th Lucas number almost instantly:

```
time(lucas(35)), time(lucas(100))
```

```
0.0, 0.0
```

Alternatively, you can enable the remember mechanism for a particular procedure by using the option `remember` for that procedure. For example, use the option `remember` to enable the remember mechanism for the procedure `lucas`:

```
lucas:= proc(n>Type::PosInt)
  option remember;
begin
  if n = 1 then
    1
  elif n = 2 then
    3
  else
    lucas(n - 1) + lucas(n - 2)
  end_if
end_proc:
```

For further computations, delete the procedure lucas:

```
delete lucas:
```

## Remember Results Without Context

By default, the remember mechanism does not consider context information of a procedure call. Thus, the remember mechanism disregards any changes in assumptions set on the arguments of a procedure call and the number of digits used for floating-point arithmetic. By default, remember tables contain only arguments and results of procedure calls. They do not store context information. For example, create the function `f` that computes the reciprocal of a number. Use `prog::remember` to enable the remember mechanism for this function:

```
f := (x)-> 1.0/x:
f := prog::remember(f):
```

The default number of significant digits for floating-point numbers is 10. Use the function `f` to compute the reciprocal of 3. The system displays the result with the 10-digits accuracy:

```
f(3)

0.3333333333
```

Now increase the number of digits to 50. Then call the function `f` with the argument 3 again. By default, MuPAD does not realize that you increased





```
f := prog::remember(f):
```

Now compute the Heaviside function for the values -10, 0, and 10. MuPAD uses the value `heaviside(0)=1/2`:

```
f(-10), f(0), f(10)
```

```
0,  $\frac{1}{2}$ , 1
```

You can define a different value for `heaviside(0)`. First, use the `unprotect` function to be able to overwrite the value of `heaviside`. Then, assign the new value to `heaviside(0)`:

```
unprotect(heaviside):
heaviside(0) := 0:
```

Despite the new value `heaviside(0) = 0`, the wrapper procedure `f` returns the old value `1/2`:

```
f(0)
```

```
 $\frac{1}{2}$ 
```

The result of the procedure call `f(0)` does not change because the system does not reevaluate this result. It finds the result in the remember table of the procedure `f` and returns that result. To display the content of the remember table, call the wrapper procedure `f` with the `Remember` option as a first argument and the `Print` option as a second argument. The value  $10^6$  in the second column is the value of `MAXEFFORT` used during computations.

```
f(Remember, Print)
```

$[f(0), 0]$	$[\frac{1}{2}, 1000000.0]$
$[f(-10), 0]$	$[0, 1000000.0]$
$[f(10), 0]$	$[1, 1000000.0]$

To force reevaluation of the procedure calls of `f`, clear the remember table of that procedure. To clear the remember table, call `f` with the `Remember` option as a first argument and the `Clear` option as a second argument:

```
f(Remember, Clear):
```

Now `f` returns the correct result:

```
f(0)
```

```
0
```

If you use the option `remember`, you also can clear the remember table and force reevaluation. For example, rewrite the procedure `f` as follows:

```
f := proc(x)
  option remember;
begin
  heaviside(x)
end:
f(0)
```

```
0
```

Now restore the `heaviside` function to its default definition:

```
heaviside(0) := 1/2:
```

To clear a remember table created by the option `remember`, use the `forget` function:



```
forget(f):  
f(0)
```

 $\frac{1}{2}$ 

Use the `protect` function with the `ProtectLevelError` option to prevent further changes to `heaviside`. Also, delete the procedure `f`:

```
protect(heaviside, ProtectLevelError):  
delete f
```

## Potential Problems Related to the Remember Mechanism

The remember mechanism is a powerful tool for improving performance of MuPAD procedures. Nevertheless, you can encounter some problems when using this mechanism:

- Remember tables are efficient only if the access time of the remember table is significantly less than the time needed to evaluate the result. If a remember table is very large, evaluation can be computationally cheaper than accessing the result stored in the remember table.
- Storing large remember tables requires a large amount of memory. Especially, remember tables created with the option `remember` can grow very large, and significantly reduce available memory. The number of entries in remember tables created by `prog::remember` is limited. When the number of entries in a remember table created by `prog::remember` reaches the maximum number, the system removes a group of older entries.
- Using `prog::remember` or the option `remember` for nonrecurring procedure calls can significantly decrease code performance. Avoid using the remember mechanism for nonrecurring procedure calls, especially if the arguments are numerical.
- If you change the properties of input arguments or modify the variables `DIGITS` or `ORDER`, the remember mechanism ignores these changes by default. See Remembering Results Without Context.
- In some cases you must clear the remember table of a procedure to enforce reevaluation and avoid incorrect results. For example, clearing

the remember table can be necessary when a procedure changes global variables or if global variables affect the results of a procedure. See [Clearing Remember Tables](#).

- Many predefined MuPAD functions have special values stored in their remember tables. Therefore, clearing the remember tables of predefined MuPAD functions is not recommended. Note that the `forget` function does not error when you call it for a predefined MuPAD function.

## History Mechanism

### In this section...

“Access the History Table” on page 4-79

“Specify Maximum Number of Entries” on page 4-82

“Clear the History Table” on page 4-83

### Access the History Table

MuPAD implements the internal *history mechanism*. The history mechanism lets you access a limited number of previously computed results, with or without the commands that generated the results. The history mechanism also helps you reduce the number of additional identifiers commonly used for storing the results of intermediate computations. Instead of assigning results of such computations to auxiliary identifiers, you can access the entries of the history table and get the previously computed result.

To access the entries of the history table, use the `last` and `history` functions. The `last` function returns previously computed results without the command that generated the results. The `history` function returns the previously computed results along with the commands that generated those results.

For the `last` function, MuPAD also provides the shortcut `%`. The function call `last(n)` (or `% n`) returns the  $n$ th entry of the history table. The `last` function counts the entries of the history table from the end of the table. Thus, when you use the `last` function, the most recent result is the first entry of the history table.

For example, compute the factorials of the numbers 10, 20, and 30:

```
10!; 20!; 30!;
```

```
3628800
```

```
2432902008176640000
```

```
26525285981219105863630848000000
```

To access the computed factorial of 30, use the function call `last(3)` or the shorter call `%3`:

```
last(3)
```

```
3628800
```

---

**Note** When you call the `last` or `history` function, MuPAD adds the result of that call to the history table.

---

Calling the `last` function or its shortcut `%` inserts a new entry in the history table. Thus, the history table now contains the results of the following evaluations: `10!`, `20!`, `30!`, and `%3` (which in this example is equal to `10!`). If you call the `last` function with the argument 3 again, the system displays the result of evaluation of `20!`:

```
last(3)
```

```
2432902008176640000
```

To access the most recent entry of the history table, you can use the shortcut `%` without parameters. For example, solve the following equation, and then simplify the result. Note that using `%` lets you avoid assigning the result of a solution to an identifier:

```
solve(log(2, x) + log(2, 5) = x + 5, x);
```

```

$$\left\{ -\frac{W_0\left(-2^{5-\log_2(5)} \ln(2)\right)}{\ln(2)} \right\}$$

```

```
simplify(%)
```

$$\left\{ \begin{array}{l} W_0\left(-\frac{32 \ln(2)}{2^{\log_2(5)}}\right) \\ -\frac{\ln(2)}{\ln(2)} \end{array} \right\}$$

The `last` function does not evaluate results. The `last` function also returns the results for which you used colons to suppress the outputs:

```
hold(2 + 2):
%
```

```
2+2
```

The `history` function displays both the result and the command that produced that result. This function counts the entries of the history table from the first result obtained in the current MuPAD session. Thus, when you use the `history` function, the most recent result is the last entry of the history table. In this section, the first entry of the history table is the computation of the factorial of 10:

```
history(1)
```

```
[10!, 3628800]
```

To find the current number of entries in the history table, call the `history` function without an argument:

```
history()
```

```
10
```

You can use the `history` function to access the most recent computation and its result:

```
a := 2:
history(history())
```

```
[a := 2, 2]
```

For the following statements, the history mechanism depends on whether you call the statement interactively or within a procedure:

- for, repeat, and while loops
- if and case conditional statements
- procedure definitions

These statements are called compound statements. At the interactive level, MuPAD stores compound statements as one unit. In procedures, MuPAD stores the statements found within a compound statement in a separate history table of the procedure. In this case, the system does not store the compound statement itself.

### Specify Maximum Number of Entries

By default, the history table for interactive computations can contain up to 20 entries. When the number of entries reaches 20, the system removes the oldest entry from the history table every time it needs to add a new entry. For interactive computations, the environment variable HISTORY determines the maximum number of entries in the history table:

```
HISTORY
```

```
20
```

To change the maximum number of entries in the history table for the current MuPAD session, assign the new value to HISTORY:

```
HISTORY := 2:
```

Now MuPAD stores only the two most recent commands and their results in the history table:

```
a := 1: b := 2: c := 3:  
%1, %2;  
%3
```

3, 2

Error: The argument is invalid. [last]

---

**Note** Within a procedure, the maximum number of entries in the local history table of the procedure is always 3, independent of the value of HISTORY.

---

For further computations, restore the default maximum number entries in the history table:

```
delete HISTORY:  
HISTORY
```

20

## Clear the History Table

Although MuPAD restricts the number of entries in the history table, it does not restrict its size. If a command returns a large result, the system stores that result in the history table. While this result remains in the history table, MuPAD cannot release the memory needed to store that result. One or more large entries in the history table can significantly reduce available memory and slow down further computations. If you know that a particular command returns memory-consuming results, avoid writing that command and its results to the history table. To avoid writing a command and its result to the history table, set the value of HISTORY to 0. The disadvantage of this approach is that you delete all previous results from the history table.

For example, set the value of HISTORY to 0 before creating a sequence of 1,000,000 random numbers:

```
HISTORY := 0:  
random() $ i = 1..10^6:
```

For further computations, restore the default maximum number entries in the history table:

```
delete HISTORY:  
HISTORY
```

```
20
```

If the history table already contains a memory-consuming result, to release the memory you also can clear the history table by setting the value of HISTORY to 0. Alternatively, you can wait until the MuPAD fills the history table with new entries. Also, you can select **Notebook > Disconnect** to restart the MuPAD engine.



## Why Test Your Code

After you debug and optimize your code, you might still need to test it. Debugging lets you catch run-time errors that appear in your code. Testing lets you catch bugs that appear when users provide unexpected combinations of input arguments, or when they run your code on different platforms or MuPAD versions. It also helps to catch bugs that can appear when you or someone else edits your code later or when you need to integrate parts of the program written by different developers. MuPAD provides tools for unit testing your code. These tools let you write and execute your own test scripts for a particular part of your code, for example, for a function or a procedure.

Suppose you create the procedure that accepts two numeric arguments, `a` and `b`, compares them, and returns the larger number:

```
f := proc(a::Numeric, b::Numeric)
begin
  if a = b or a > b then
    return(a)
  else
    return(b)
  end_if
end_proc:
```

The type `Type::Numeric` includes integers, rationals, floating-point numbers, and also complex numbers. Therefore, any complex numbers are valid arguments of the procedure `f`. The procedure `f` has a flaw in its design because you cannot compare two complex numbers. If you call this procedure for two different complex numbers, the procedure call results in an error. Nevertheless, the procedure works if you call it for equal complex numbers:

```
f(I, I)

i
```

Suppose you decide to keep the procedure `f` as it is. It works for equal complex arguments. The error only occurs when the complex arguments are not equal. Later, you or somebody else forgets about the issue with complex numbers, but sees that the procedure can be improved as follows:

```
f := proc(a::Numeric, b::Numeric)
begin
  if a >= b then
    return(a)
  else
    return(b)
  end_if
end_proc:
```

This code looks shorter, and takes advantage of the `>=` operator. However, if some users relied on the procedure `f` to recognize equal complex numbers, their code breaks when they use the updated version:

```
f(I, I)
```

```
Error: Cannot evaluate to Boolean. [_leequal]
Evaluating: f
```

If you do not create and use a test script for the procedure `f`, you might never realize that the procedure stopped working for the particular choices of arguments. Even if you tested this choice of arguments before, you might forget to test it for the updated version. Writing a test script and running it every time when you (or somebody else) update your code helps to avoid unexpected loss in functionality.

## Write Single Tests

The `prog::test` function is the basic testing tool in MuPAD. This function compares the actual result of computations with the expected result that you specify. For example, create the procedure `f`:

```
f := proc(a::Type::Numeric, b::Type::Numeric)
begin
  if a = b or a > b then
    return(a)
  else
    return(b)
  end_if
end_proc:
```

To test the procedure, use the `prog::test` function. If the test does not reveal any problems, `prog::test` returns the void object `null()` and does not print anything:

```
prog::test(f(I, I), I)
```

If the procedure call tested by `prog::test` errors or if actual results differ from expected results, `prog::test` prints information about the test execution. For example, if your test compares two different complex numbers, `prog::test` returns the following message:

```
prog::test(f(2*I, I), I)
```

```
Error in test 2
```

```
Input: f(2*I, I)
```

```
Expected: I
```

```
Got:      TrapError = [1003, "Error: Cannot evaluate to Boolean. [_1
```

```
Near line: 1
```

If the error is expected, you can rewrite the test using the `TrapError` option:

```
prog::test(f(2*I, I), TrapError = 1003)
```

When you call `prog::test`, MuPAD evaluates actual and expected results before comparing them:

```
prog::test(f(x^2 | x = 2, 5), 2*2)
```

```
Error in test 4
```

```
Input: f(x^2 | x = 2, 5)
```

```
Expected: 4
```

```
Got:      5
```

```
Near line: 1
```

Evaluation of actual and expected results can take a long time. To avoid long evaluations, the `prog::test` function lets you specify the time limit for evaluation of the test. To limit the evaluation time for a particular test, use the `Timeout` option of the `prog::test` function. For example, set the time limit to 2 seconds:

```
prog::test(f([i! $ i = 1..1000000], [i! $ i = 1..1000000]),  
           [i! $ i = 1..1000000], Timeout = 2)
```

```
Error in test interactive
```

```
5
```

```
Input: f([i! $ i = 1..100000],  
        [i! $ i = 1..100001])
```

```
Expected:  
FAIL
```

```
Got:      TrapError = [1320,  
"Error: Execution time exceeded"]
```

```
Timeout:  
2.0 (5.106*prog::ntime())
```

In this example, the time limit measurement depends on your hardware configuration. The test report also shows the hardware-independent time in terms of the `prog::ntime` function.

By default, `prog::test` tests the strict equality between actual and expected results. Testing equality of floating-point values can be confusing when the display precision differs from the internal precision. In this case, different floating-point numbers can look identical. Thus, with the default values of `DIGITS` and `Pref::outputDigits`, the floating-point approximation of  $1/3$  and the number `0.3333333333` look identical:

```
prog::test(float(1/3), 0.3333333333)
```

```
Error in test 5
```

```
Input: float(1/3)
```

```
Expected: 0.3333333333
```

```
Got:      0.3333333333
```

```
Near line: 1
```

Internally, MuPAD uses more than 10 digits to approximate  $1/3$  with the floating-point number. The system adds guard digits for increased precision. To see how many guard digits the system uses, increase the number of output digits using the `Pref::outputDigits` function. Then, test the equality of the numbers again:

```
Pref::outputDigits(20):  
prog::test(float(1/3), 0.3333333333)
```

```
Error in test 6
```

```
Input: float(1/3)
```

```
Expected: 0.3333333333
```

```
Got:      0.3333333333333333333304
```

### Near line: 2

When you test equality of floating-point numbers, it can be helpful to test the approximate equality. The approximate equality operator in MuPAD is `~='`. The corresponding function is `_approx`. The `prog::test` function lets you choose the method for comparing actual and expected results. For example,  $1/3$  is approximately equal to `0.3333333333` within the default 10-digits precision:

```
prog::test(float(1/3), 0.3333333333, Method= `~='`)
```

Also, using the `Method` option lets you specify more than one acceptable solution. For example, if you randomly pick one solution of the following equation, you can get any of its four valid solutions:

```
i := random(1..4):
prog::test(solve(x^4 - 16 = 0, x)[i()],
           {-2, 2, -2*I, 2*I}, Method= _in)
```

For further computations, restore the default output precision:

```
Pref::outputDigits(UseDigits):
```

## Write Test Scripts

If you write your code in collaboration with other developers or intend to extend or update it later, you need to test your code more than once. Testing a part of code every time when you or somebody else change it helps you ensure that this part works properly. When you plan to test some part of code repeatedly, it is helpful to write a test script and execute it every time when you need to test the code. For example, create the following procedure `f`. This procedure accepts two numeric arguments, `a` and `b`, compares them, and returns the larger number:

```
f := proc(a::Numeric, b::Numeric)
begin
  if a = b or a > b then
    return(a)
  else
    return(b)
  end_if
end_proc:
```

Suppose you will likely update this procedure in the future. To ensure that the procedure works as expected after all possible updates, write the test script and execute it after any update. The test script in MuPAD includes a set of single tests created by the `prog::test` function. See [Writing Single Tests](#) for more information.

Each test script starts with the `prog::testinit` function and ends with the `prog::testexit` function. To specify the name of the tested procedure, use `print(Unquoted, "testname")` after `prog::testinit`. This name does not affect the tested procedure itself. It only appears in the test reports generated by your test script.

To test the procedure `f` for different choices of parameters, write the following test script and save it to the `test-f.tst` file. The test does not find any unexpected results or errors. After MuPAD executes the test script, it generates the test report. The test script does not require any particular file extension. You can use any file extension that you like. The test report shows the number of executes tests, the number of errors encountered while executing the test script, and the time and memory used to execute the test script:

```
//test-f.tst
prog::testinit("f");
print(Unquoted, "function f that compares two numbers")

prog::test(f(1, 1), 1):
prog::test(f(1, 2), 2):
prog::test(f(2, 1), 2):
prog::test(f(100, 0.01), 100):
prog::test(f(0.01, 100), 100):
prog::test(f(-10, 10), 10):
prog::test(f(2*I, 2*I), 2*I):
prog::test(f(2 + I, 2 + I), 2 + I):
prog::test(error(f(2 + I, 3 + I)), TrapError=1003):
prog::test(error(f(x, y)), TrapError=1202):
prog::test(error(f(x, x)), TrapError=1202):

prog::testexit(f)

Info:
11 tests, 0 errors, runtime factor 0.0 (nothing expected)

Info: CPU time: 12.7 s

Info: Memory allocation 20452460 bytes [prog::testexit]
```

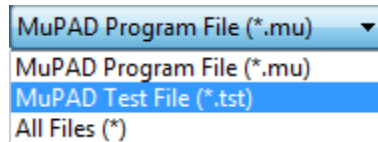
If you change the original procedure `f`, run the test script to catch any unexpected results:

```
f := proc(a::Type::Numeric, b::Type::Numeric)
begin
  if a >= b then
    return(a)
  else
    return(b)
  end_if
end_proc:
```

You do not need to copy the test script to the notebook. Instead, you can execute the test script that you saved to a file without opening the file. To execute a test script:



- 1 Select **Notebook > Read Commands** to open the Read Commands dialog box.
- 2 Change the file filter to show MuPAD test files or all files.



- 3 Navigate to the test file that contains the script and click **OK**.

Alternatively, use the `READPATH` variable to specify the path to the folder that contains the file. Then use the `read` function to find and execute the test file. For example, if you saved the test file `test-f.tst` in the folder `C:/MuPADfiles/TestScripts`, use the following commands:

```
READPATH := "C:/MuPADfiles/TestScripts":
read("test-f.tst")
```

```
Error
in test function f that compares two numbers 7
```

```
Input: f(2*I, 2*I)
```

```
Expected:
2*I
```

```
Got:      TrapError = [1003,
"Error: Can't evaluate to boolean [_leequal];\r\n Evaluating: f"]
```

```
Near line: 9
```

```
Error in test function f that compares two
numbers 8
```

```
Input: f(2 + I, 2 + I)
```

```
Expected: 2 + I
```

```
Got:      TrapError = [1003, "Error: Can't evaluate to boolean [_leequal]
          Evaluating: f"]
```

```
Near line: 10
```

```
Info: 11
tests, 2 errors, runtime factor 0.0 (nothing expected)
```

```
Info: CPU time: 12.7 s
```

```
Info: Memory allocation 20452460 bytes [prog::testexit]
```

Although the change seems reasonable and safe, the test report shows that the procedure does not work for equal complex numbers anymore. Instead, the procedure throws an error. If you do not test the code, you can miss this change in procedure behavior. If this behavior is expected, correct the test script. Otherwise, correct the procedure.

## Code Verification

Even if your code executes without errors, and all your tests run without failures, the code can still have some flaws. For example, it can:

- Modify global variables, protected identifiers, environment variables, and formal parameters.
- Declare local variables or formal parameters and not use them afterwards.
- Contain undefined entries of domains or domain interfaces.

To ensure that your code does not introduce such flaws, use the `prog::check` function to verify it. Use this function to check your procedures, domains, and function environments. Suppose you wrote the following procedure:

```
f := proc(x, n)
  local a, b, c;
begin
  a := m; b := a;
  if x > 0 then
    x := level(b, 2)
  else
    x := -level(b, 2)
  end_if;
end:
```

When you call this procedure, it does not error:

```
f(42, 24)
```

```
m
```

To check `f` for common programming flaws, use `prog::check`. When calling `prog::check`, you can specify how detailed the report must be. This setting is called the information level of the report. The second argument controls the information level of the report generated by `prog::check`. Use options to see specific flaws or call `prog::check` without options to see all common flaws that MuPAD finds in the procedure. For the procedure `f`, `prog::check` with the information level 3 reports these flaws:

```
prog::check(f, 3)
```

```
Critical usage of 'level' on local variable '
```

```
Critical usage of 'level' on local variable '
```

```
Global idents: {m} in [f]
```

```
Unused local variables: {c} in [f]
```

```
Function 'level' applied to variables: {b} in [f]
```

```
Warnings: 3 [f]
```

For the list of all available options, see the `prog::check` help page.

## Protect Function and Option Names

The names of the built-in MuPAD functions and options are protected. If you try to assign a value to a MuPAD function or option, the system throws an error. This approach ensures that you will not overwrite a built-in function or option accidentally.

If you create a new MuPAD procedure, it is recommended to protect the procedure and all its options, especially if you often use that procedure. For example, MuPAD does not provide a function for computing Lucas numbers. You can write your own procedure for computing Lucas numbers, and then protect the procedure name.

The Lucas numbers are a sequence of integers. The recursion formula that defines the  $n$ th Lucas number is similar to the definition of the Fibonacci numbers:

$$L_0 = 1, L_1 = 3, L_{n+2} = L_n + L_{n+1}.$$

Create the following procedure that computes the  $n$ th Lucas number:

```
lucas := proc(n:Type::PosInt)
  option remember;
begin
  if n = 1 then
    1
  elif n = 2 then
    3
  else
    lucas(n - 1) + lucas(n - 2)
  end_if
end_proc:
lucas(i) $ i = 1..5
```

1, 3, 4, 7, 11

Now protect the procedure name, `lucas`, using `protect` with the `ProtectLevelError` option:

```
protect(lucas, ProtectLevelError):
```

ProtectLevelError lets you set full protection for the identifier. Now, trying to assign any value to lucas results in error:

```
lucas := 0
```

```
Error: The identifier 'lucas' is protected. [_assign]
```

Alternatively, you can use the ProtectLevelWarning option. In this case, you can still assign a value to the protected identifier, but a warning appears, for example:

```
protect(lucas, ProtectLevelWarning):
```

You can assign any value to lucas now, but such assignment triggers a warning:

```
lucas := 0
```

```
Warning: The protected variable 'lucas' is overwritten. [_assign]
```

```
0
```

For further computations, remove protection from the identifier lucas:

```
unprotect(lucas):
```

## Data Collection

### In this section...

“Parallel Collection” on page 4-99

“Fixed-Length Collection” on page 4-102

“Known-Maximum-Length Collection” on page 4-103

“Unknown-Maximum-Length Collection” on page 4-104

### Parallel Collection

Suppose the data that you want to collect is generated element-by-element and you know in advance how many elements will be generated. The intuitive approach for collecting such data is to create an empty list and append each new element to the end of the list. For example, this procedure uses this approach to collect random integers generated by `random`:

```
col :=  
proc(n)  
  local L, i;  
begin  
  L := [];  
  for i from 1 to n do  
    L := L.[random()];  
  end_for;  
end:
```

The procedure generates random integers and collects them in a list:

```
col(5)
```

```
[427419669081, 321110693270, 343633073697, 474256143563, 558458718976]
```

To estimate the performance of this approach, use the procedure `col` to generate a list of 50,000 random numbers:

```
time(col(50000))
```

9828.063

The `time` function returns results measured in milliseconds.

Now, check how much time the procedure actually spends generating random numbers:

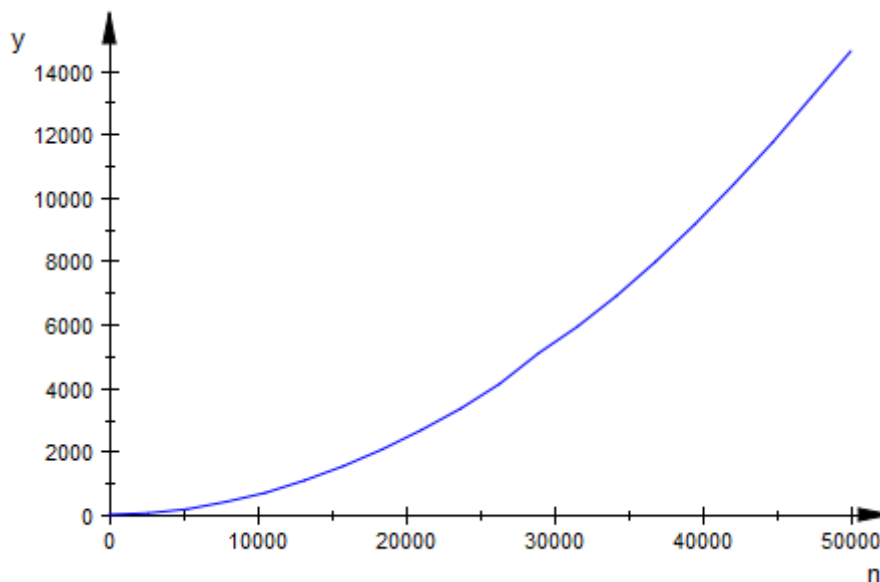
```
time(random() $ i = 1..50000)
```

312.02

Thus, the procedure spends most of the time appending the newly generated numbers to a list. In MuPAD, appending a new entry to a list of  $n$  entries takes time proportional to  $n$ . Therefore, run time of `col(n)` is proportional to  $n^2$ . You can visualize this dependency by plotting the times that `col(n)` spends when creating lists of 1 to 50,000 entries:

```
plotfunc2d(n -> time(col(n)), n = 1..50000,  
           Mesh = 20, AdaptiveMesh = 0)
```





When appending a new entry to a list, MuPAD allocates space for the new, longer list. Then it copies all entries of the old list plus a new entry to this new list. The faster approach is to create the entire list at once, without adding each new entry separately. This approach is called *parallel collection* because you create and collect data simultaneously. Use the sequence operator `$` to implement this approach:

```
col := proc(n)
  local i;
begin
  [random() $ i = 1..n];
end:
```

This procedure spends most of its time generating random numbers:

```
time(col(50000))
```

```
312.03
```

## Fixed-Length Collection

Suppose you know how many elements you will generate, but you cannot generate them all at once. In this case, the best strategy is to create a list of the required length filling it with some constant, such as 0 or NIL. Then you can replace any entry of this list with the generated value. In this case, you do not need to generate elements in the order in which you want them to appear in the list.

For example, use this procedure to generate the list of the first  $n$  Lucas numbers. The procedure creates a list of  $n$  entries, where each entry is 0. Then it assigns the values to the first two entries. To replace all other entries of the list with the Lucas numbers, the procedure uses the `for` loop:

```
lucas :=  
proc(n)  
  local L, i;  
begin  
  L := [0 $ n];  
  L[1] := 1;  
  L[2] := 3;  
  for i from 3 to n do  
    L[i] := L[i-1] + L[i-2];  
  end_for;  
  L  
end:
```

Measure the time needed to create a list of 10,000 Lucas numbers:

```
time(lucas(10000))
```

```
62.4004
```

If you use the procedure that creates an empty list and appends each generated Lucas number to this list, then creating a list of 10,000 Lucas numbers takes much longer:

```
lucas :=  
proc(n)  
  local L, i;
```

```

begin
  L := [];
  L := L.[1];
  L := L.[3];
  for i from 3 to n do
    L := L.[L[i-1] + L[i-2]];
  end_for;
  L
end:

```

```
time(lucas(10000))
```

```
421.2027
```

## Known-Maximum-Length Collection

If you cannot predict the number of elements that you will generate, but have a reasonable upper limit on this number, use this strategy:

- 1 Create a list with the number of entries equal to or greater than the upper limit.
- 2 Generate the data and populate the list.
- 3 Discard the unused part of the list.

For example, use the following procedure to create a list. The entries of this list are modular squares of a number  $a$  ( $a^2 \bmod n$ ). You cannot predict the number of entries in the resulting list because it depends on the parameters  $a$  and  $n$ . Nevertheless, you can see that in this case the number of entries in the list cannot exceed  $n$ :

```

modPowers :=
proc(a, n)
  local L, i;
begin
  L := [0 $ n];
  L[1] := a;
  L[2] := a^2 mod n;
  i := 2;

```

```
while L[i] <> a do
    L[i + 1] := a*L[i] mod n;
    i := i + 1;
end_while;
L := L[1..i - 1];
end:
```

When you call `modPowers` for `a = 3` and `a = 2`, it creates two lists of different lengths:

```
modPowers(3, 14);
modPowers(2, 14)
```

```
[3, 9, 13, 11, 5, 1]
```

```
[2, 4, 8]
```

### Unknown-Maximum-Length Collection

Often, you cannot predict the number of elements and cannot estimate the upper limit on this number before you start generating actual data. One way of dealing with this problem is to choose some upper limit, and use the strategy described in Known Maximum Length Collection. If that limit is reached, then:

- 1 Choose a larger limit.
- 2 Create a new list with the number of elements corresponding to the new limit.
- 3 Copy existing collected data to the new list.

Typically, increasing the list length by a constant factor results in better performance than increasing it by a constant number of entries:

```
rndUntil42 :=
proc()
    local L, i;
begin
    i := 1;
```

```

L := [random()];
while L[i] mod 42 <> 0 do
  if i = nops(L) then
    L := L.L;
  end_if;
  i := i+1;
  L[i] := random();
end_while;
L[1..i];
end:

```

```

SEED := 123456789:
rndUntil42()

```

```

[900763287358, 105114186275, 873298641118, 648455050747, 234588728784, 92685732612,
986862526343, 604554099645, 670181021599, 362027535788, 418959155834, 993175789760,
149559951993, 321479945512, 428426485680, 895682234176, 447822957763, 41591770379,
80666499751, 874251833750, 992824008008, 119331599241, 55782166770, 716820986241,
814491390573, 864609020354, 219333389722, 422468038876, 665245836081, 381920617859,
300482942701, 159259847066, 589167468787, 999975884731, 405723681406, 785149905992,
770372676363, 739767410836, 909396123045, 713889761414, 242358421991, 526049482068,
209286390536, 38034535073, 730171970247, 627506313885, 915109441004, 131065916285,
948970540311, 807086241612]

```

```

SEED := 123456789:
time(rndUntil42() $ i = 1..500)

```

220.013

Alternatively, if you cannot predict the number of elements that you will need to collect, then use a table that grows automatically (a hash table):

```

rndUntil42 :=
proc()
  local L, i, j;
begin
  i := 1;
  L := table(1 = random());
  while L[i] mod 42 <> 0 do

```

```
        i := i+1;
        L[i] := random();
    end_while;
    [L[j] $ j=1..i];
end:

SEED := 123456789:
time(rndUntil42() $ i = 1..500)
```

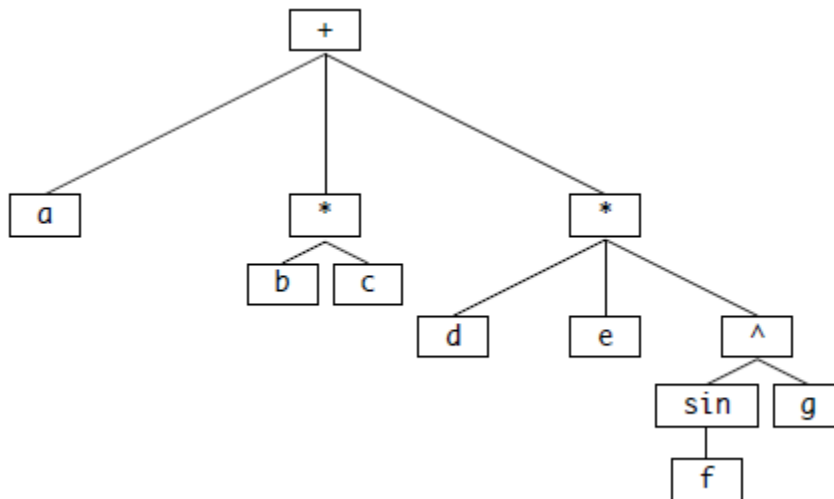
216.014

For this example, using the table is slightly faster. If you change the value 42 to another value, using the list might be faster. In general, tables are preferable when you collect large amounts of data. Choose the approach that works best for solving your problem.

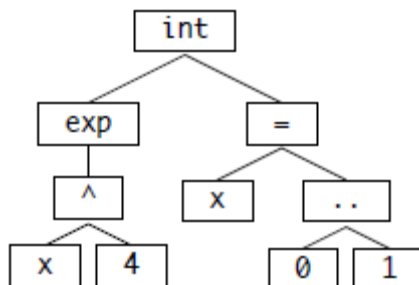
## Visualize Expression Trees

A useful model for visualizing a MuPAD expression is the *expression tree*. It reflects the internal representation of an expression. The operators or their corresponding functions are the vertices, and the arguments are sub-trees. The lowest precedence operator is always at the root of an expression tree.

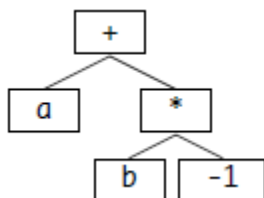
For example, you can represent the expression  $a + b * c + d * e * \sin(f)^g$  using this expression tree.



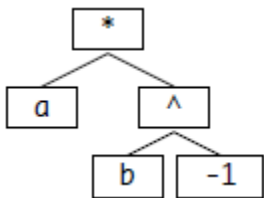
The following expression tree represents the expression  $\text{int}(\exp(x^4), x = 0..1)$ .



MuPAD internally represents the difference  $a - b$  as  $a + b * (-1)$ . Therefore, MuPAD represents the difference using this expression tree.



Similarly, a quotient  $a/b$  has the internal representation  $a * b^{(-1)}$ .



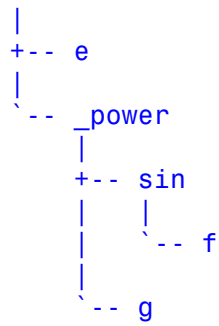
To display expression trees in a MuPAD notebook, use the `prog::exptree` function. It replaces operators with the names of the corresponding system functions:

```
prog::exptree(a + b * c + d * e * sin(f)^g):
```

```

_plus
|
+-- a
|
+-- _mult
|   |
|   +-- b
|   |
|   -- c
|
-- _mult
|
+-- d
  
```





## Modify Subexpressions

### In this section...

“Find and Replace Subexpressions” on page 4-110

“Recursive Substitution” on page 4-113

### Find and Replace Subexpressions

Operations on symbolic expressions often involve finding subexpressions and replacing them with values or with other subexpressions. MuPAD provides the following functions for this task:

- `evalAt` and its shortcut `|`
- `subs`
- `subsex`
- `subsop`
- `prog::find`

`evalAt` replaces specified objects in the expression tree with the specified values or subexpressions, and then evaluates the expression. This function replaces only entire branches of expression trees. It cannot replace a part of the branch, for example, two terms in the sum of three terms:

```
a*(b + c) | b = d,  
a*(b + c) | b + c = d,  
a*(b + c + 1) | b + c = d
```

```
a(c+d), a d, a(b+c+1)
```

`evalAt` replaces only free identifiers:

```
int(f(x), x = 0..1) + f(x) | x = 100
```

$$f(100) + \int_0^1 f(x) dx$$

`subs` replaces specified objects in the expression tree with the specified values or subexpressions. This function cannot replace a part of the branch in the expression tree:

```
subs(a*(b + c), b + c = d),
subs(a*(b + c + 1), b + c = d)
```

$a d, a(b+c+1)$

After substitution, `subs` does not evaluate the resulting expression (although it can simplify the expression). You can enforce evaluation of the modified subexpressions by using the `EvalChanges` option:

```
subs(ln(x), x = E),
subs(ln(x), x = E, EvalChanges)
```

$\ln(e), 1$

`subs` replaces both free and dependent identifiers. In some cases, replacing dependent identifiers can cause invalid results:

```
subs(int(f(x), x = 0..1) + f(x), x = 100)
```

$$f(100) + \int_0^1 f(100) d100$$

`subsex` analyzes associative system operators and can replace part of the branch in the expression tree:

```
subsex(a*(b + c), b + c = d),
subsex(a*(b + c + 1), b + c = d)
```

$a d, a (d+1)$

subsex does not evaluate the resulting expression:

`subsex(ln(x + a + 1), x + a = E - 1)`

$\ln(e)$

Use `eval` to evaluate results returned by `subsex`:

`eval(%)`

1

`subsop` replaces only entire branches in the expression tree of an expression, the same way as `subs`. When using `subsop`, you must know the position (index) of the branch inside the expression in internal order that might differ from the output order used to represent the expression on screen. To find the internal index of a particular subexpression, use the `op` function:

`ex := sin(a*x + b) + cos(a*x + b):`  
`op(ex);`

$\cos(b + a x), \sin(b + a x)$

`op(ex, 2);`

$\sin(b + a x)$

`op(ex, [2, 1]);`

$b + a x$

`op(ex, [2, 1, 2]);`

$ax$ `op(ex, [2, 1, 2, 1])` $a$ 

Now you can use `subsop` to replace the parameter `a` with some value. For example, replace it with the value 3:

`subsop(ex, [2, 1, 2, 1] = 3)` $\sin(b + 3x) + \cos(b + ax)$ 

`prog::find` helps you find all occurrences of a specific value in the expression. For example, find all sums in this expression:

```
ex := (x + 1)/(x^2 + 2*x - 2) - 1/x + 1/(x + 1):
pos := [prog::find(ex, hold(_plus))];
map(pos, p -> op(ex, p));
map(pos, p -> op(ex, p[1..-2]))
```

[[0], [1, 1, 0], [2, 1, 0], [2, 2, 1, 0]]

[\_plus, \_plus, \_plus, \_plus]

$$\left[ \frac{1}{x+1} + \frac{x+1}{x^2+2x-2} - \frac{1}{x}, x+1, x+1, x^2+2x-2 \right]$$

## Recursive Substitution

You also can find all subexpressions of a particular type (for example, all Bessel functions or all branches not containing `x`), execute some code for these subexpressions and replace them with the return value of that code. For this task, use the `misc::maprec` function.

Suppose you want to rewrite all terms that contain the sine and tangent functions in terms of cosine functions. (In this example, do not use  $\sin(x)^2 = 1 - \cos(x)^2$  and similar identities.) First, create the functions `sin2cos` and `tan2cos` that rewrite expressions in terms of the cosine function. These functions access the operands of the sine and tangent functions using `op(ex)`:

```
sin2cos := ex -> cos(op(ex) - PI/2):
tan2cos := ex -> cos(op(ex) - PI/2)/cos(op(ex)):
```

Now you can use these functions when replacing all occurrences of sine and tangent functions in an expression. To replace subexpressions of the original expression, use `misc::maprec`. The `misc::maprec` function uses the syntax `misc::maprec(expression, selector = replacement)`, where:

- `expression` is the original expression (subexpressions of which you want to replace).
- `selector` is the selection criterion (for example, a set of types of subexpressions that you want to replace).
- `replacement` is the procedure that you want to use to replace subexpressions of the original expression.

MuPAD applies `misc::maprec` recursively to all subexpressions of the original expression. For example, in this call `misc::maprec` replaces all occurrences of `sin` and `tan`, including the sine function in `tan(sin(x))`:

```
misc::maprec(sin(x) + tan(x^2) - tan(sin(x)),
  {"sin"} = sin2cos,
  {"tan"} = tan2cos)
```

$$\cos\left(x - \frac{\pi}{2}\right) + \frac{\cos\left(x^2 - \frac{\pi}{2}\right)}{\cos(x^2)} - \frac{\cos\left(\cos\left(x - \frac{\pi}{2}\right) - \frac{\pi}{2}\right)}{\cos\left(\cos\left(x - \frac{\pi}{2}\right)\right)}$$

Besides data types or types of expressions, such as `"sin"` or `"tan"`, you can use procedures to represent selection criteria in `misc::maprec`. In this example, the selection criterion of `misc::maprec` is the procedure `ex -> bool(freeIndets(ex) = {})` that excludes free identifiers and selects all constants of an expression. Using the procedure `f` as a replacement,

`misc::maprec` replaces all nonrational constants of an expression with new identifiers:

```
f :=
proc(x)
  option remember;
begin
  if testtype(x, Type::Rational) then x
  else genident();
  end_if;
end:
misc::maprec(a = 5*b + PI*sqrt(2)*c + PI,
  (ex -> bool(freeIndets(ex) = {})) = f)
```

$a = X6 + 5 b + X6 X7 c$

`option remember` in `f` ensures that constants appearing multiple times always get the same identifier. Moreover, you can access the remember table of the procedure `f` and select which substitutions you want to make:

```
select([op(op(f,5))], _not@bool)
```

$[\pi = X6, \sqrt{2} = X7]$

## Variables Inside Procedures

In this section...
“Closures” on page 4-116
“Static Variables” on page 4-118

### Closures

When you call a procedure, MuPAD allocates memory for the local variables, marks them as uninitialized, and evaluates the body of the procedure. At the end of a procedure call, MuPAD destroys local variables freeing the allocated memory. Now suppose that the result of a procedure call refers to local variables of that procedure. For example, the returned value of this procedure refers to its local variable `z`:

```
f :=  
proc(x, y)  
  local z;  
begin  
  z := x + y;  
  return(z);  
end:
```

In this case, the variable `z` is replaced by its value at the end of the procedure call. Therefore, the returned value of the procedure is the value of the variable `z`, not the variable `z` itself:

```
f(1, 2)
```

```
3
```

Use `hold` to suppress evaluation of the variable `z`. Now the procedure returns an object of type `DOM_VAR`:

```
f :=  
proc(x, y)  
  local z;  
begin
```



```

    z := x + y;
    return(hold(z));
end:
f(1, 2)

```

`DOM_VAR(0, 2)`

Objects of type `DOM_VAR` represent local variables and can only be used inside procedures. An object of type `DOM_VAR` returned as a result of a procedure call is useless because it does not have any connection to the procedure.

You can access local variables of a procedure if you either declare them in that procedure or declare them in a lexically enclosing procedure. For example, in the following code the procedure `g` can access and modify the variable `x` of the enclosing procedure `f`:

```

f :=
proc(x)
  local g;
begin
  g := proc()
    begin
      x := x+1;
    end:
  g();
end:
f(2)

```

`3`

Instead of returning the result of the procedure call `g()`, you can return `g` itself. In this case, the returned value retains a link to the variable `x` of the procedure call. For reasons of memory management, `f` must declare that it will return something holding a reference to a local variable. To declare it, use option `escape`:

```

f :=
proc(x)
  local g;

```

```
    option escape;
begin
  g := proc()
  begin
    x := x+1;
  end:
  g;
end:
h := f(2):
i := f(17):
h(); h(); i(); h()
```

3

4

18

5

This programming construct is called a *closure*. It is supported in many programming languages.

## Static Variables

### Alternative to Static Variables in MuPAD

Many programming languages support the concept of static variables. Static variables are local variables the values of which are not reset in each call to a procedure. The value of a static variable is initialized during the first call to a procedure. In each subsequent call, a procedure remembers the value of a static variable from the previous call.

Although MuPAD does not let you declare a variable inside a procedure as a static variable, you can still use the concept of static variables while programming in MuPAD.

When defining a procedure with `proc`, you often assign the procedure to an identifier. However, MuPAD lets you use anonymous procedures. Also, you can define one procedure inside another procedure. Thus, you can implement the alternative to a static variable in MuPAD as a nested procedure where:

- 1 The outer procedure has a local variable. The outer procedure can be anonymous.
- 2 The inner procedure uses the local variable of the outer procedure. For the inner procedure that variable is not local, and therefore it does not reset its value in each call.

For example, this code implements `cnt` as a static variable in MuPAD:

```
proc()
  local cnt;
  option escape;
begin
  cnt := 0;
  f :=
  proc()
  begin
    cnt := cnt + 1;
  end:
end():
f(); f(); f()
```

1

2

3

### Shared Static Variables

The technique of creating static variables in MuPAD lets you create shared static variables by creating several inner procedures. The inner procedures use the same local variable of the outer procedure. For inner procedures that variable is not local, and therefore it does not reset its value:

```
proc()
  local x, y;
  option escape;
begin
  x := 0;
  y := 0;
  f := () -> (x := x + y; [x, y]);
  g := n -> (y := y + n; [x, y]);
end_proc():
f();
g(2);
f();
f()
```

[0, 0]

[0, 2]

[2, 2]

[4, 2]

## Utility Functions

### In this section...

“Utility Functions Inside Procedures” on page 4-121

“Utility Functions Outside Procedures” on page 4-122

“Utility Functions in Closures” on page 4-123

### Utility Functions Inside Procedures

You can define utility functions inside a procedure. For example, define the utility function, `helper`, inside the procedure `f`:

```
f :=  
proc(arguments)  
  local helper, ...;  
begin  
  helper :=  
  proc(...)  
  begin  
    ...  
  end:  
  
  ... code using helper(...) ...  
end:
```

The `helper` function is not visible or accessible outside `f`. Your users cannot see the `helper` function, and therefore, they do not rely on a particular implementation of this procedure. You can change its implementation without breaking their code. At the same time, `helper` can access and modify arguments of `f`.

The major disadvantage of this approach is that your test files cannot access `helper` directly. Since it is typically recommended to start testing at the smallest possible building blocks, this is a real disadvantage. Nevertheless, for many tasks the benefits of this approach prevail over this disadvantage, especially if the utility function must be able to modify the arguments of the calling function.

## Utility Functions Outside Procedures

You can define utility functions outside a procedure. For example, define the utility function, `helper`, in the function environment `f`:

```
f := funcenv(  
  proc(arguments)  
    local ...;  
  begin  
    ... code using f::helper(...) ...  
  end):  
f::helper :=  
  proc(...)  
  begin  
    ...  
  end:
```

This approach does not require you to define the utility function in the function environment of the procedure that uses it. Defining a utility function in the function environment only helps you clarify to people reading your code that you intend to call `f::helper` primarily or solely within `f`. If you later decide to use `f::helper` in another procedure, you can move the utility function to a more generic utility library. Again, this recommendation only helps you improve readability of your code.

Defining utility functions outside the procedure that uses them does not hide utility functions. Therefore, this approach lets you:

- Test utility functions directly.
- Define a utility function and a function that uses it in different source files.
- Use the same utility function for different procedures.

Defining utility functions outside the procedure that uses them has the following disadvantages:

- Your users can access utility functions. If they rely on a particular implementation of the utility function, changing that implementation might affect their code.

- The utility function cannot access local variables of the procedure that uses that utility function. The workaround is to pass these local variables as arguments to the utility function.
- The utility function does not have privileged access to the arguments of the procedure that uses that utility function.
- Defining the utility function far from the code line where you call it reduces readability of the code.

Be careful when defining utility functions in slots of a function environment because MuPAD uses these slots for overloading. Do not define utility functions with such names as `f::print`, `f::diff`, `f::evaluate`, or `f::simplify` unless you want to use these utility functions for overloading.

## Utility Functions in Closures

You can define a utility function and all procedures that use it inside one procedure. In this case, you must also define the utility function as a local variable of that outer procedure. The outer procedure can be anonymous. For example, create the anonymous procedure that has a local variable `helper` and includes the utility function `helper` and two other procedures, `f` and `g`, that use the utility function:

```
proc()
  local helper;
  option escape;
begin
  helper :=
  proc(...)
    ...
  end:

  f :=
  proc(arguments)
    local ...;
  begin
    ... code using helper(...) ...
  end:

  g :=
```

```
proc(arguments)
  local ...;
begin
  ... code using helper(...) ...
end:
end():
```

For details about such structures, see Closures and Static Variables.

If you define a utility function in a closure, that function is inaccessible to any external code. Your users cannot see and, therefore, rely on a particular implementation of that utility function. Changing it will not break their code. At the same time, this approach lets you create more than one procedure that can access the utility function. In the example, both `f` and `g` can access `helper`.

The disadvantage of this approach is that the helper function cannot access the local variables of the procedures that use it. To overcome this limitation, you can use the `context` function or shared static variables.

---

**Note** Using `context` or shared static variables to make local variables of the calling procedure accessible for the utility function is not recommended.

---

Using `context` to overcome this limitation typically leads to unreadable and difficult to maintain code. The problems with shared static variables resemble the problems with global variables, especially for recursive calls. The helper procedure can access and modify such variables, but all other procedures inside the same outer procedure can access and modify them too.



## Private Methods

Although MuPAD does not let you declare a method as private, you can create private methods by using closures.

MuPAD uses a fundamentally simple object and name lookup model. Objects are data that belong to a particular domain type, and domains have named entries (called *slots*). If the value of a slot is a function, this entry is called a *method*. Therefore, MuPAD lets you use the same techniques for hiding method calls as you use for hiding utility functions. For details, see Utility Functions in Closures.

This example creates the private method `f` of the domain `d`. This method is not accessible from methods in inherited domains and from category methods:

```
domain d
  local f;
  inherits Dom::BaseDomain;
  g := proc()
    begin
      print("g");
      f();
    end;
begin
  f := proc()
    begin
      print("f");
    end;
end:

d::f

FAIL

d::g()

"g"
```

"f"

## Calls by Reference and Calls by Value

### In this section...

“Calls by Value” on page 4-127

“Calls by Reference” on page 4-128

### Calls by Value

When calling a procedure with some arguments, you expect the procedure to assign these values for its local variables and perform some computations with those variables. For example, this procedure divides any number that you pass to it by 10:

```
f := x -> (x := x/10):
```

In this example,  $x$  is a local variable of  $f$ . When you call  $f$  with any value, the procedure assigns that value to the local variable  $x$ , uses it to compute the result, and then destroys the local variable  $x$  freeing allocated memory:

```
x := 10:
f(x), x
```

```
1, 10
```

Although the value of the local variable  $x$  changes to 1 inside the procedure and then gets destroyed, the value of the global variable  $x$  remains the same. Therefore, you can conclude that the procedure does not access the actual memory block that contains the value of  $x$ .

When you call this procedure, MuPAD allocates a new memory block and copies the value of  $x$  to that block. While the procedure executes, the system associates the local variable  $x$  with this new memory block. At the end of the procedure call, it frees this memory block. The memory block that contains the value of the global variable  $x$  does not change.

The strategy of copying values of procedure arguments to new memory blocks and referring to these blocks during the procedure calls is known as *calling by value*. Most MuPAD functions use this strategy.

Since calling by value creates extra copies of data, it can be inefficient in terms of memory management. Creating extra copies of large data sets, especially when your procedure calls are recursive or nested can significantly reduce free memory on your computer. In many programming languages, calling by value always means copying data, even when a procedure does not change that data.

MuPAD uses *lazy copying* to avoid creating unnecessary copies of data. When you call a procedure, MuPAD does not allocate new memory blocks right away. Instead, it links local variables to memory blocks where the arguments of the procedure are stored. The system always counts how many objects are linked to the same memory block. When a procedure modifies the value of a local variable, MuPAD checks if there are other objects linked to the same memory block, and creates a copy only if necessary.

For example, when you call  $f(x)$ , MuPAD points both global variable  $x$  (DOM\_IDENT) and local (DOM\_VAR) variable  $x$  to the same memory block. Only when the local variable  $x$  changes its value, MuPAD allocates a new memory block for it.

### Calls by Reference

Typically, when you call a MuPAD procedure with some arguments, the system uses the *calling-by-value* approach and creates copies of the values of these arguments. This approach prevents procedures from modifying objects passed to a procedure as arguments. For some functions, such as assignment, MuPAD uses the *calling-by-reference* approach. In a call by reference, the system does not copy the values of arguments to new memory blocks. Instead, it copies references (*pointers*) to these arguments.

Some programming languages let you specify which approach to use for each particular function. MuPAD does not offer specific language constructs for calling by reference. Nevertheless, you can still call by reference in MuPAD.

---

**Note** For experienced MuPAD users, objects with reference semantics can behave unexpectedly. Be careful when exposing reference semantics to your users because it can be confusing.

---

Suppose your task requires a function call to be able to change the values of its arguments. The simple strategy is to return a modified copy of the arguments and overwrite the original object by using assignment. For example, replace matrix A with its upper row echelon form:

```
A := linalg::hilbert(3)
```

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

```
A := linalg::gaussElim(A)
```

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{pmatrix}$$

When working with large data sets and deep nested calls, this strategy can cause memory problems. Check the profiling report to see if your implementation has such problems. For details, see [Profiling Your Code](#).

You also can achieve the calling-by-reference effect in MuPAD using:

- Lexical Scoping
- Closures in Objects
- Domains in Objects
- Context Switching

### Lexical Scoping

Instead of passing data as arguments of a procedure, you can use a local variable of the outer procedure to store the data. For the inner procedure, this variable is not local. Therefore, the inner procedure can change the value of

that variable, and the variable is not destroyed after the inner procedure is executed:

```
f :=  
proc(x)  
  local n, incr;  
begin  
  n := 0;  
  incr := () -> (n := n + 1);  
  while x > n do  
    incr();  
  end_while;  
end_proc:
```

This approach does not fit many programming tasks, but it is recommended wherever you can apply it. It is the simplest and most readable approach to get the calling-by-reference effect in MuPAD.

### **Closures in Objects**

When working with domains, you also can use the approach of having the actual data in a closure. For example, instead of storing the actual data in the objects, you can store functions that access the data:

```
domain d  
  local get, set;  
  inherits Dom::BaseDomain;  
  
  new := proc(x)  
    option escape;  
  begin  
    new(dom, () -> x, y -> (x := y));  
  end;  
  
  incr := x -> set(x, get(x) + 1);  
  print := x -> get(x);  
  
begin  
  get := x -> extop(x, 1)();  
  set := (x, y) -> extop(x, 2)(y);
```

```
end_domain:
```

```
e := d(4)
```

```
4
```

```
d::incr(e)
```

```
5
```

```
e
```

```
5
```

See Closures for more details.

## Domains in Objects

You can implement the calling-by-reference approach in your code by using domains as tables with reference effects. (Using domains as tables is unrelated to object-oriented programming.) The following example demonstrates this strategy:

```
domain dd
  local get, set;
  inherits Dom::BaseDomain;

  new := proc(x)
    local d;
    begin
      d := newDomain(genident());
      d::number := x;
      new(dom, d);
    end;

  get := (x, entry) -> slot(extop(x, 1), entry);
  set := (x, entry, value) -> slot(extop(x, 1), entry, value);
```

```
incr := x -> (dom::set(x, "number",
    dom::get(x, "number") + 1); x);
print := x -> dom::get(x, "number");

end_domain:

e := dd(4)

4

dd::incr(e)

5

e

5
```

The primitives of the `plot` library use this strategy. For example, when you execute the following code, the domain `plot::Function2d` overloads the `slot` function:

```
f := plot::Function2d(sin(x), x=-PI..PI):
f::Color := RGB::Green:
```

### Context Switching

The context function and option `hold` also let you implement the calling be reference effect in your procedures. `option hold` prevents the procedure from evaluating its arguments before executing the code in the procedure body. The context function lets you perform operations as if they occur in the calling procedure.

For example, in this code `option hold` prevents the `incr` procedure from evaluating its argument. Therefore, the system does not copy the value of `x` to a new memory block:

```
incr :=
```



```

proc(x)
  option hold;
begin
  context(hold(_assign)(x, x + 1));
end_proc:
operator("++", incr, Prefix, 500):

```

While executing this procedure, the system performs the assignment operation in the context of the procedure `f` (the calling procedure for `incr`). Thus, `incr` changes the value of the argument, `n`, with which it was called:

```

f :=
proc(x)
  local n;
begin
  n := 0;
  while x > n do
    ++n;
  end_while;
end_proc:

```

If you use the `++` operator on an unchangeable object, MuPAD throws an error. For example, you cannot assign the value 2 to the value 1:

```
++1
```

Error: The left side is invalid. [\_assign]

This error message does not mention `incr` because the error occurs in the assignment which is performed in a different evaluation context. The `incr` procedure behaves essentially like a dynamic macro.

## Integrate Custom Functions into MuPAD

MuPAD provides a variety of tools for handling built-in mathematical functions such as `sin`, `cos`, `exp`, and so on. These tools implement the mathematical properties of these functions. Typical examples are the `float` conversion routine, the `diff` differentiation function, or the `expand` function, which you use to manipulate expressions:

```
float(sin(1));  
diff(sin(x), x, x, x);  
expand(sin(x + 1))
```

```
0.8414709848
```

```
-cos(x)
```

```
cos(1) sin(x) + sin(1) cos(x)
```

You can say that the mathematical knowledge about the built-in functions is distributed over several system functions: `float` knows how to compute numerical approximations of the sine function, `diff` knows the derivative of the sine function, and `expand` knows the addition theorems of the trigonometric functions.

When you implement your own function, you can integrate it into the MuPAD system, making other functions aware how to work with this new function correctly. If the new function consists only of built-in MuPAD functions, then you do not need to take extra steps. All MuPAD functions interact correctly with the new function:

```
f := x -> (x*sin(x)):  
diff(f(x), x)
```

```
sin(x) + x cos(x)
```

However, if you implement a function that is not composed of the standard MuPAD objects (for example, a new special function), you must distribute the

knowledge about the mathematical meaning of the new function to standard MuPAD functions, such as `diff`, `expand`, `float`, and so on. This extra task is necessary for integrating the new function with the rest of the system. For example, you might want to differentiate an expression that contains both the new function and some built-in functions, and such differentiation is only possible via the MuPAD differentiation routine. Therefore, this routine must know how to handle the new symbol.

MuPAD uses *function environments* (domain type `DOM_FUNC_ENV`) to integrate functions into the system. A function environment stores special function attributes (`slots`) in an internal table. Whenever an overloadable system function, such as `diff`, `expand`, or `float`, encounters an object of type `DOM_FUNC_ENV`, it searches the function environment for a corresponding slot. If a system function finds the appropriate slot, it calls that slot and returns the value produced by the slot. All built-in MuPAD functions are implemented as function environments:

```
domtype(sin), domtype(exp)
```

```
DOM_FUNC_ENV, DOM_FUNC_ENV
```

You can call a function environment as you would call any MuPAD function or procedure:

```
sin(1.7), exp(1.0)
```

```
0.9916648105, 2.718281828
```

Suppose you implement the complete elliptic integral functions of the first and second kind,  $K(z)$  and  $E(z)$ . These functions appear in different contexts, such as calculating the perimeter of an ellipse, the gravitational or electrostatic potential of a uniform ring, and the probability that a random walk in three dimensions ever goes through the origin. The elliptic integrals have the following special values:

$$E(0) = K(0) = \frac{\pi}{2}, E(1) = 1, K\left(\frac{1}{2}\right) = \frac{8\pi^{3/2}}{\Gamma\left(-\frac{1}{4}\right)^2}, K\left(\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}}$$

MuPAD provides the built-in functions `ellipticE` and `ellipticK` for computing these elliptic integrals. However, you can implement your own functions for the same task. For example, write the procedures `ellipE` and `ellipK`. These procedures define the values of the elliptic integrals for special values of  $x$ . For all other argument values, the values of elliptic integrals are unknown, and the procedures return the symbolic expressions `ellipE(x)` and `ellipK(x)`. Use `procname` to return symbolic expressions:

```
ellipE :=
proc(x) begin
  if x = 0 then PI/2
  elif x = 1 then 1
  else procname(x) end_if
end_proc:

ellipK :=
proc(x) begin
  if x = 0 then PI/2
  elif x = 1/2 then 8*PI^(3/2)/gamma(-1/4)^2
  elif x = -1 then gamma(1/4)^2/4/sqrt(2*PI)
  else procname(x) end_if
end_proc:
```

`ellipE` and `ellipK` return special values for particular arguments. For all other arguments, they return symbolic expressions:

```
ellipE(0), ellipE(1/2),
ellipK(12/17), ellipK(x^2 + 1)
```

$$\frac{\pi}{2}, \text{ellipE}\left(\frac{1}{2}\right), \text{ellipK}\left(\frac{12}{17}\right), \text{ellipK}(x^2 + 1)$$

The first derivatives of these elliptic integrals are as follows:

$$E'(z) = \frac{E(z) - K(z)}{2z}, \quad K'(z) = \frac{E(z) - (1-z)K(z)}{2(1-z)z}.$$

The standard MuPAD differentiation function `diff` does not know about these rules. Therefore, trying to differentiate `ellipE` and `ellipK` simply returns the symbolic notations of the derivatives:

```
diff(ellipE(x), x),
diff(ellipK(x), x)
```

$$\frac{\partial}{\partial x} \text{ellipE}(x), \frac{\partial}{\partial x} \text{ellipK}(x)$$

To make `diff` work with the new functions, create function environments from the procedures `ellipE` and `ellipK`. In addition, function environments let you control the appearance of the symbolic function calls in outputs.

A function environment consists of three operands.

- The first operand is a procedure that computes the return value of a function call.
- The second operand is a procedure for printing a symbolic function call on the screen.
- The third operand is a table that specifies how the system functions handle symbolic function calls.

To create function environments, use `funcenv`. For example, create function environments `ellipE` and `ellipK`. Use the second argument to specify that symbolic calls to `ellipE` and `ellipK` must appear as `E` and `K` outputs:

```
output_E := f -> hold(E)(op(f)):
ellipE := funcenv(ellipE, output_E):
```

```
output_K := f -> hold(K)(op(f)):
ellipK := funcenv(ellipK, output_K):
```

Although `ellipE` and `ellipK` are now function environments, you can call them as you would call any other MuPAD function:

```
ellipE(0), ellipE(1/2),
ellipK(12/17), ellipK(x^2+1)
```

$$\frac{\pi}{2}, E\left(\frac{1}{2}\right), K\left(\frac{12}{17}\right), K(x^2+1)$$

The third argument `funcenv` is a table of function attributes. It tells the system functions (such as `float`, `diff`, `expand`, and so on) how to handle symbolic calls of the form `ellipE(x)` and `ellipK(x)`. You can update this table specifying the rules for the new function. For example, specify the new differentiation rules by assigning the appropriate procedures to the `diff` slot of the function environments:

```
ellipE::diff :=
proc(f,x)
local z;
begin
z := op(f);
(ellipE(z) - ellipK(z))/(2*z) * diff(z, x)
end_proc:

ellipK::diff :=
proc(f,x)
local z;
begin
z := op(f);
(ellipE(z) - (1-z)*ellipK(z))/
(2*(1-z)*z) * diff(z, x)
end_proc:
```

Now, whenever `f = ellipE(z)`, and `z` depends on `x`, the call `diff(f, x)` uses the procedure assigned to `ellipE::diff`:

```
diff(ellipE(z), z);
diff(ellipE(y(x)), x);
diff(ellipE(x*sin(x)), x)
```

$$\frac{E(z) - K(z)}{2z}$$

$$\frac{(E(y(x)) - K(y(x))) \frac{\partial}{\partial x} y(x)}{2y(x)}$$

$$\frac{(E(x \sin(x)) - K(x \sin(x))) (\sin(x) + x \cos(x))}{2 x \sin(x)}$$

The new differentiation routine also finds higher-order derivatives:

`diff(ellipE(x), x, x)`

$$\frac{\frac{E(x)-K(x)}{2x} + \frac{E(x)+K(x)(x-1)}{2x(x-1)}}{2x} - \frac{E(x)-K(x)}{2x^2}$$

Since the `taylor` function internally calls `diff`, the new differentiation routine also lets you compute Taylor expansions of the elliptic integrals:

`taylor(ellipK(x), x = 0, 6)`

$$\frac{\pi}{2} + \frac{\pi x}{8} + \frac{9 \pi x^2}{128} + \frac{25 \pi x^3}{512} + \frac{1225 \pi x^4}{32768} + \frac{3969 \pi x^5}{131072} + O(x^6)$$

If a derivative of a function contains the function itself, the integration routine has a good chance of finding symbolic integrals after you implement the `diff` attributes. For example, `int` now computes the following integrals:

`int(ellipE(x), x)`

$$\frac{2 E(x)}{3} - \frac{2 K(x)}{3} + x \left( \frac{2 E(x)}{3} + \frac{2 K(x)}{3} \right)$$

`int(ellipK(x), x)`

$$2 E(x) - 2 K(x) + 2 x K(x)$$





# Graphics and Animations

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- “Gallery” on page 5-2
- “Easy Plotting: Graphs of Functions” on page 5-25
- “Advanced Plotting: Principles and First Examples” on page 5-84
- “The Full Picture: Graphical Trees” on page 5-100
- “Viewer, Browser, and Inspector: Interactive Manipulation” on page 5-105
- “Primitives” on page 5-110
- “Attributes” on page 5-115
- “Layout of Canvas and Scenes” on page 5-129
- “Animations” on page 5-139
- “Groups of Primitives” on page 5-170
- “Transformations” on page 5-172
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- “Fonts” on page 5-182
- “Colors” on page 5-185
- “Save and Export Pictures” on page 5-191
- “Import Pictures” on page 5-195
- “Cameras in 3D” on page 5-197
- “Possible Strange Effects in 3D” on page 5-208

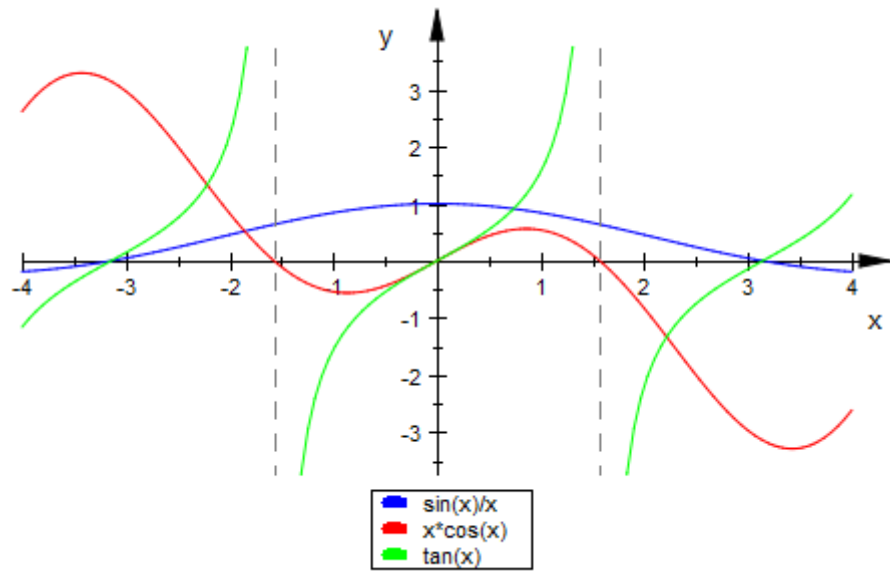
## Gallery

In this section...
“2D Function and Curve Plots” on page 5-2
“Other 2D examples” on page 5-7
“3D Functions, Surfaces, and Curves” on page 5-17

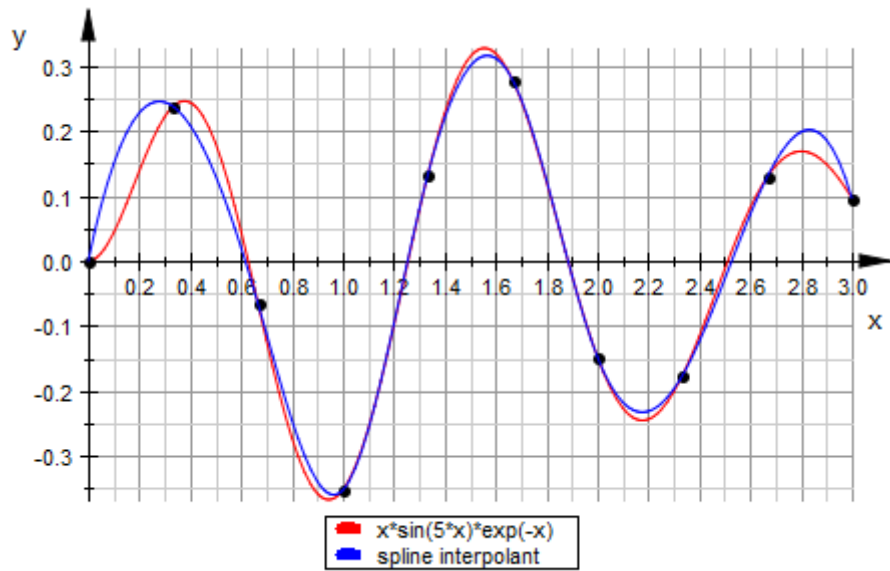
We present a collection of pictures illustrating the capabilities of the present MuPAD graphics system. These pictures are created at various places in this document where they are used to demonstrate certain features of the graphics system. A reference to the location of detailed documentation is provided along with each picture in this gallery. There, further details including the MuPAD commands for generating the picture can be found.

### 2D Function and Curve Plots

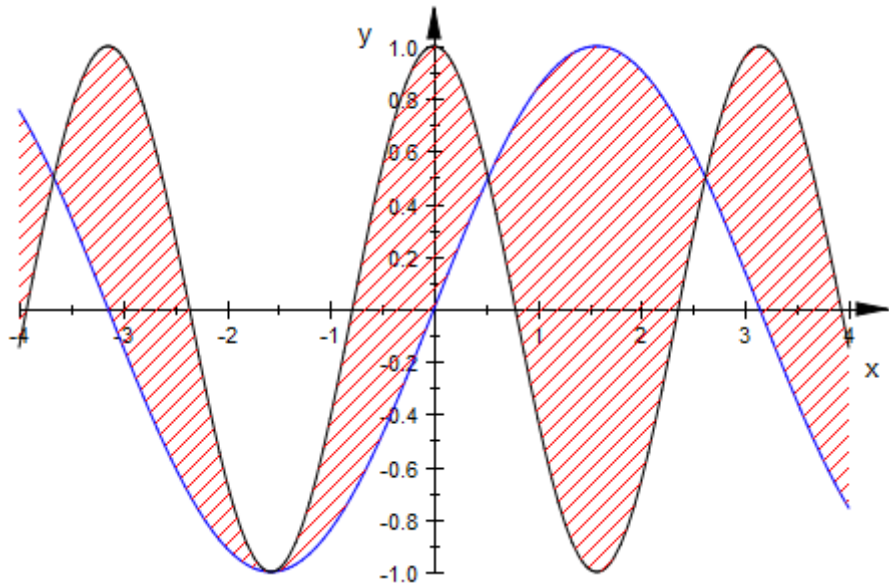
The following picture shows a plot of several functions. Singularities are highlighted by “vertical asymptotes.” See 2D Function Graphs: plotfunc2d for details:



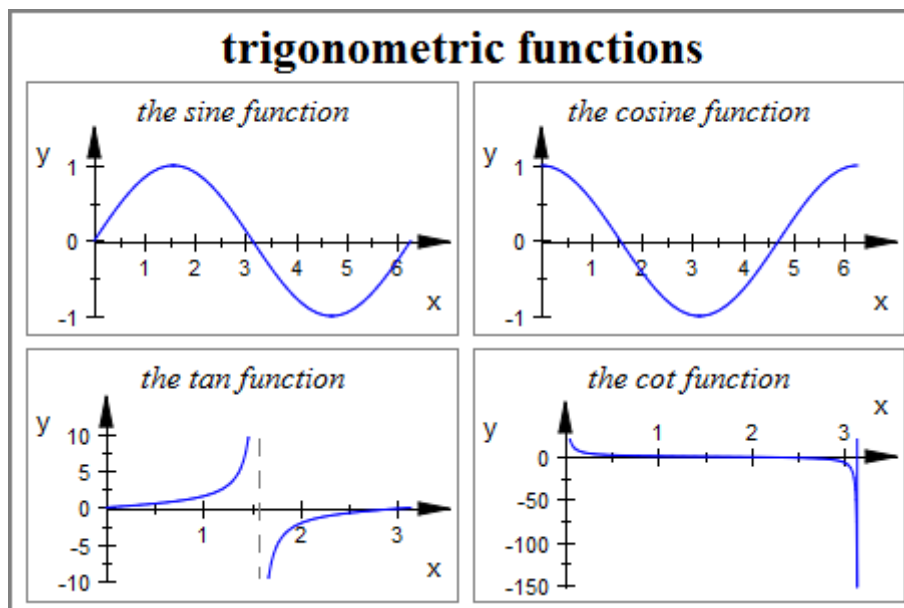
The following picture shows a function plot together with a spline interpolation through a set of sample points. See section Some Examples for details:



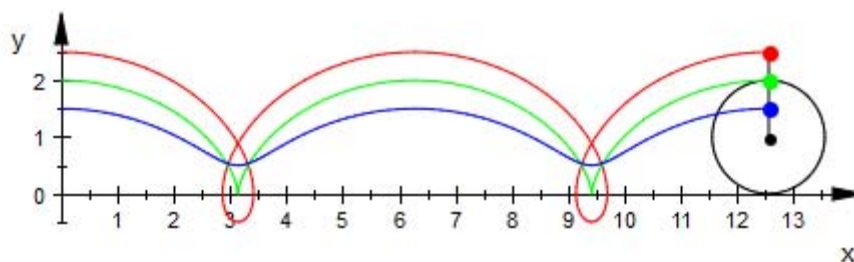
The following picture shows a hatched area between functions. See the examples on the help page of `plot::Hatch` for details:



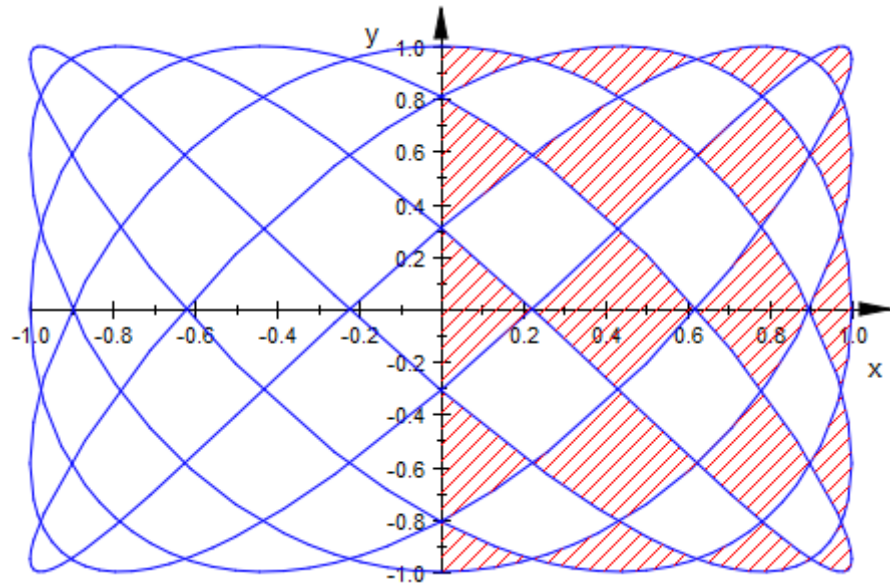
The following picture demonstrates some layout possibilities. See the examples on the help page of Layout for details:



The following picture demonstrates the construction of cycloids via points fixed to a rolling wheel. See section Some Examples for an animated version and details:

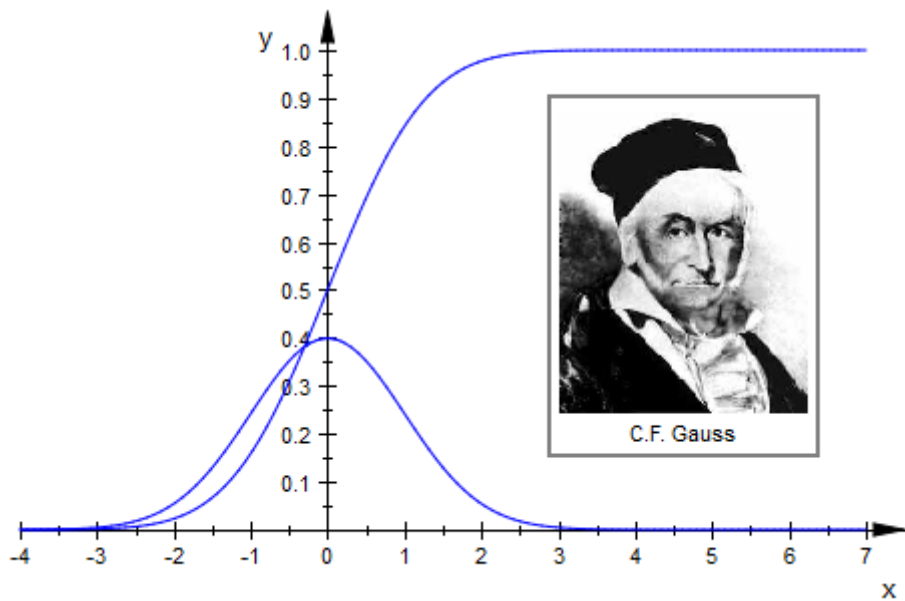


The following picture demonstrates hatched areas inside curves. See the examples on the help page of `plot::Hatch` for details:



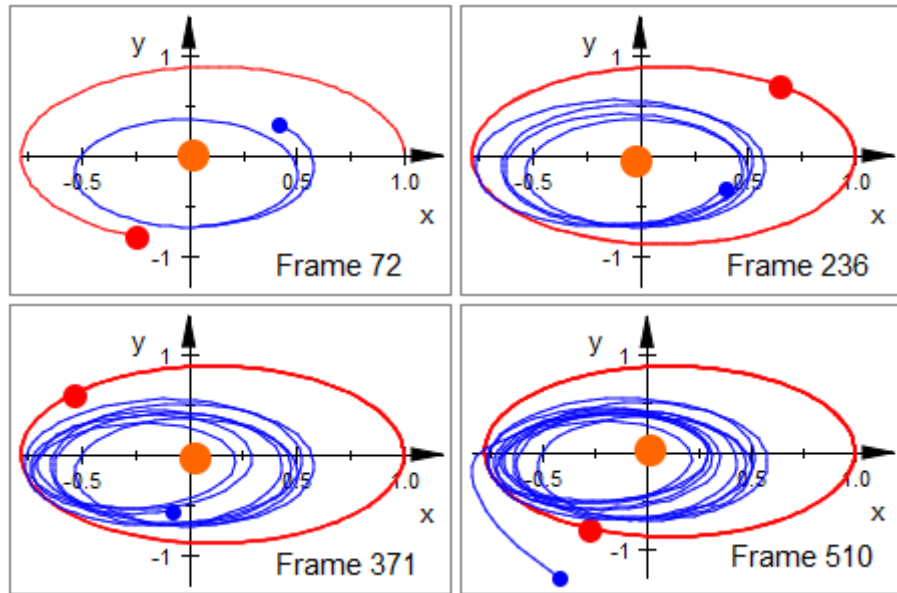
### Other 2D examples

The following picture shows an imported bitmap inside function plots. See section Importing Pictures for details:

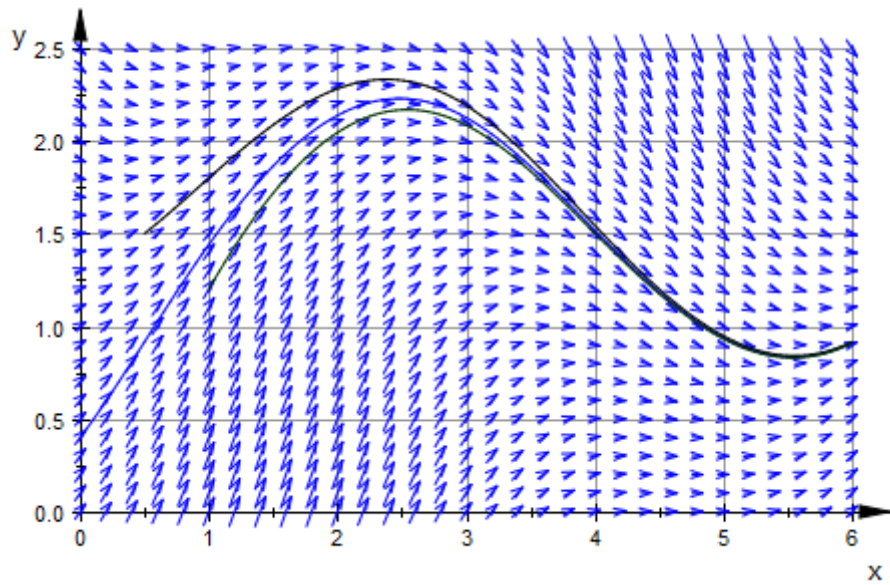


The following picture shows some frames of an animation of the perturbed orbit of a small planet kicked out of a solar system by a giant planet after a near-collision. See section Example 3 for details of the animation:

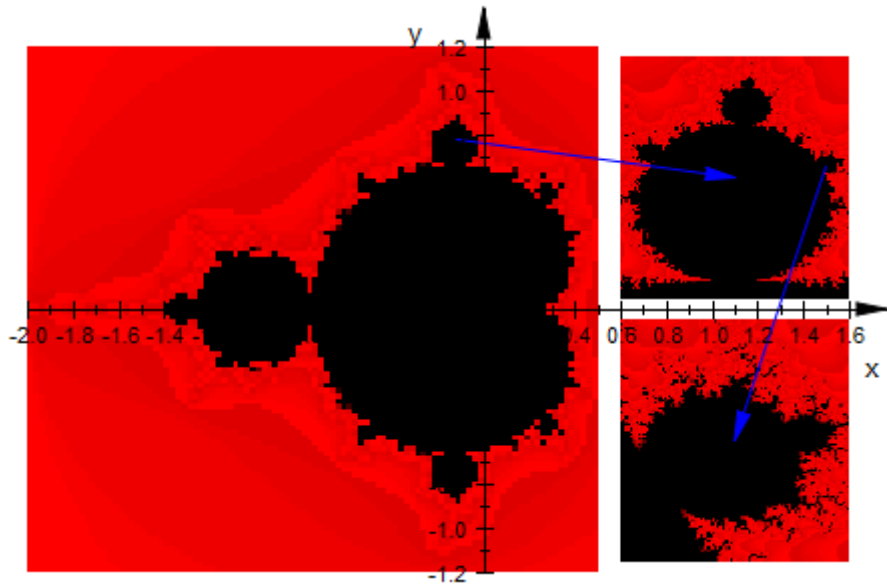




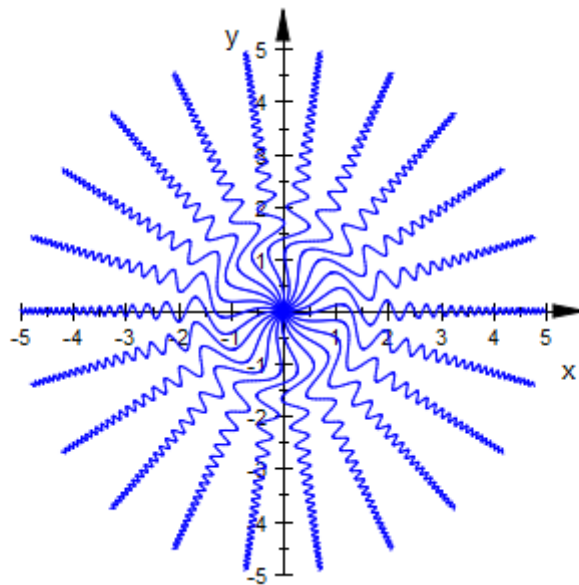
The following picture shows three solution curves of an ODE inside the directional vector field associated with the ODE. See the examples on the help page of `plot::VectorField2d` for details:



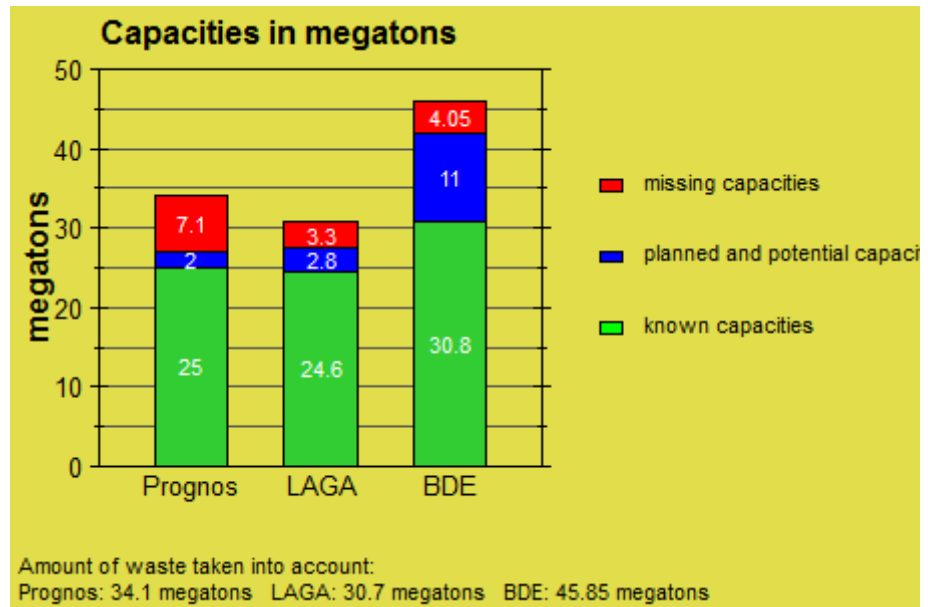
The following picture shows the Mandelbrot set together with two blow ups of regions of special interest. See the examples on the help page of `plot::Density` for details:



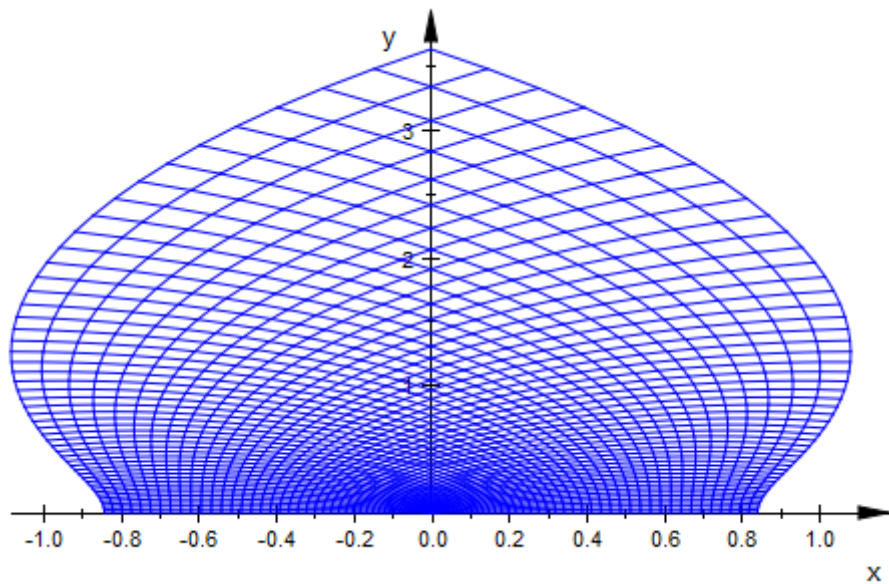
The following picture shows several rotated copies of a function graph. See the examples on the help page of `plot::Rotate2d` for details:



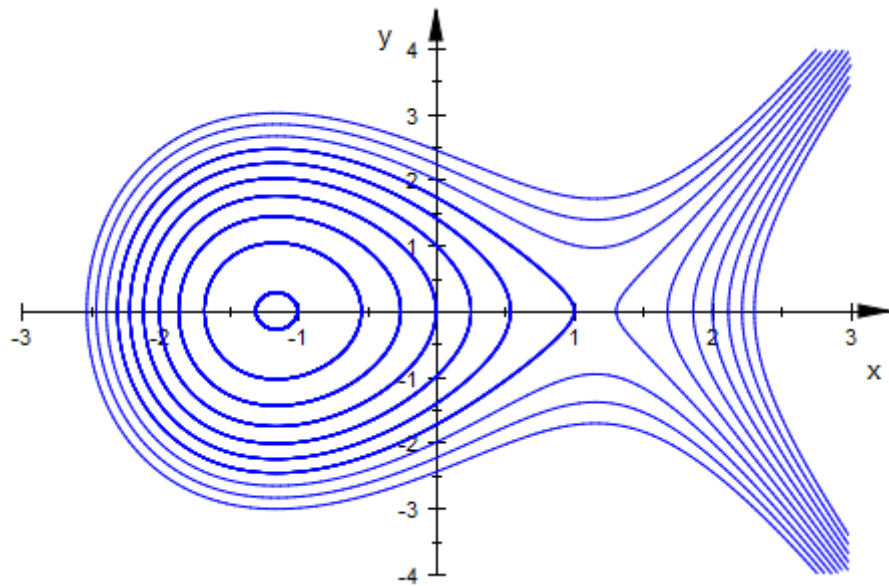
The following picture shows a data plot of type `plot::Bars2d`. See the examples on the help page of `plot::Bars2d` for details:



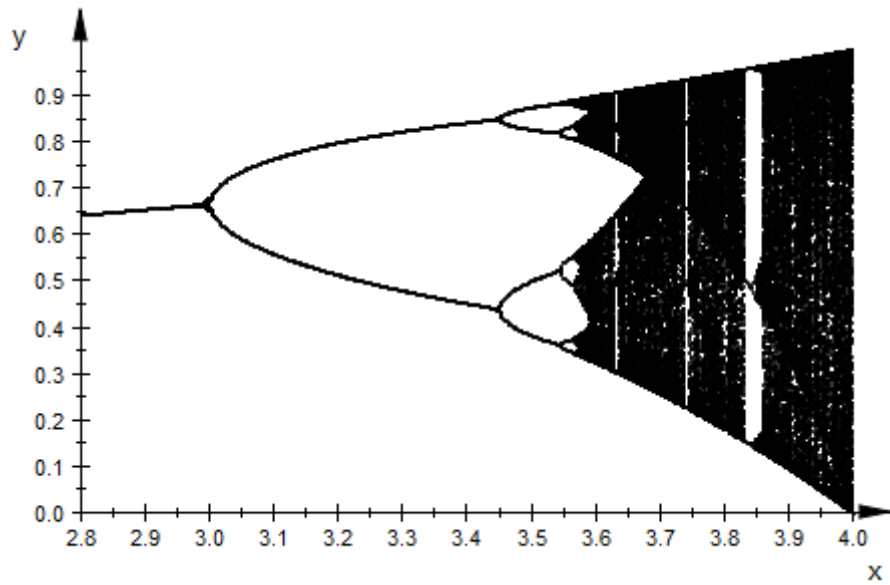
The following picture shows the image of a rectangle in the complex plane under the map  $z \rightarrow \sin(z^2)$ . See the examples on the help page of `plot::Conformal` for details:



The following picture shows some elliptic curves generated as a contour plot. See the examples on the help page of `plot::Implicit2d` for details:

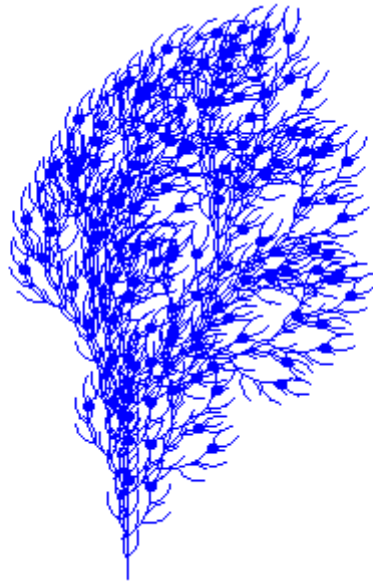


The following picture shows the Feigenbaum diagram of the logistic map. See the examples on the help page of `plot::PointList2d` for details:



The following picture shows a fractal object generated by a turtle plot of a Lindenmayer system. See the examples on the help page of `plot::Lsys` for details:

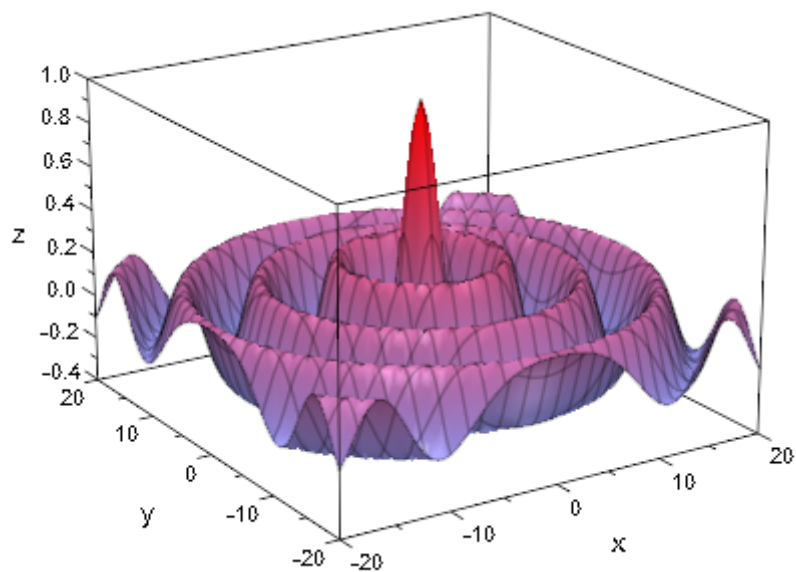




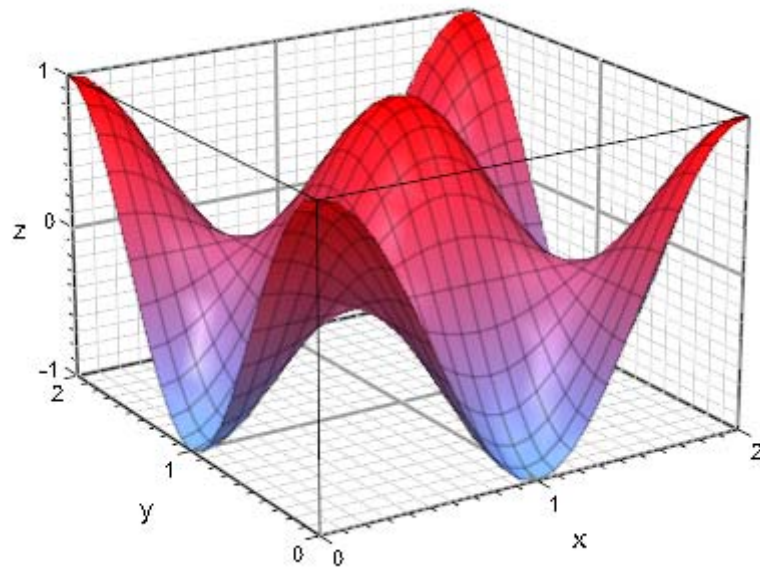
### 3D Functions, Surfaces, and Curves

The following picture demonstrates a 3D function plot of

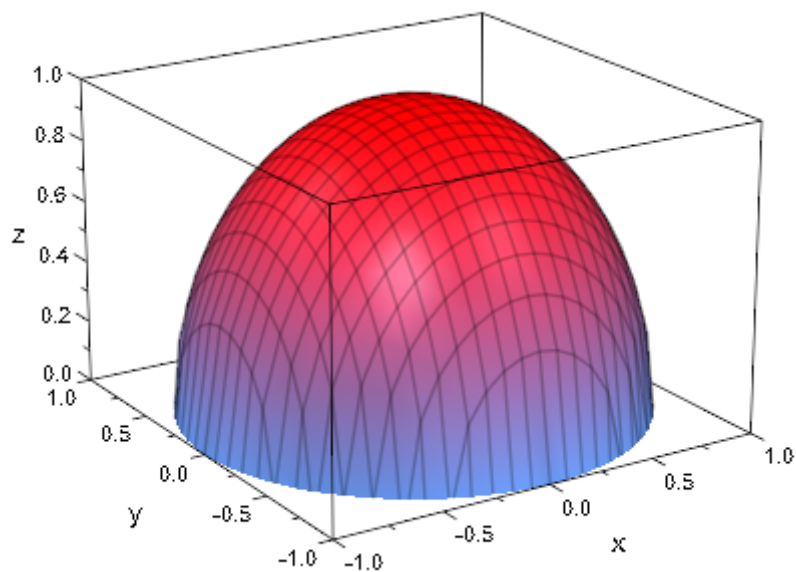
$f(x, y) = J_0(\sqrt{x^2 + y^2})$ , where  $J_0(z)$  is the Bessel function of the first kind. See the examples on the help page of `plot::Function3d` for details:



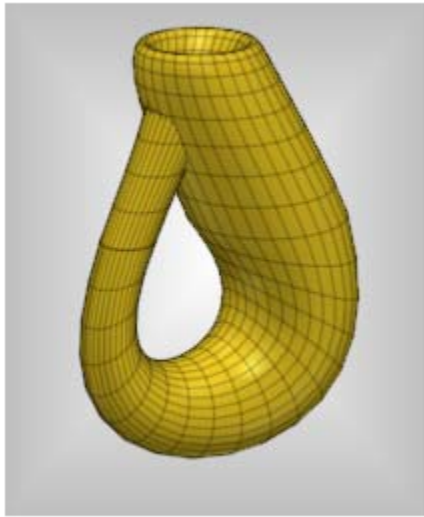
The following picture demonstrates a 3D function plot enhanced by a coordinate grid. See the examples on the help page of `GridVisible` for details:



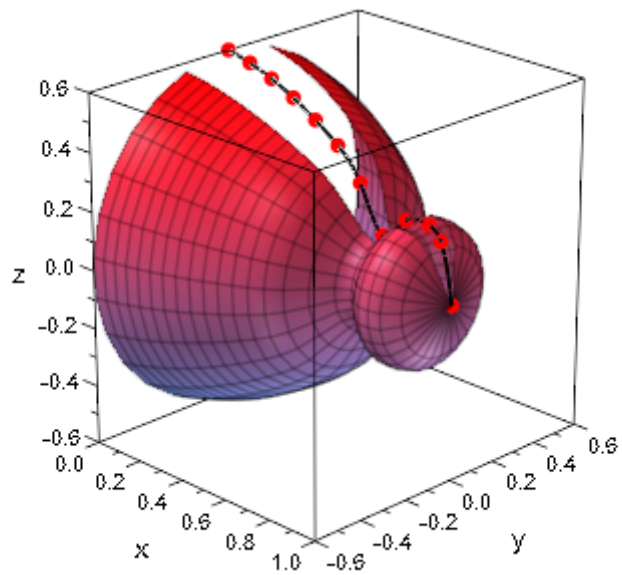
The following picture demonstrates a 3D function plot of  $\sqrt{1-x^2-y^2}$ , which is not real for some parts of the parameter space. See the documentation of `plot::Function3d` for details:



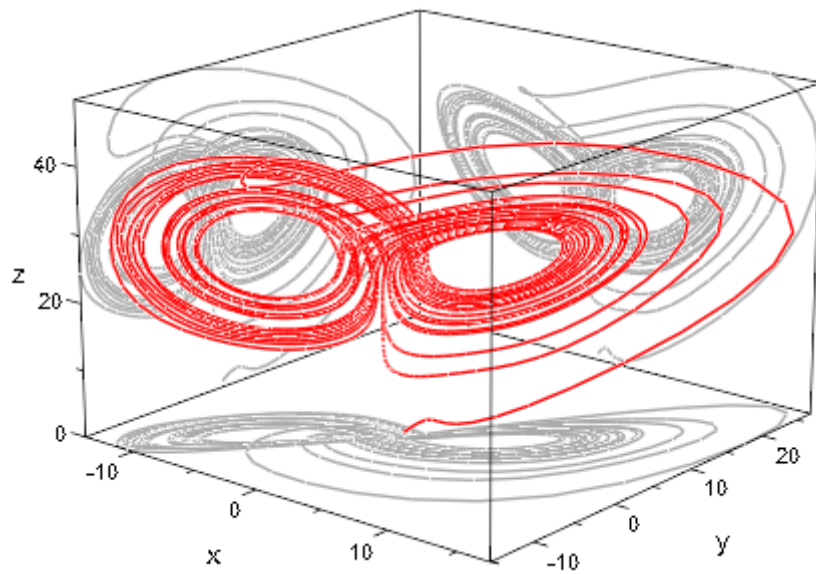
The following picture shows “Klein’s bottle” (a famous topological object). This surface does not have an orientation; there is no “inside” and no “outside” of this object. See the examples on the help page of `plot::Surface` for details:



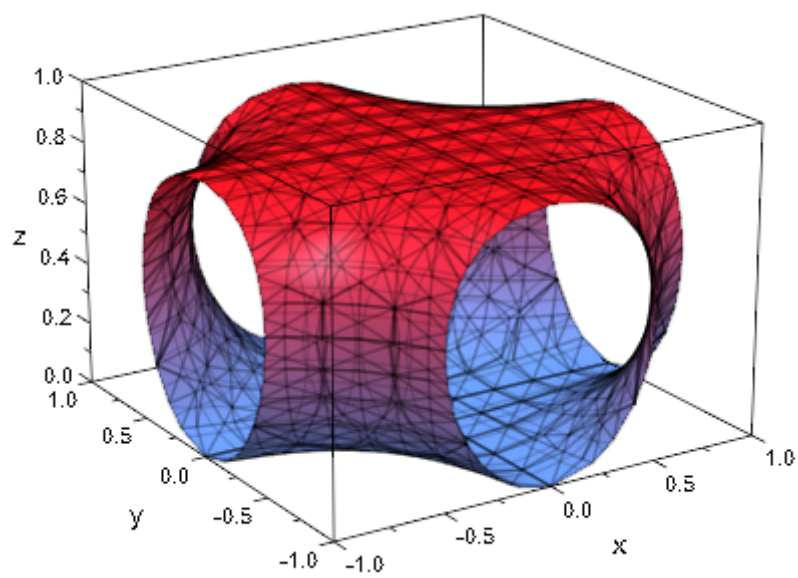
The following picture demonstrates the reconstruction of an object with rotational symmetry from measurements of its radius at various points. See section Some Examples for details:



The following picture shows the “Lorenz attractor.” See section Cameras in 3D for an animated version and details:



The following picture shows a 3D level surface of a function (the solution set of  $z^2 = \sin(z - x^2y^2)$ ). See the examples on the help page of `plot::Implicit3d` for details:





## Easy Plotting: Graphs of Functions

### In this section...

“2D Function Graphs: `plotfunc2d`” on page 5-25

“3D Function Graphs: `plotfunc3d`” on page 5-41

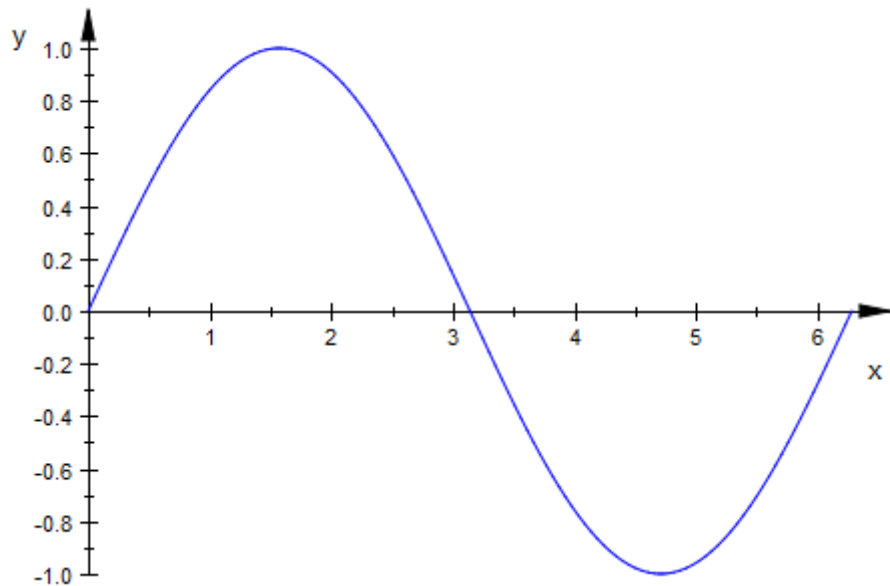
“Attributes for `plotfunc2d` and `plotfunc3d`” on page 5-57

The probably most important graphical task in a mathematical context is to visualize function graphs, i.e., to plot functions. There are two graphical routines `plotfunc2d` and `plotfunc3d` which allow to create 2D plots of functions with one argument (such as  $f(x) = \sin(x)$ ,  $f(x) = x \ln(x)$  etc.) or 3D plots of functions with two arguments (such as  $f(x, y) = \sin(x^2 + y^2)$ ,  $f(x, y) = y \ln(x) - x \ln(y)$  etc.). The calling syntax is simple: just pass the expression that defines the function and, optionally, a range for the independent variable(s).

### 2D Function Graphs: `plotfunc2d`

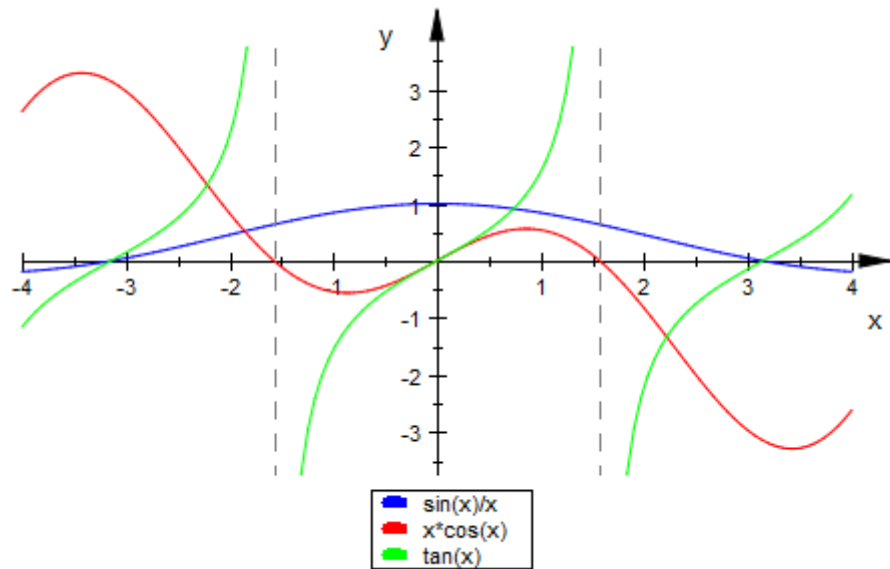
We consider 2D examples, i.e., plots of univariate functions  $y = f(x)$ . Here is one period of the sine function:

```
plotfunc2d(sin(x), x = 0..2*PI):
```



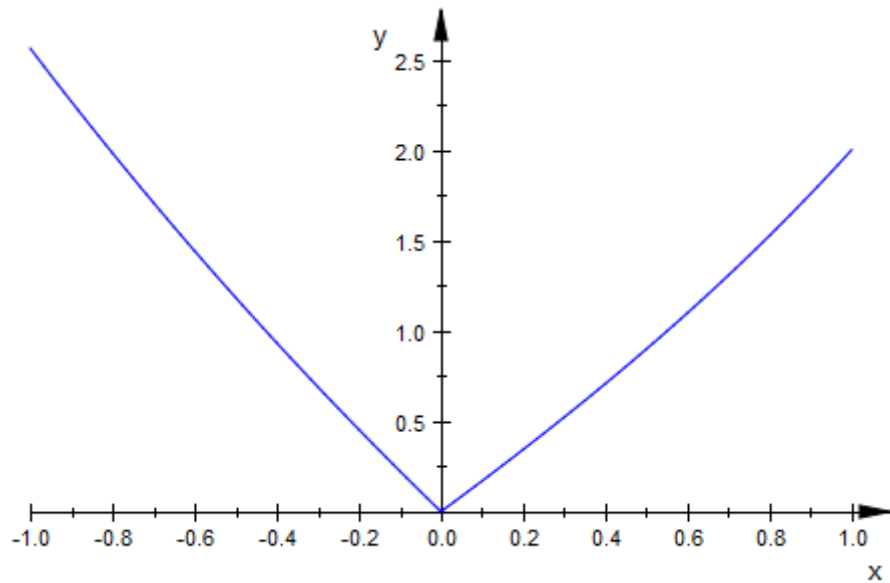
If several functions are to be plotted in the same graphical scene, just pass a sequence of function expressions. All functions are plotted over the specified common range:

```
plotfunc2d(sin(x)/x, x*cos(x), tan(x), x = -4..4):
```



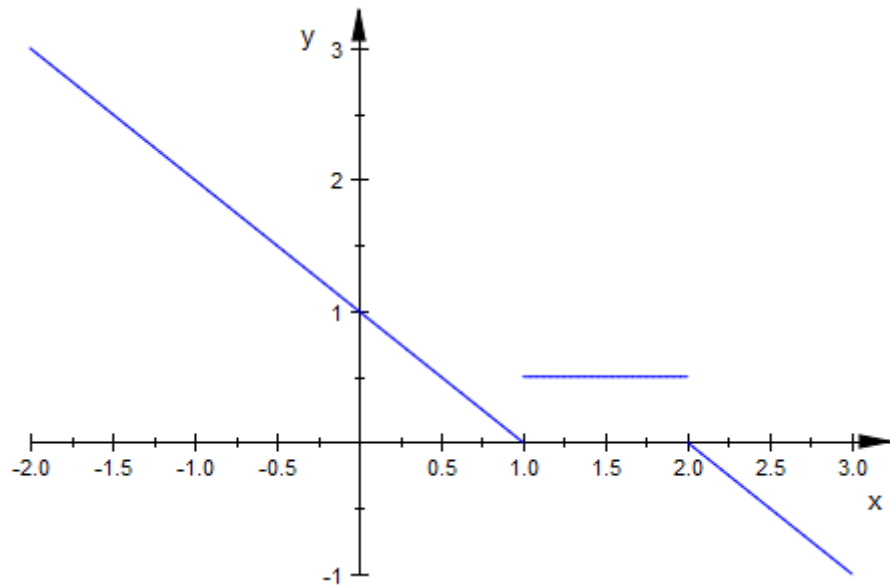
Functions that do not allow a simple symbolic representation by an expression can also be defined by a procedure that produces a numerical value  $f(x)$  when called with a numerical value  $x$  from the plot range. In the following example we consider the largest eigenvalue of a symmetric  $3 \times 3$  matrix that contains a parameter  $x$ . We plot this eigenvalue as a function of  $x$ :

```
f := x -> max(numeric::eigenvalues(matrix([[ -x, x, -x ],
                                           [ x, x, x ],
                                           [ -x, x, x^2]]))):
plotfunc2d(f, x = -1..1):
```



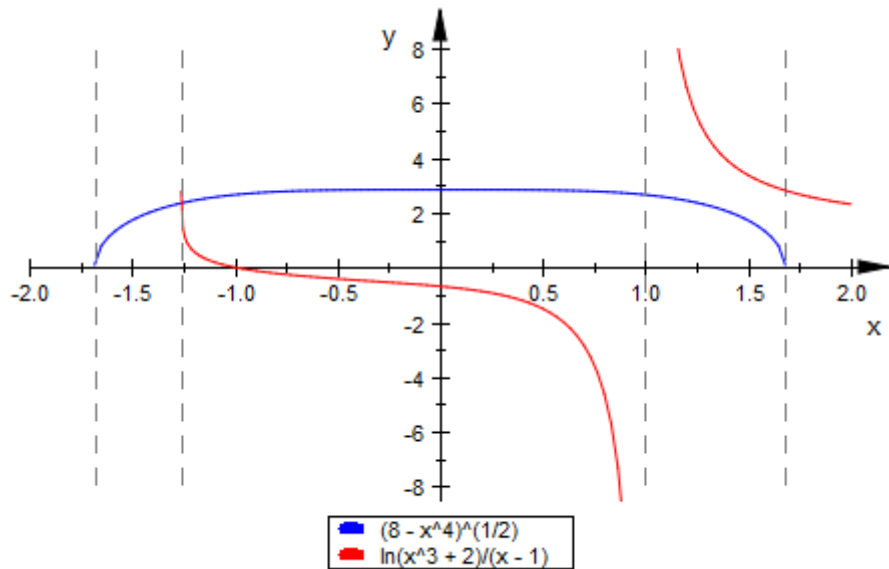
The name  $x$  used in the specification of the plotting range provides the name that labels the horizontal axis. Functions can also be defined by `piecewise` objects:

```
plotfunc2d(piecewise([x < 1, 1 - x],  
                    [1 < x and x < 2, 1/2],  
                    [x > 2, 2 - x]),  
           x = -2..3)
```



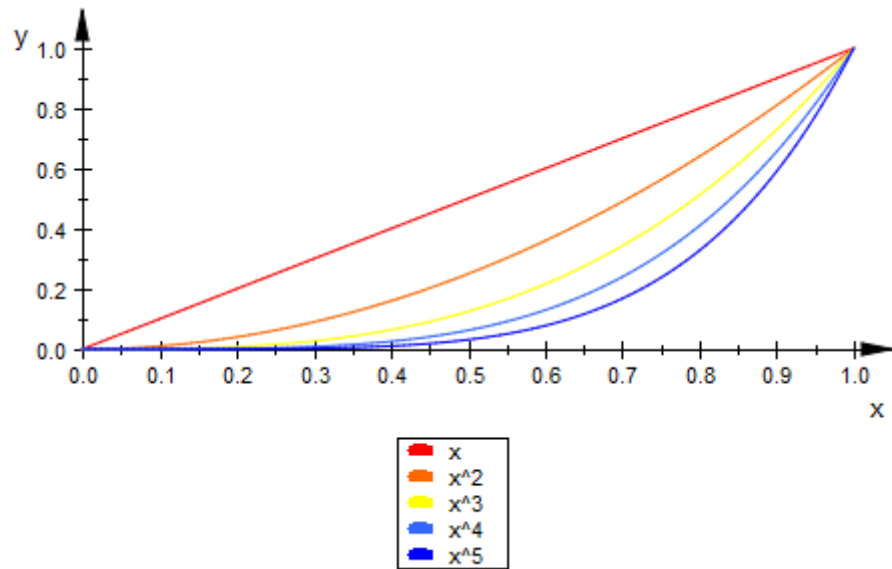
Note that there are gaps in the definition of the function above: no function value is specified for  $x = 1$  and  $x = 2$ . This does not cause any problem, because `plotfunc2d` simply ignores all points that do not produce real numerical values. Thus, in the following example, the plot is automatically restricted to the regions where the functions produce real values:

```
plotfunc2d(sqrt(8 - x^4), ln(x^3 + 2)/(x - 1), x = -2 ..2):
```



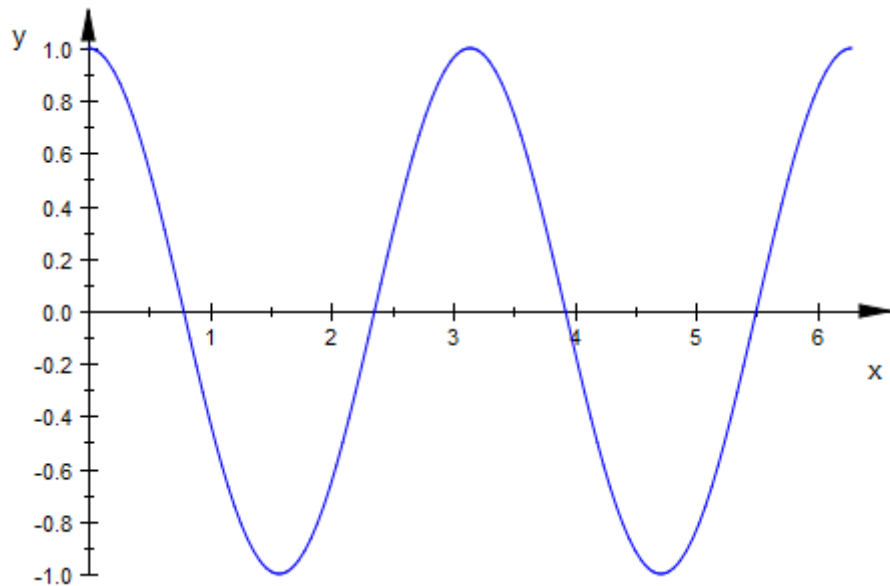
When several functions are plotted in the same scene, they are drawn in different colors that are chosen automatically. With the `Colors` attribute one may specify a list of RGB colors that `plotfunc2d` shall use:

```
plotfunc2d(x, x^2, x^3, x^4, x^5, x = 0..1,  
           Colors = [RGB::Red, RGB::Orange, RGB::Yellow,  
                    RGB::BlueLight, RGB::Blue]):
```



Animated 2D plots of functions are created by passing function expressions depending on a variable ( $x$ , say) and an animation parameter ( $a$ , say) and specifying a range both for  $x$  and  $a$ :

```
plotfunc2d(cos(a*x), x = 0..2*PI, a = 1..2):
```

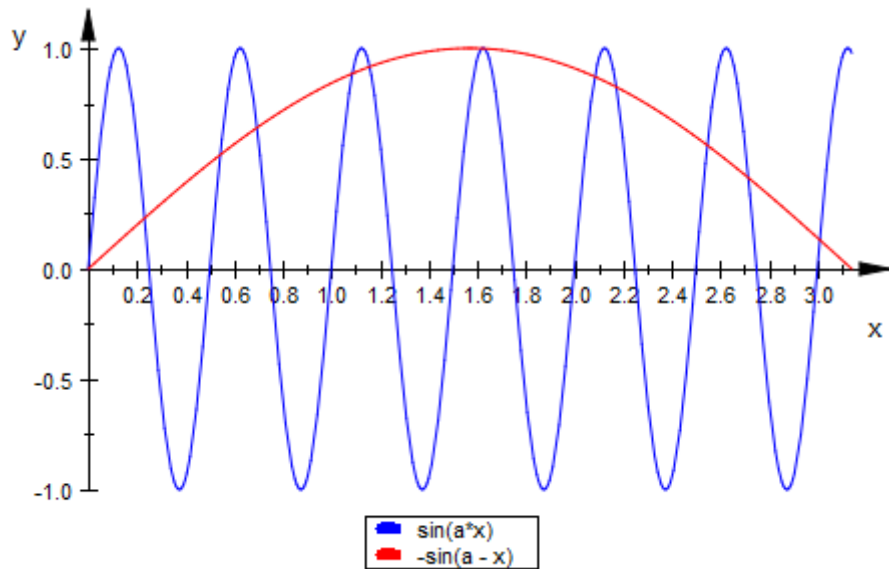


Once the plot is created, the first frame of the picture appears as a static plot. After clicking on the picture, the graphics tool starts playing the animation. There are the usual controls to stop, start, and fast-forward/rewind the animation.

The default number of frames of the animation is 50. If a different value is desired, just pass the attribute `Frames = n`, where  $n$  is the number of frames that shall be created:

```
plotfunc2d(sin(a*x), sin(x - a), x = 0..PI, a = 0..4*PI,  
           Colors = [RGB::Blue, RGB::Red], Frames = 200):
```





Apart from the color specification or the Frames number, there is a large number of further attributes that may be passed to `plotfunc2d`. Each attribute is passed as an equation `AttributeName = AttributeValue` to `plotfunc2d`. Here, we only present some selected attributes. See the section on attributes for `plotfunc` for further tables with more attributes.

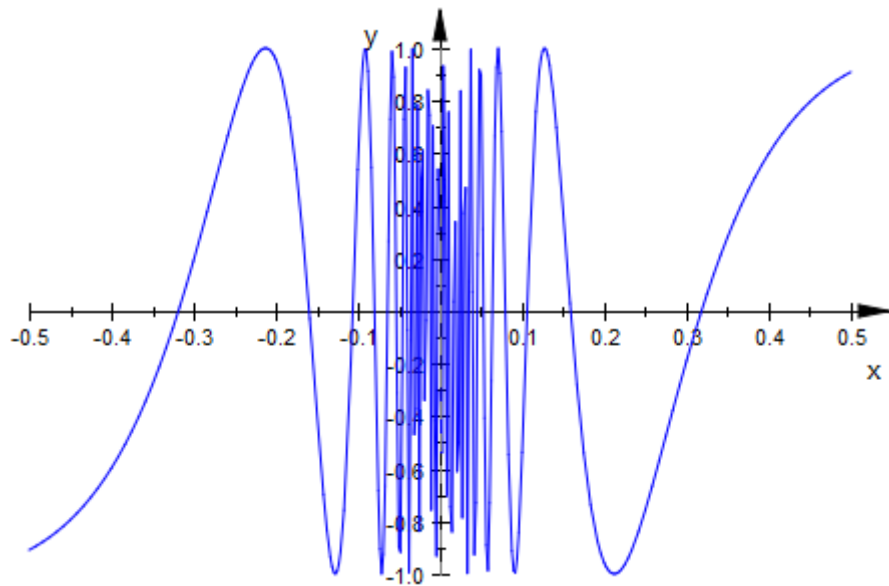
attribute name	possible values/example	meaning	default
Height	<code>8*unit::cm</code>	physical height of the picture	<code>80*unit::mm</code>
Width	<code>12*unit::cm</code>	physical width of the picture	<code>120*unit::mm</code>
Footer	string	footer text	" " (no footer)
Header	string	header text	" " (no header)
Title	string	title text	" " (no title)

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
TitlePosition	[real value, real value]	coordinates of the lower left corner of the title	
GridVisible	TRUE, FALSE	visibility of “major” grid lines in all directions	FALSE
SubgridVisible	TRUE, FALSE	visibility of “minor” grid lines in all directions	FALSE
AdaptiveMesh	integer $\geq 2$	number of sample points of the numerical mesh	121
Axes	None, Automatic, Boxed, Frame, Origin	axes type	Automatic
AxesVisible	TRUE, FALSE	visibility of all axes	TRUE
AxesTitles	[string, string]	titles of the axes	["x", "y"]
CoordinateType	LinLin, LinLog, LogLin, LogLog	linear-linear, linear-logarithmic, logarithmic-linear, log-log	LinLin
Colors	list of RGB values	line colors	first 10 entries of RGB::ColorList
Frames	integer $\geq 0$	number of frames of an animation	50
LegendVisible	TRUE, FALSE	legend on/off	TRUE

attribute name	possible values/example	meaning	default
LineColorType	Dichromatic, Flat, Functional, Monochrome, Rainbow	color scheme	Flat
Mesh	integer $\geq 2$	number of sample points of the numerical mesh	121
Scaling	Automatic, Constrained, Unconstrained	scaling mode	Unconstrained
TicksNumber	None, Low, Normal, High	number of labeled ticks at all axes	Normal
VerticalAsymptotes	TRUE, FALSE	vertical asymptotes on/off	TRUE
ViewingBoxYRange	ymin..ymax	restricted viewing range in y direction	Automatic
YRange	ymin..ymax	restricted viewing range in y direction (equivalent to ViewingBoxYRange)	Automatic

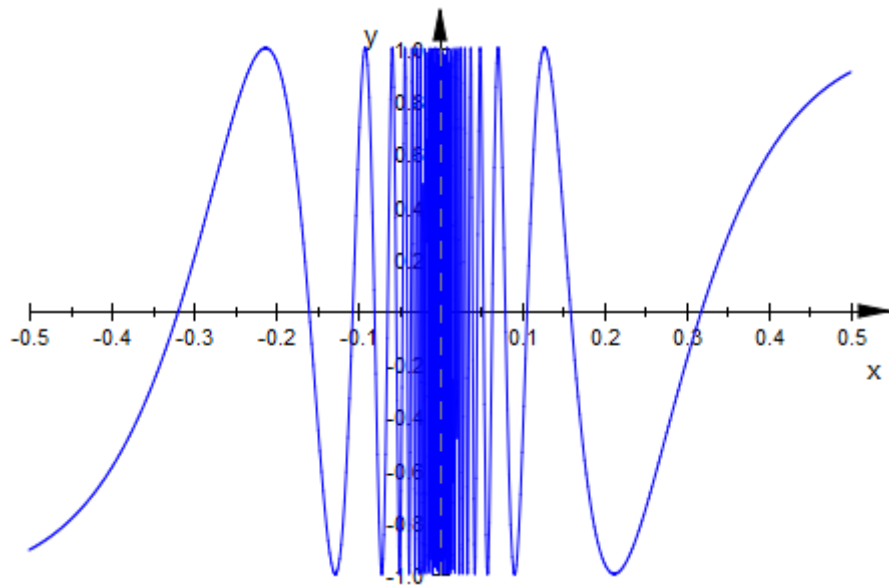
The following plot example features the notorious function  $\sin\left(\frac{1}{x}\right)$  that oscillates wildly near the origin:

```
plotfunc2d(sin(1/x), x = -0.5..0.5):
```



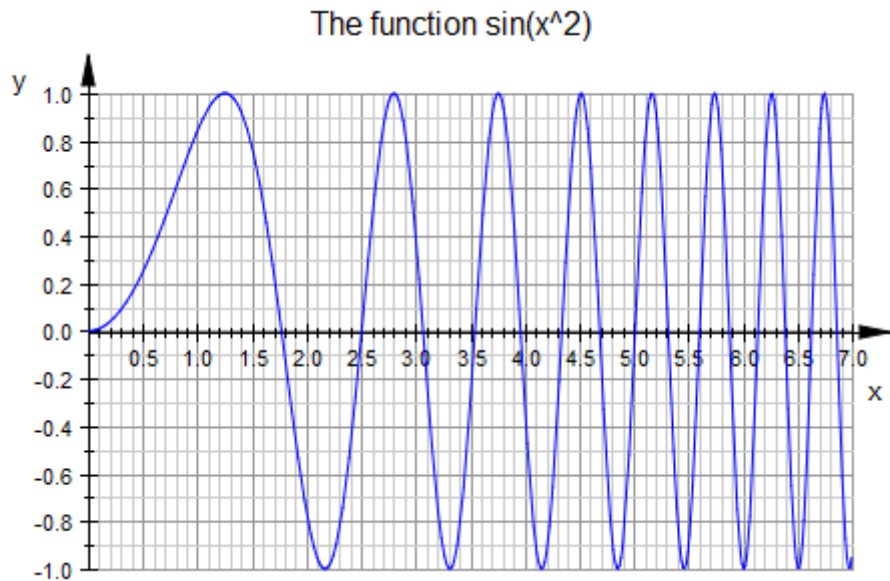
Clearly, the default of 121 sample points used by `plotfunc2d` does not suffice to create a sufficiently resolved plot. We increase the number of numerical mesh points via the `Mesh` attribute. Additionally, we increase the resolution depth of the adaptive plotting mechanism from its default value `AdaptiveMesh = 2` to `AdaptiveMesh = 4`:

```
plotfunc2d(sin(1/x), x = -0.5..0.5, Mesh = 500,  
           AdaptiveMesh = 4):
```



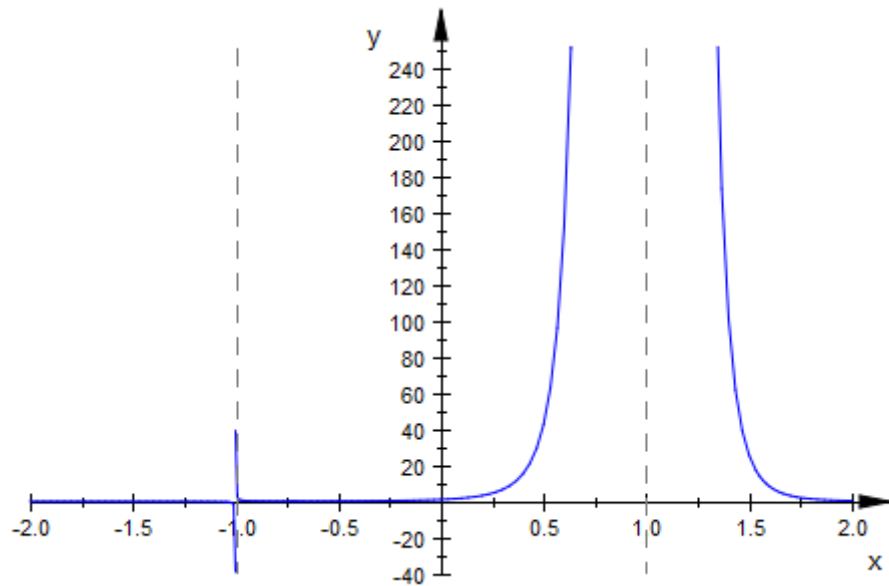
The following call specifies a header via `Header = "The function  $\sin(x^2)$ "`. The distance between labeled ticks is set to 0.5 along the  $x$  axis and to 0.2 along the  $y$  axis via `XTicksDistance = 0.5` and `YTicksDistance = 0.2`, respectively. Four additional unlabeled ticks between each pair of labeled ticks are set in the  $x$  direction via `XTicksBetween = 4`. One additional unlabeled tick between each pair of labeled ticks in the  $y$  direction is requested via `YTicksBetween = 1`. Grid lines attached to the ticks are "switched on" by `GridVisible = TRUE` and `SubgridVisible = TRUE`:

```
plotfunc2d(sin(x^2), x = 0..7,
  Header = "The function  $\sin(x^2)$ ",
  XTicksDistance = 0.5, YTicksDistance = 0.2,
  XTicksBetween = 4, YTicksBetween = 1,
  GridVisible = TRUE, SubgridVisible = TRUE):
```



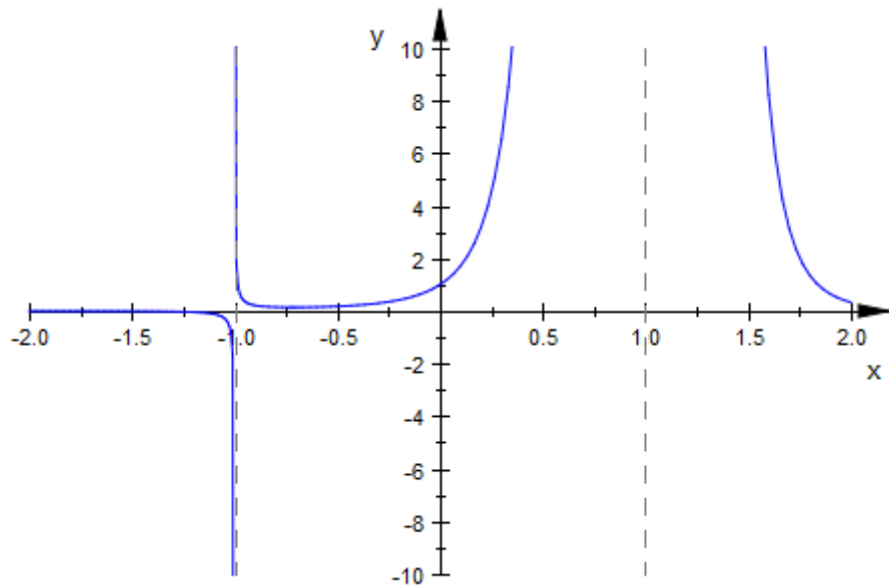
When singularities are found in the function, an automatic clipping is called trying to restrict the vertical viewing range in some way to obtain a “reasonably” scaled picture. This is a heuristic approach that sometimes needs a helping adaptation “by hand”. In the following example, the automatically chosen range between  $y \approx -1$  and  $y \approx 440$  in vertical direction is suitable to represent the 6th order pole at  $x = 1$ , but it does not provide a good resolution of the first order pole at  $x = -1$ :

```
plotfunc2d(1/(x + 1)/(x - 1)^6, x = -2..2):
```



There is no good viewing range that is adequate for both poles because they are of different order. However, some compromise can be found. We override the automatic viewing range suggested by `plotfunc2d` and request a specific viewing range in vertical direction via `ViewingBoxYRange`:

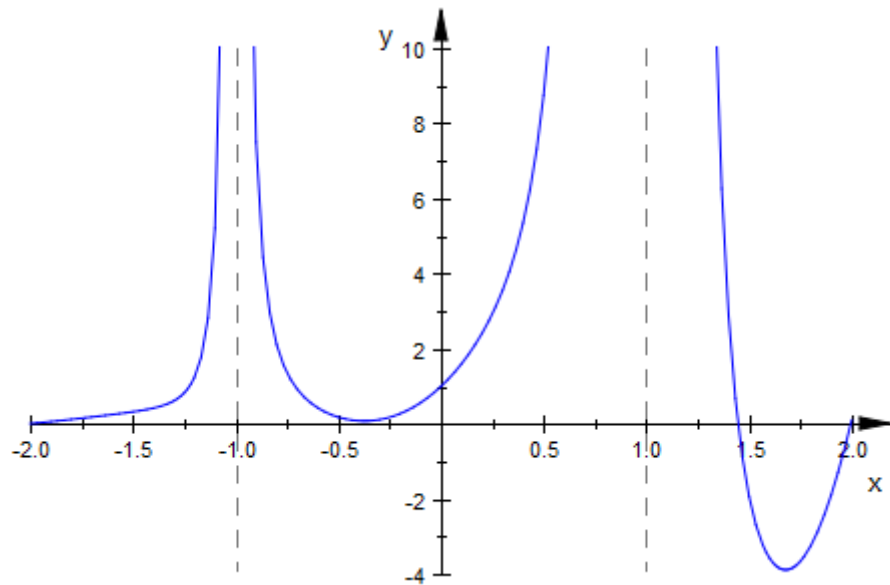
```
plotfunc2d(1/(x + 1)/(x - 1)^6, x = -2..2,  
           ViewingBoxYRange = -10..10):
```



The values of the following function have a lower bound but no upper bound. We use the attribute `ViewingBoxYRange = Automatic..10` to let `plotfunc2d` find a lower bound for the viewing box by itself whilst requesting a specific value of 10 for the upper bound:

```
plotfunc2d(exp(x)*sin(PI*x) + 1/(x + 1)^2/(x - 1)^4, x = -2..2,  
           ViewingBoxYRange = Automatic..10):
```

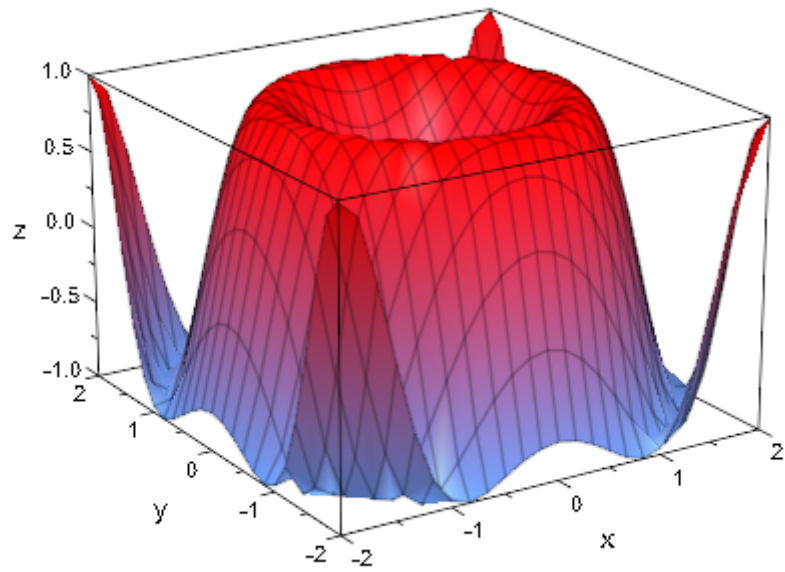




### 3D Function Graphs: `plotfunc3d`

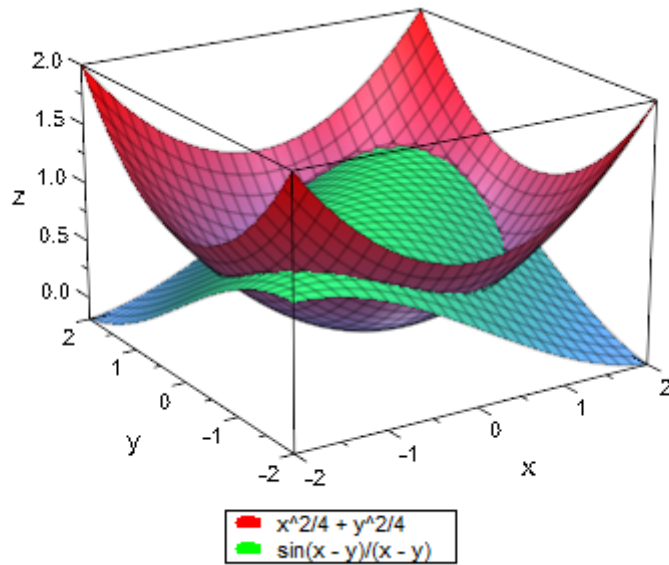
We consider 3D examples, i.e., plots of bivariate functions  $z = f(x, y)$ . Here is a plot of the function  $\sin(x^2 + y^2)$ :

```
plotfunc3d(sin(x^2 + y^2), x = -2..2, y = -2..2):
```



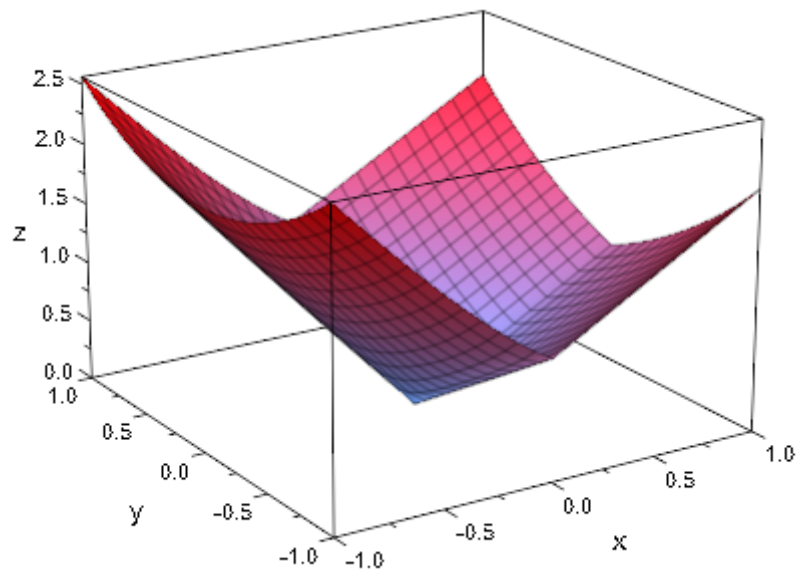
If several functions are to be plotted in the same graphical scene, just pass a sequence of function expressions; all functions are plotted over the specified common range:

```
plotfunc3d((x^2 + y^2)/4, sin(x - y)/(x - y),  
           x = -2..2, y = -2..2):
```



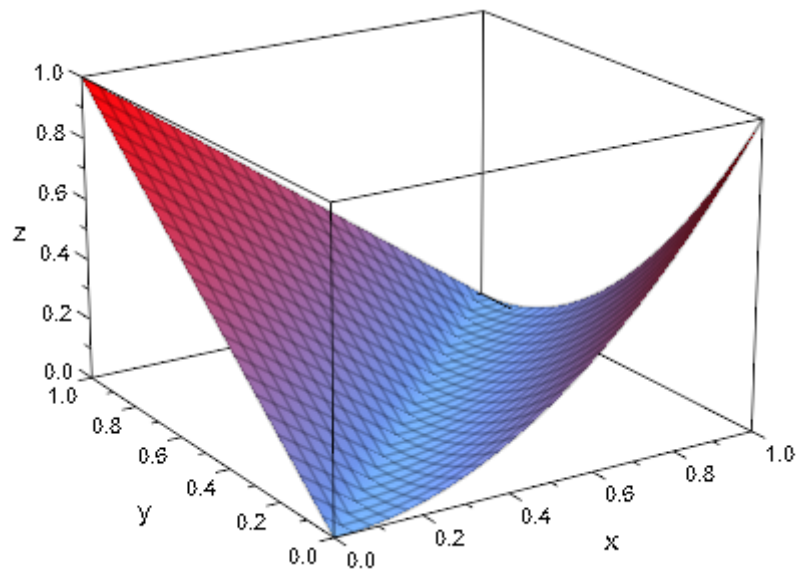
Functions that do not allow a simple symbolic representation by an expression can also be defined by a procedure that produces a numerical value  $f(x, y)$  when called with numerical values  $x, y$  from the plot range. In the following example we consider the largest eigenvalue of a symmetric  $3 \times 3$  matrix that contains two parameters  $x, y$ . We plot this eigenvalue as a function of  $x$  and  $y$ :

```
f := (x, y) -> max(numeric::eigenvalues(
    matrix([[ -y, x, -x],
           [ x, y, x],
           [-x, x, y^2]]))):
plotfunc3d(f, x = -1..1, y = -1..1):
```



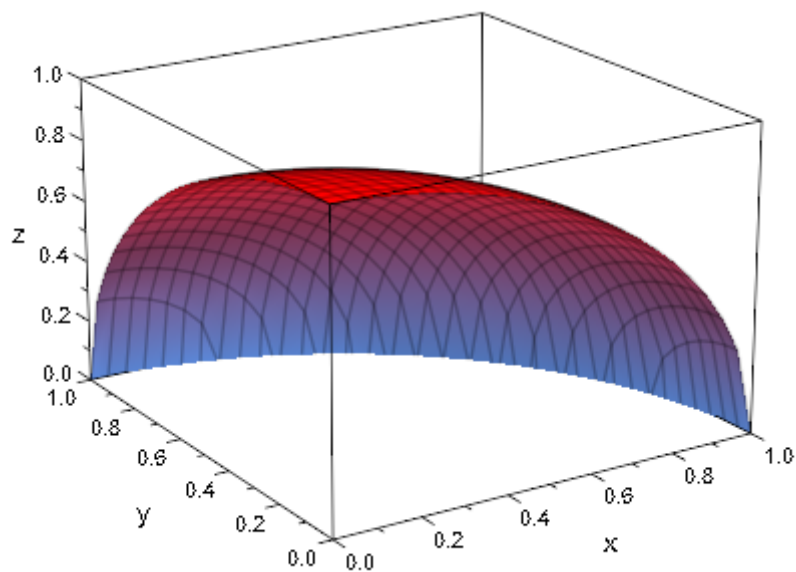
The names  $x, y$  used in the specification of the plotting range provide the labels of the corresponding axes. Functions can also be defined by piecewise objects:

```
plotfunc3d(piecewise([x < y, y - x], [x > y, (y - x)^2]),  
           x = 0..1, y = 0..1)
```



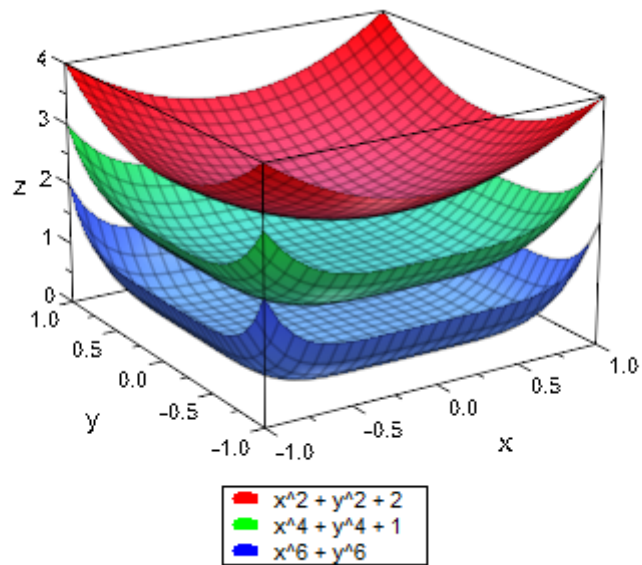
Note that there are gaps in the definition of the function above: no function value is specified for  $x = y$ . This does not cause any problem, because `plotfunc3d` simply ignores points that do not produce real numerical values if it finds suitable values in the neighborhood. Thus, missing points do not show up in a plot if these points are isolated or are restricted to some 1-dimensional curve in the  $x$ - $y$  plane. If the function is not real valued in regions of nonzero measure, the resulting plot contains holes. The following function is real valued only in the disk  $x^2 + y^2 \leq 1$ :

```
plotfunc3d(sqrt(1 - x^2 - y^2), x = 0..1, y = 0..1):
```



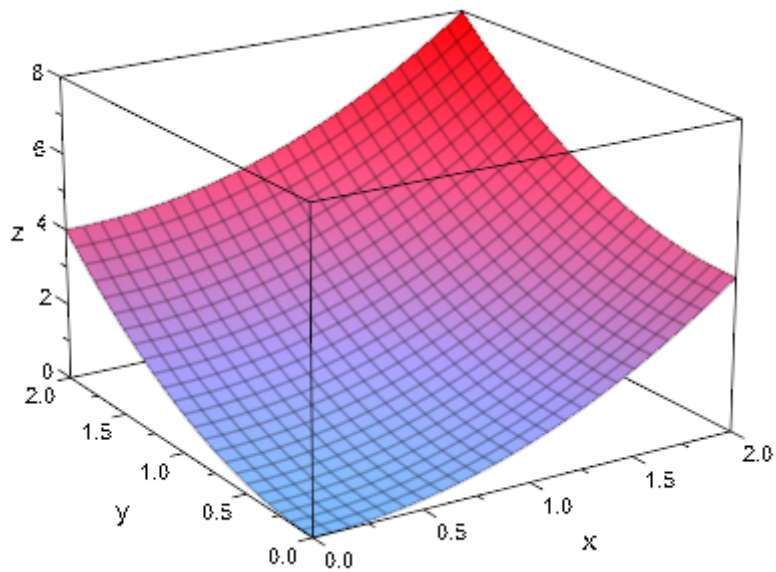
When several functions are plotted in the same scene, they are drawn in different colors that are chosen automatically. With the `Colors` attribute one may specify a list of RGB colors that `plotfunc3d` shall use:

```
plotfunc3d(2 + x^2 + y^2, 1 + x^4 + y^4, x^6 + y^6,  
           x = -1..1, y = -1..1,  
           Colors = [RGB::Red, RGB::Green, RGB::Blue]):
```



Animated 3D plots of functions are created by passing function expressions depending on two variables ( $x$ ,  $y$ , say) and an animation parameter ( $a$ , say) and specifying a range for  $x$ ,  $y$ , and  $a$ :

```
plotfunc3d(x^a + y^a, x = 0..2, y = 0..2, a = 1..2):
```

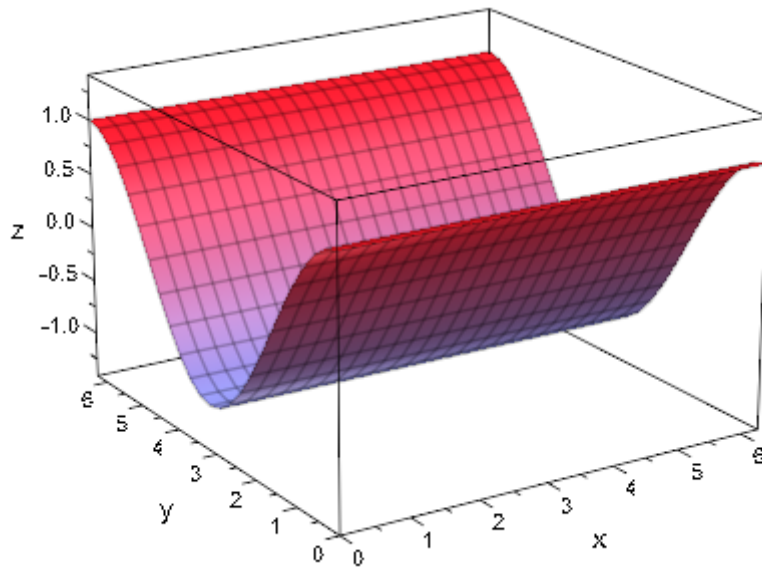


Once the plot is created, the first frame of the picture appears as a static plot. After double-clicking on the picture, the animation starts. The usual controls for stopping, going to some other point in time etc. are available.

The default number of frames of the animation is 50. If a different value is desired, just pass the attribute `Frames = n`, where  $n$  is the number of frames that shall be created:

```
plotfunc3d(sin(a)*sin(x) + cos(a)*cos(y),  
           x = 0..2*PI, y = 0..2*PI,  
           a = 0..2*PI, Frames = 32):
```





Apart from the color specification or the Frames number, there is a large number of further attributes that may be passed to `plotfunc3d`. Each attribute is passed as an equation `AttributeName = AttributeValue` to `plotfunc3d`. Here, we only present some selected attributes. Section [Attributes for plotfunc2d and plotfunc3d](#) provides further tables with more attributes.

### **plotfunc3d**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
Height	<code>8*unit::cm</code>	physical height of the picture	<code>80*unit::mm</code>
Width	<code>12*unit::cm</code>	physical width of the picture	<code>120*unit::mm</code>
Footer	string	footer text	<code>" "</code> (no footer)
Header	string	header text	<code>" "</code> (no header)

**plotfunc3d (Continued)**

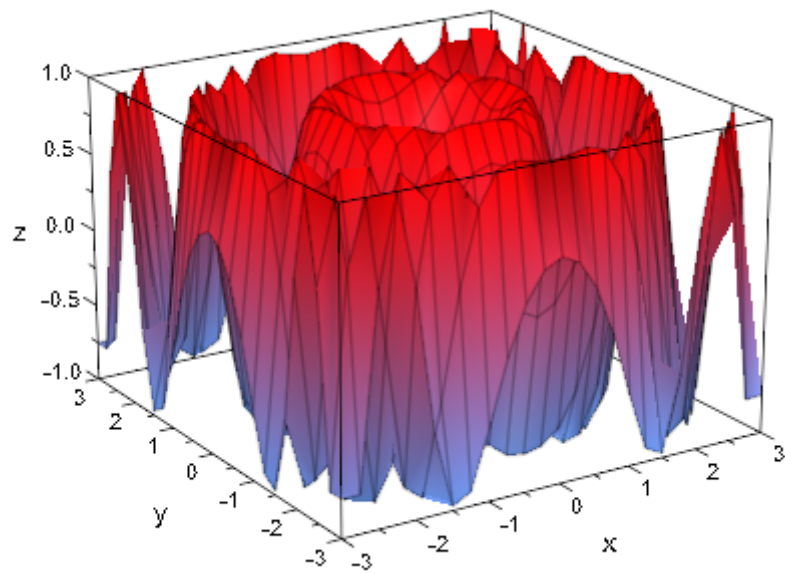
<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
Title	string	title text	" " (no title)
TitlePosition	[real value, real value]	coordinates of the lower left corner of the title	
GridVisible	TRUE, FALSE	visibility of "major" grid lines in all directions	FALSE
SubgridVisible	TRUE, FALSE	visibility of "minor" grid lines in all directions	FALSE
AdaptiveMesh	integer $\geq 0$	depth of the adaptive mesh	0
Axes	Automatic, Boxed, Frame, Origin	axes type	Boxed
AxesVisible	TRUE, FALSE	visibility of all axes	TRUE
AxesTitles	[string, string, string]	titles of the axes	["x", "y", "z"]
CoordinateType	LinLinLin, ..., LogLogLog	linear-linear-linear, linear-logarithmic, logarithmic-linear, log-log plot	LinLinLin
Colors	list of RGB values	fill colors	
Frames	integer $\geq 0$	number of frames of the animation	50
LegendVisible	TRUE, FALSE	legend on/off	TRUE

**plotfunc3d (Continued)**

attribute name	possible values/example	meaning	default
FillColorType	Dichromatic, Flat, Functional, Monochrome, Rainbow	color scheme	Dichromatic
Mesh	[integer $\geq 2$ , integer $\geq 2$ ]	number of “major” mesh points	[25, 25]
Submesh	[integer $\geq 0$ , integer $\geq 0$ ]	number of “minor” mesh points	[0, 0]
Scaling	Automatic, Constrained, Unconstrained	scaling mode	Unconstrained
TicksNumber	None, Low, Normal, High	number of labeled ticks at all axes	Normal
ViewingBoxZRange	zmin..zmax	restricted viewing range in $z$ direction	Automatic
ZRange	zmin..zmax	restricted viewing range in $z$ direction (equivalent to ViewingBoxZRange)	Automatic

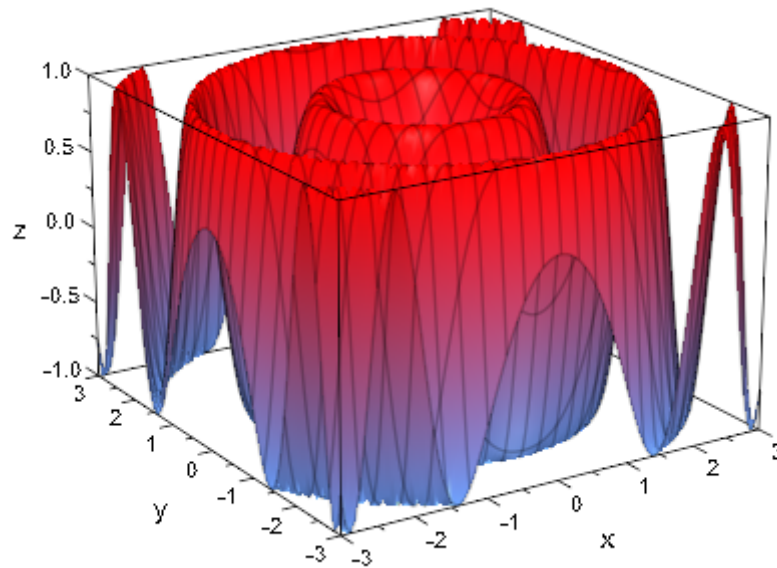
In the following example, the default mesh of  $25 \times 25$  sample points used by `plotfunc3d` does not suffice to create a sufficiently resolved plot:

```
plotfunc3d(sin(x^2 + y^2), x = -3..3, y = -3..3):
```



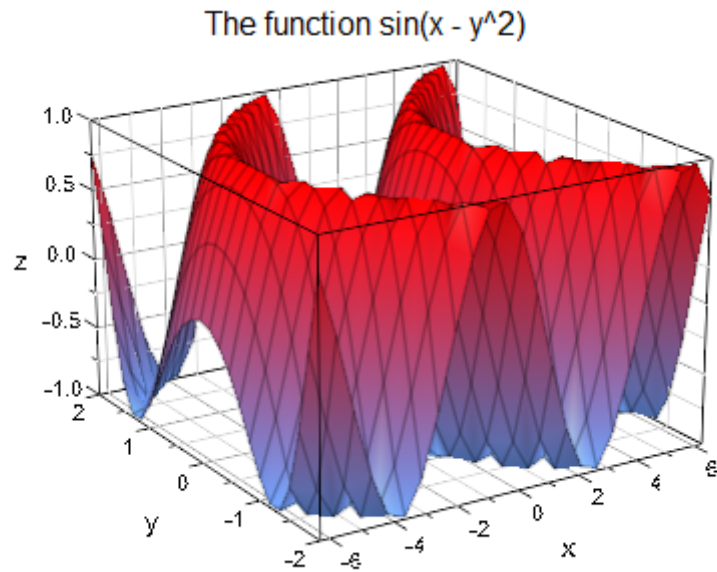
We increase the number of numerical mesh points via the Submesh attribute:

```
plotfunc3d(sin(x^2 + y^2), x = -3..3, y = -3..3, Submesh = [3, 3])
```



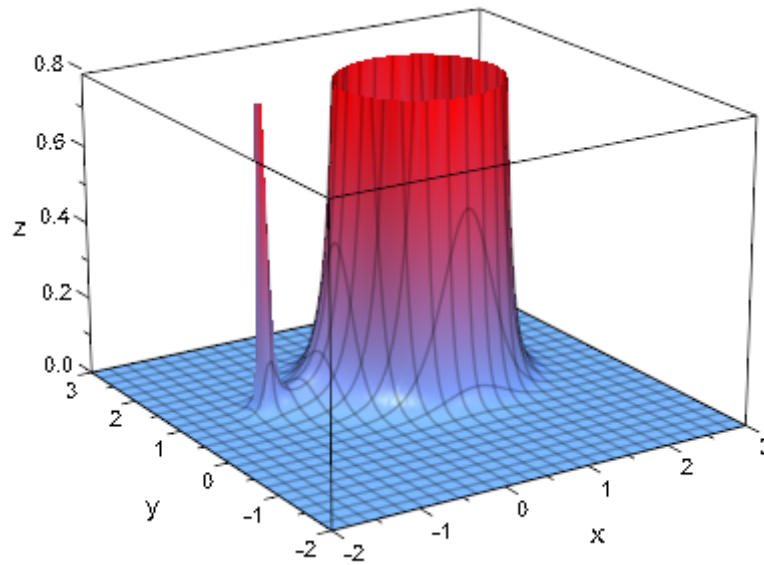
The following call specifies a header via `Header = "The function  $\sin(x - y^2)$ "`. Grid lines attached to the ticks are "switched on" by `GridVisible = TRUE` and `SubgridVisible = TRUE`:

```
plotfunc3d(sin(x - y^2), x = -2*PI..2*PI, y = -2..2,  
           Header = "The function  $\sin(x - y^2)$ ",  
           GridVisible = TRUE, SubgridVisible = TRUE):
```



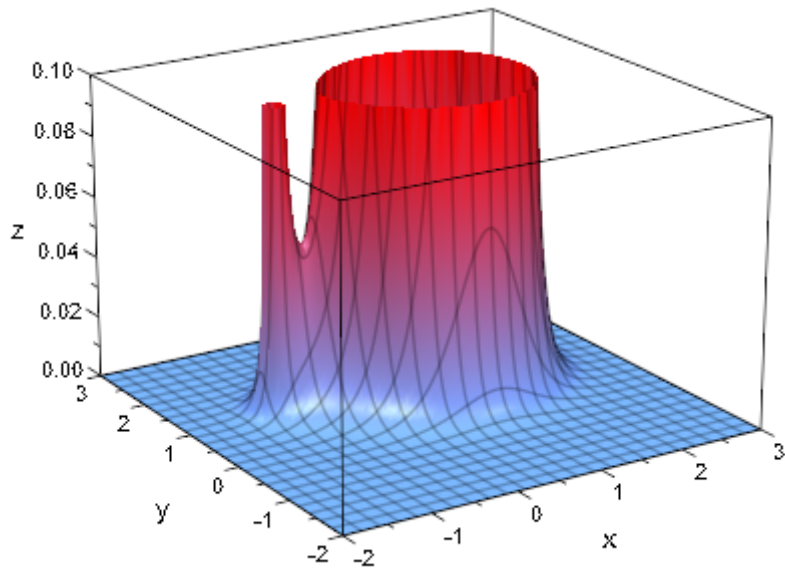
When singularities are found in the function, an automatic clipping is called trying to restrict the vertical viewing range in some way to obtain a “reasonably” scaled picture. This is a heuristic approach that sometimes needs a helping adaptation “by hand”. In the following example, the automatically chosen range between  $z \approx 0$  and  $z \approx 0.8$  in vertical direction is suitable to represent the pole at  $x = 1, y = 1$ , but it does not provide a good resolution of the pole at  $x = -1, y = 1$ :

```
plotfunc3d(1/((x + 1)^2 + (y - 1)^2)/((x - 1)^2 + (y - 1)^2)^5,
           x = -2..3, y = -2..3, Submesh = [3, 3]):
```



There is no good viewing range that is adequate for both poles because they are of different order. We override the automatic viewing range suggested by `plotfunc3d` and request a specific viewing range in the vertical direction via `ViewingBoxZRange`:

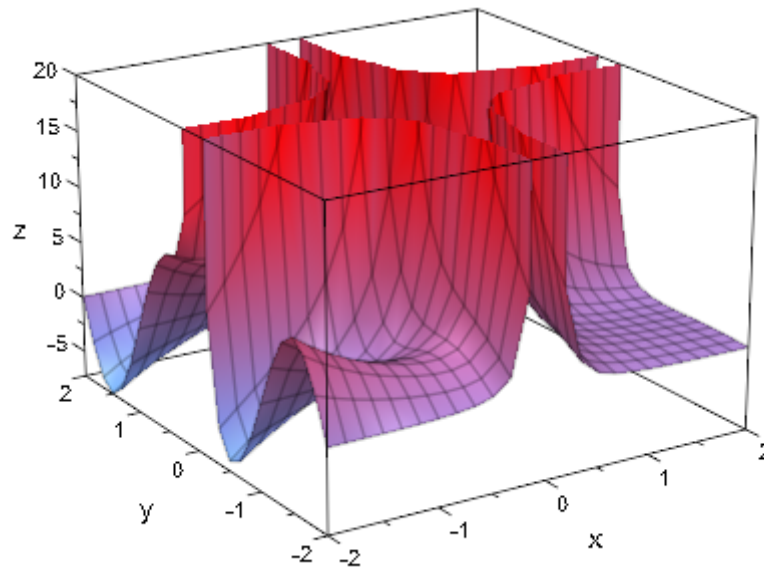
```
plotfunc3d(1/((x + 1)^2 + (y - 1)^2)/((x - 1)^2 + (y - 1)^2)^5,
           x = -2..3, y = -2..3,
           Submesh = [3, 3], ViewingBoxZRange = 0..0.1):
```



The values of the following function have a lower bound but no upper bound. We use the attribute `ViewingBoxZRange = Automatic..20` to let `plotfunc2d` find a lower bound for the viewing box by itself whilst requesting a specific value of 20 for the upper bound:

```
plotfunc3d(1/x^2/y^2 + exp(-x)*sin(PI*y),  
           x = -2..2, y = -2..2,  
           ViewingBoxZRange = Automatic..20):
```





## Attributes for `plotfunc2d` and `plotfunc3d`

The function plotters `plotfunc2d` and `plotfunc3d` accept a large number of attributes (options). In this section we give an overview over the most important attributes. There is a help page for each attribute that provides more detailed information and examples.

Attributes are passed as equations `AttributeName = AttributeValue` to `plotfunc2d` and `plotfunc3d`. Several attributes can be passed simultaneously as a sequence of such equations.

The attributes can be changed interactively in the property inspector. Click on the plot to make subwindows appear for the “object browser” and the “property inspector” (see section Viewer, Browser, and Inspector: Interactive Manipulation). The functions plotted by `plotfunc2d` and `plotfunc3d` appear as plot objects of type `plot::Function2d` and `plot::Function3d`, respectively. They are embedded in a coordinate system inside a graphical scene. The scene is embedded in a viewing area called the ‘Canvas.’ In the viewer, the various plot attributes are associated with the different objects of this graphical hierarchy. Typically, layout parameters and titles are set

within the canvas, whilst axes, grid lines, viewing boxes etc. are associated with the coordinate system. Some attributes such as colors, line width, the numerical mesh size etc. belong to the function graphs and can be set separately for each function plotted by `plotfunc2d/plotfunc3d`.

The last entry in the following tables provides the location of the attribute in the graphical hierarchy of the object browser. For example, for changing the background color of the picture, select the scene by double clicking the 'Scene2d'/'Scene3d' entry in the object browser. Now, the property inspector provides a tree of attributes with the nodes 'Annotation,' 'Layout,' and 'Style.' Opening the 'Style' sub-tree, one finds an entry for `BackgroundColor` which allows to change the background color interactively.

Here is a table of the most important attributes for setting the layout and the background of the picture:

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
Width	12*unit::cm	physical width of the picture	120*unit::mm	Canvas
Height	8*unit::cm	physical height of the picture	80*unit::mm	Canvas
BackgroundColor	RGB color	color of the background	RGB::White	Scene2d/3d
BorderColor	RGB color	color of the border	RGB::Grey50	Scene2d/3d
BorderWidth	1*unit::mm	width of the border	0	Scene2d/3d
Margin	1*unit::mm	common width for all margins: BottomMargin, LeftMargin, etc.	1*unit::mm	Scene2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
BottomMargin	1*unit::mm	width of bottom margin	1*unit::mm	Scene2d/3d
LeftMargin	1*unit::mm	width of left margin	1*unit::mm	Scene2d/3d
RightMargin	1*unit::mm	width of right margin	1*unit::mm	Scene2d/3d
TopMargin	1*unit::mm	width of top margin	1*unit::mm	Scene2d/3d
BackgroundStyle	Flat, LeftRight, TopBottom, Pyramid	background style of 3D scenes	Flat	Scene3d
BackgroundColor	RGB color	secondary color of the background (used for color blends)	RGB::Grey75	Scene3d
BackgroundTransparent	TRUE/FALSE	transparent background?	FALSE	Scene2d

An overall title can be set as a footer and/or a header. Here is a table of the attributes determining the footer and/or header of the picture:

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
Footer	string	footer text	" " (no footer)	Scene2d/3d
Header	string	header text	" " (no header)	Scene2d/3d

(Continued)

attribute name	possible values/example	meaning	default	browser entry
FooterAlignment	left, Center, Right	horizontal alignment	Center	Scene2d/3d
HeaderAlignment	left, Center, Right	horizontal alignment	Center	Scene2d/3d
FooterFont	see section Fonts	font for the footer	sans-serif 12	Scene2d/3d
HeaderFont	see section Fonts	font for the header	sans-serif 12	Scene2d/3d

Apart from footers and/or headers of scenes and canvas, there are titles associated with the functions. In contrast to footer and header, function titles can be placed anywhere in the coordinate system via the attribute `TitlePosition`. Typically, titles are associated with individual objects rather than with entire scenes. Thus, when using `plotfunc2d` or `plotfunc3d`, a title attribute will usually only be used when a single function is displayed. However, several titles with separate positions can be set interactively in the property inspector for each of the functions:

attribute name	possible values/example	meaning	default	browser entry
TitleText	string	title text	" " (no title)	Function2d/3d
TitlePosition	[real value]	coordinates of the lower left corner of the title		Function2d
TitlePosition	[real value, real value]	coordinates of the lower left corner of the title		Function3d
TitlePositionX	real value	x coordinate of the lower left corner of the title		Function2d/3d
TitlePositionY	real value	y coordinate of the lower left corner of the title		Function2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
TitlePositionZ	z	z coordinate of the lower left corner of the title		Function3d
TitleFont	see section Fonts	font for the titles	sans-serif 11	Function2d/3d

If several functions are drawn simultaneously in one picture, it is useful to display a legend indicating which color is used for which function. See section Legends for further details on legends. Here is a table of the most important attributes determining the form of the legend. The attributes LegendEntry, LegendText, and LegendVisible = TRUE are set automatically by plotfunc2d/plotfunc3d if more than one function is plotted. The property inspector (Viewer, Browser, and Inspector: Interactive Manipulation) allows to reset the legend entry for each function:

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
LegendEntry	TRUE, FALSE	add this function to the legend?	TRUE	Function2d/3d
LegendText	string	legend text		Function2d/3d
LegendVisible	TRUE, FALSE	legend on/off	TRUE	Scene2d/3d
LegendPlacement	Top, Bottom	vertical placement	Bottom	Scene2d/3d
LegendAlignment	Left, Center, Right	horizontal alignment	Center	Scene2d/3d
LegendFont	see section Fonts	font for the legend text	sans-serif 8	Scene2d/3d

When singular functions are plotted, it is often useful to request a specific viewing range. Here is a table of the most important attributes for setting

viewing ranges. In the interactive object browser, you will find them under CoordinateSystem2d (CS2d) and CoordinateSystem3d (CS3d), respectively:

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
ViewingBox	xmax, ymin..ymax], [Automatic, Automatic]	viewing range in $x$ and $y$ direction	[Automatic, Automatic]	CS2d
ViewingBox	xmax, ymin..ymax, zmin..zmax], [Automatic, Automatic, Automatic]	viewing range in $x$ , $y$ , $z$ direction	[Automatic, Automatic, Automatic]	CS3d
ViewingBoxXRange		viewing range in $x$ direction	Automatic.. Automatic	CS2d/3d
ViewingBoxYRange		viewing range in $y$ direction	Automatic.. Automatic	CS2d/3d
ViewingBoxZRange		viewing range in $z$ direction	Automatic.. Automatic	CS3d
ViewingBoxXMin	value or Automatic	lowest viewing value in $x$ direction	Automatic	CS2d/3d
ViewingBoxXMax	value or Automatic	highest viewing value in $x$ direction	Automatic	CS2d/3d
ViewingBoxYMin	value or Automatic	lowest viewing value in $y$ direction	Automatic	CS2d/3d
ViewingBoxYMax	value or Automatic	highest viewing value in $y$ direction	Automatic	CS2d/3d

**(Continued)**

attribute name	possible values/example	meaning	default	browser entry
ViewingBoxZMin	ZMin value or Automatic	lowest viewing value in z direction	Automatic	CS3d
ViewingBoxZMax	ZMax value or Automatic	highest viewing value in z direction	Automatic	CS3d

In contrast to the routines of the plot library, `plotfunc2d` and `plotfunc3d` also accept the attributes `YMin`, `YMax`, `YRange` and `ZMin`, `ZMax`, `ZRange`, respectively, as shortcuts for the somewhat clumsy attribute names `ViewingBoxYMin` etc. E.g.,

```
plotfunc2d(f(x), x = xmin..xmax, YRange = ymin..ymax)
```

is equivalent to

```
plotfunc2d(f(x), x = xmin..xmax,
           ViewingBoxYRange = ymin..ymax)
```

and

```
plotfunc3d(f(x, y), x = xmin..xmax, y = ymin..ymax,
           ZRange = zmin..zmax)
```

is equivalent to

```
plotfunc3d(f(x, y), x = xmin..xmax, y = ymin..ymax,
           ViewingBoxZRange = zmin..zmax)
```

Here is a table of the most important attributes for arranging coordinate axes. In the interactive object browser, you will find them under CoordinateSystem2d (CS2d) and CoordinateSystem3d (CS3d), respectively:

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
Axes	Automatic, Boxed, Frame, Origin	axes type	Automatic	CS2d/3d
AxesVisible	TRUE, FALSE	visibility of all axes	TRUE	CS2d/3d
XAxisVisible	TRUE, FALSE	visibility of the <i>x</i> axis	TRUE	CS2d/3d
YAxisVisible	TRUE, FALSE	visibility of the <i>y</i> axis	TRUE	CS2d/3d
ZAxisVisible	TRUE, FALSE	visibility of the <i>z</i> axis	TRUE	CS3d
AxesTitles	[string, string]	titles of the axes (2D)	[ "x", "y" ]	CS2d
AxesTitles	[string, string, string]	titles of the axes (3D)	[ "x", "y", "z" ]	CS3d
XAxisTitle	string	title of the <i>x</i> axis	"x"	CS2d/3d
YAxisTitle	string	title of the <i>y</i> axis	"y"	CS2d/3d



**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
ZAxisTitle	string	title of the z axis	"z"	CS3d
AxesTitleAlign	Begin, Center, End	alignment for all axes titles	End	CS2d
AxesTitleAlign	Begin, Center, End	alignment for all axes titles	Center	CS3d
XAxisTitleAlign	Begin, Center, End	alignment for the x axis title	End	CS2d
XAxisTitleAlign	Begin, Center, End	alignment for the x axis title	Center	CS3d
YAxisTitleAlign	Begin, Center, End	alignment for the y axis title	End	CS2d
YAxisTitleAlign	Begin, Center, End	alignment for the y axis title	Center	CS3d
ZAxisTitleAlign	Begin, Center, End	alignment for the z axis title	Center	CS3d

**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
YAxisTitleOrientation	Vertical, Horizontal	orientation of the y axis title	Horizontal	CS2d
AxesTips	TRUE, FALSE	axes with tips?	TRUE	CS2d/3d
AxesOrigin	[real value, real value]	crosspoint of the axes (2D)	[0, 0]	CS2d
AxesOrigin	[real value, real value, real value]	crosspoint of the axes (3D)	[0, 0, 0]	CS3d
AxesOriginX	real value	x value of AxesOrigin	0	CS2d/3d
AxesOriginY	real value	y value of AxesOrigin	0	CS2d/3d
AxesOriginZ	real value	z value of AxesOrigin	0	CS3d
AxesLineColor	RGB color	color of the axes	RGB::Black	CS2d/3d
AxesLineWidth	0.18*unit::mm	physical width of the axes lines	0.18*unit::mm	CS2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
AxesInFront	TRUE, FALSE	axes in front of the objects?	FALSE	CS2d
AxesTitleFont	see section Fonts	font for the axes titles	sans-serif 10	CS2d/3d

Here is a table of the most important attributes for setting tick marks and tick labels along the axes. In the interactive object browser, you will find them under `CoordinateSystem2d` (CS2d) and `CoordinateSystem3d` (CS3d), respectively:

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
TicksVisible	TRUE, FALSE	visibility of ticks along all axes	TRUE	CS2d/3d
XTicksVisible	TRUE, FALSE	visibility of ticks along the $x$ axis	TRUE	CS2d/3d
YTicksVisible	TRUE, FALSE	visibility of ticks along the $y$ axis	TRUE	CS2d/3d
ZTicksVisible	TRUE, FALSE	visibility of ticks along the $z$ axis	TRUE	CS3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
TicksDistance	positive real value	distance between labeled ticks along all axes		CS2d/3d
XTicksDistance	positive real value	distance between labeled ticks along the <i>x</i> axis		CS2d/3d
YTicksDistance	positive real value	distance between labeled ticks along the <i>y</i> axis		CS2d/3d
ZTicksDistance	positive real value	distance between labeled ticks along the <i>z</i> axis		CS3d
TicksAnchor	real value	the position of a labeled tick to start with	0	CS2d/3d
XTicksAnchor	real value	the position of a labeled tick to start with	0	CS2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
YTicksAnchor	real value	the position of a labeled tick to start with	0	CS2d/3d
ZTicksAnchor	real value	the position of a labeled tick to start with	0	CS3d
TicksNumber	None, Low, Normal, High	number of labeled ticks along all axes	Normal	CS2d/3d
XTicksNumber	None, Low, Normal, High	number of labeled ticks along the $x$ axis	Normal	CS2d/3d
YTicksNumber	None, Low, Normal, High	number of labeled ticks along the $y$ axis	Normal	CS2d/3d
ZTicksNumber	None, Low, Normal, High	number of labeled ticks along the $z$ axis	Normal	CS3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
TicksBetween	integer $\geq 0$	number of smaller unlabeled ticks between labeled ticks along all axes	1	CS2d/3d
XTicksBetween	integer $\geq 0$	number of smaller unlabeled ticks between labeled ticks along the $x$ axis	1	CS2d/3d
YTicksBetween	integer $\geq 0$	number of smaller unlabeled ticks between labeled ticks along the $y$ axis	1	CS2d/3d
ZTicksBetween	integer $\geq 0$	number of smaller unlabeled ticks between labeled ticks along the $z$ axis	1	CS3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
TicksLabelStyle	Diagonal, Horizontal, Shifted, Vertical	orientation and style of tick labels along all axes	Horizontal	CS2d/3d
XTicksLabelStyle	Diagonal, Horizontal, Shifted, Vertical	orientation and style of tick labels along the $x$ axes	Horizontal	CS2d/3d
YTicksLabelStyle	Diagonal, Horizontal, Shifted, Vertical	orientation and style of tick labels along the $y$ axis	Horizontal	CS2d/3d
ZTicksLabelStyle	Diagonal, Horizontal, Shifted, Vertical	orientation and style of tick labels along the $z$ axis	Horizontal	CS3d
TicksAt	[tick1, tick2, ...], where tick.i is a real value (the position) or an equation position = "label string" (such as $3.14 = \text{"pi"}$ )	ticks set by the user, valid for all axes		CS2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
XTicksAt	see TicksAt	ticks along the <i>x</i> axis set by the user		CS2d/3d
YTicksAt	see TicksAt	ticks along the <i>y</i> axis set by the user		CS2d/3d
ZTicksAt	see TicksAt	ticks along the <i>z</i> axis set by the user		CS3d
TicksLength	2*unit::mm	length of the tick marks	2*unit::mm	CS2d
TicksLabelFont	Font section Fonts	font for all axes titles	sans-serif 8	CS2d/3d

Coordinate grid lines can be drawn in the background of a graphical scene (corresponding to the rulings of lined paper). They are attached to the tick marks along the axes. There are grid lines attached to the “major” labeled tick marks which are referred to as the “Grid.” There are also grid lines associated with the “minor” unlabeled tick marks set by the attribute TicksBetween. These “minor” grid lines are referred to as the “Subgrid.” The two kinds of grid lines can be set independently. In the interactive object browser, you will find the following attributes under CoordinateSystem2d (CS2d) and CoordinateSystem3d (CS3d), respectively:



**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
GridVisible	TRUE, FALSE	visibility of “major” grid lines in all directions	FALSE	CS2d/3d
SubgridVisible	TRUE, FALSE	visibility of “minor” grid lines in all directions	FALSE	CS2d/3d
XGridVisible	TRUE, FALSE	visibility of “major” grid lines in $x$ direction	FALSE	CS2d/3d
XSubgridVisible	TRUE, FALSE	visibility of “minor” grid lines in $x$ direction	FALSE	CS2d/3d
YGridVisible	TRUE, FALSE	visibility of “major” grid lines in $y$ direction	FALSE	CS2d/3d
YSubgridVisible	TRUE, FALSE	visibility of “minor” grid lines in $y$ direction	FALSE	CS2d/3d
ZGridVisible	TRUE, FALSE	visibility of “major” grid lines in $z$ direction	FALSE	CS3d
ZSubgridVisible	TRUE, FALSE	visibility of “minor” grid lines in $z$ direction	FALSE	CS3d

**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
GridLineColor	RGB color	color of all “major” grid lines	RGB::Grey75	CS2d/3d
SubgridLineColor	RGB color	color of all “minor” grid lines	RGB::Grey	CS2d/3d
GridLineWidth	0.1*unit::mm	width of all “major” grid lines	0.1*unit::mm	CS2d/3d
SubgridLineWidth	0.1*unit::mm	width of all “minor” grid lines	0.1*unit::mm	CS2d/3d
GridLineStyle	Dashed, Dotted, Solid	drawing style of all “major” grid lines	Solid	CS2d/3d
SubgridLineStyle	Dashed, Dotted, Solid	drawing style of all “minor” grid lines	Solid	CS2d/3d
GridInFront	TRUE, FALSE	grid lines in front of all objects?	FALSE	CS2d

Animations require that plotting ranges  $x = x_{min}..x_{max}$  (and  $y = y_{min}..y_{max}$ ) are fully specified in `plotfunc2d` (or `plotfunc3d`, respectively). Animations are triggered by passing an additional range such as  $a = a_{min}..a_{max}$  to `plotfunc2d/plotfunc3d`. The animation parameter  $a$  may turn up in the expression of the functions that are to be plotted as well as in various other places such as the coordinates of titles etc. See section Graphics and Animations for details.

**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
Frames	integer $\geq 0$	number of frames of the animation	50	Function2d/3d
ParameterName	symbolic name	name of the animation parameter		Function2d/3d
ParameterRange	min..amax	range of the animation parameter		Function2d/3d
ParameterBegin	min: real value	lowest value of the animation parameter		Function2d/3d
ParameterEnd	max: real value	highest value of the animation parameter		Function2d/3d
TimeRange	start..end	physical time range for the animation	0..10	Function2d/3d
TimeBegin	start: real value	physical time when the animation begins	0	Function2d/3d
TimeEnd	end: real value	physical time when the animation ends	10	Function2d/3d
VisibleBefore	real value	physical time when the object becomes invisible		Function2d/3d

(Continued)

attribute name	possible values/example	meaning	default	browser entry
VisibleAfter	real value	physical time when the object becomes visible		Function2d/3d
VisibleFromTo	range of real values	physical time range when the object is visible		Function2d/3d
VisibleBeforeBegin	TRUE, FALSE	visible before animation begins?	TRUE	Function2d/3d
VisibleAfterEnd	TRUE, FALSE	visible after animation ends?	TRUE	Function2d/3d

Functions are plotted as polygons consisting of straight line segments between points of the “numerical mesh.” The number of points in this numerical mesh are set by various “mesh” attributes:

attribute name	possible values/example	meaning	default	browser entry
Mesh	integer $\geq 2$	number of “major” mesh points in $x$ direction. The same as XMesh.	121	Function2d
Mesh	[integer $\geq 2$ , integer $\geq 2$ ]	number of “major” mesh points in $x$ and $y$ direction. Corresponds to XMesh, YMesh.	[25,25]	Function3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
Submesh	integer $\geq 0$	number of “minor” mesh points between the “major” mesh points set by Mesh. The same as XSubmesh.	0	Function2d
Submesh	[integer $\geq 0$ , integer $\geq 0$ ]	number of “minor” mesh points between the “major” mesh points set by Mesh. Corresponds to XSubmesh, YSubmesh.	[0, 0]	Function3d
XMesh	integer $\geq 2$	number of “major” mesh points in the $x$ direction	121	Function2d
XMesh	integer $\geq 2$	number of “major” mesh points in the $x$ direction	25	Function3d
XSubmesh	integer $\geq 0$	number of “minor” mesh points between the “major” mesh points set by XMesh	0	Function2d/3d

(Continued)

attribute name	possible values/example	meaning	default	browser entry
YMesh	integer $\geq 2$	number of “major” mesh points in the $y$ direction	121	Function2d
YMesh	integer $\geq 2$	number of “major” mesh points in the $y$ direction	25	Function3d
YSubmesh	integer $\geq 0$	number of “minor” mesh points between the “major” mesh points set by YMesh	0	Function3d
AdaptiveMesh	integer $\geq 0$	depth of the adaptive mesh	2	Function2d
AdaptiveMesh	integer $\geq 0$	depth of the adaptive mesh	0	Function3d

In 2D pictures generated by `plotfunc2d`, singularities of a function are indicated by vertical lines (“vertical asymptotes”), unless `DiscontinuitySearch = FALSE` is set. Here is a table with the attributes for setting the style of the vertical asymptotes:

attribute name	possible values/example	meaning	default	browser entry
VerticalAsymptoteVisible	TRUE, FALSE	visibility	TRUE	Function2d
VerticalAsymptoteColor	RGB color	color	RGB::Grey50	Function2d

**(Continued)**

attribute name	possible values/example	meaning	default	browser entry
VerticalAsymptotes	Dashed, Dotted, Solid	drawing style	Dashed	Function2d
VerticalAsymptoteWidth	0.2*unit::mm	physical width	0.2*unit::mm	Function2d

The colors of the functions plotted by `plotfunc2d` are chosen automatically. The property inspector (see section Viewer, Browser, and Inspector: Interactive Manipulation) allows to change these attributes:

attribute name	possible values/example	meaning	default	browser entry
LinesVisible	TRUE, FALSE	visibility of lines (switch this function on/off)	TRUE	Function2d
LineWidth	0.2*unit::mm	physical line width	0.2*unit::mm	Function2d
LineColor	RGB color	color		Function2d
LineColor2	RGB color			Function2d
LineStyle	Solid, Dotted, Solid	drawing style of line objects	Solid	Function2d
LineColorType	Color, Flat, Functional, Monochrome, Rainbow	color scheme for lines	Flat	Function2d
LineColorFunction	Color function	user defined coloring		Function2d

Setting `LinesVisible = FALSE` and `PointsVisible = TRUE`, the functions plotted by `plotfunc2d` will not be displayed as polygons but as sequences of points. Here is a table of the attributes to set the presentation style of points:

(Continued)

attribute name	possible values/example	meaning	default	browser entry
PointsVisible	TRUE, FALSE	visibility of points	FALSE	Function2d
PointSize	1.5*unit::mm	physical size of points	1.5*unit::mm	Function2d
PointStyle	Circles, Crosses, Diamonds, FilledCircles, FilledDiamonds, FilledSquares, Squares, Stars, XCrosses	presentation style of points	FilledCircles	Function2d

The colors and surface styles of the functions plotted by `plotfunc3d` are chosen automatically. The property inspector (see section Viewer, Browser, and Inspector: Interactive Manipulation) allows to the change these attributes:

attribute name	possible values/example	meaning	default	browser entry
Filled	TRUE, FALSE	display as a surface or as a wireframe model?	TRUE	Function3d
FillColor	RGB or RGBA color	main color (for flat coloring)		Function3d
FillColor2	RGB or RGBA color	secondary color (for Dichromatic and Monochrome coloring)		Function3d



**(Continued)**

attribute name	possible values/examples	meaning	default	browser entry
FillColorType	Dichromatic, Flat, Functional, Monochrome, Rainbow	color scheme	Dichromatic	Function3d
FillColorFunction	procedure	user defined coloring		Function3d
Shading	Smooth, Flat	smooth or flat shading?	Smooth	Function3d
XLinesVisible	TRUE, FALSE	visibility of $x$ parameter lines	TRUE	Function3d
YLinesVisible	TRUE, FALSE	visibility of $y$ parameter lines	TRUE	Function3d
MeshVisible	TRUE, FALSE	visibility of the internal triangulation	FALSE	Function3d
LineWidth	0.35*unit::mm	physical line width	0.35*unit::mm	Function3d
LineColor	RGB or RGBa color	color of parameter lines	RGB::Black. (0,0,0)	Function3d

Besides the usual linear plots, logarithmic plots are also possible by choosing an appropriate `CoordinateType`.

With `Scaling = Constrained`, the unit box in model coordinates (a square in 2D, a cube in 3D) is displayed as a unit box. With `Scaling = Unconstrained`, the renderer applies different scaling transformation in the coordinate directions to obtain an optimal fit of the picture in the display window. This,

however, may distort a circle to an ellipse. With `Scaling = Constrained` a 2D circle appears on the screen like a circle, a 3D sphere appears like a sphere.

2D functions are preprocessed by a semi-symbolic search for discontinuities to improve the graphical representation near singularities and to avoid graphical artifacts. If continuous functions are plotted, one may gain some speed up by switching off this search with `DiscontinuitySearch = FALSE`.

When very time consuming plots are to be created, it may be useful to create the plots in “batch mode.” With the attribute `OutputFile = filename`, the graphical output is not rendered to the screen. An external file with the specified name containing the graphical data is created instead. It may contain xml data that may be viewed later by opening the file with the MuPAD graphics tool ‘VCam.’ Alternatively, bitmap files in various standard bitmap formats such as bmp, jpg etc. can be created that may be viewed by other standard tools. See section Batch Mode for further details.

<b>attribute name</b>	<b>possible values/examples</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
CoordinateType	linLin, LinLog, LogLin, LogLog	linear-linear, linear-logarithmic, logarithmic-linear, log-log	LinLin	Coord.Sys.2d
CoordinateType	linLinLin, ..., LogLogLog	linear-linear-linear, ..., log-log-log	linLinLin	Coord.Sys.3d
Scaling	Automatic, Constrained, Unconstrained	scaling mode	Unconstrained	Coord.Sys.2d/3d
YXRatio	positive real value	aspect ratio y : x (only for Scaling = Unconstrained)	1	Scene2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>	<b>browser entry</b>
ZXRatio	positive real value	aspect ratio $z : x$ (only for Scaling = Unconstrained)	2/3	Scene3d
DiscontinuitySearch	TRUE, FALSE	enable/disable semi-symbolic search for discontinuities	TRUE	Function2d
OutputFile	string	save the plot data in a file		

## Advanced Plotting: Principles and First Examples

### In this section...

“General Principles” on page 5-84

“Some Examples” on page 5-91

In the previous section, we introduced `plotfunc2d` and `plotfunc3d` that serve for plotting functions in 2D and 3D with simple calls in easy syntax. Although all plot attributes accepted by function graphs (of type `plot::Function2d` or `plot::Function3d`, respectively) are also accepted by `plotfunc2d/3d`, there remains a serious restriction: the attributes are used for all functions simultaneously.

If attributes are to be applied to functions individually, one needs to invoke a (slightly) more elaborate calling syntax to generate the plot:

```
plot(
  plot::Function2d(f1, x1 = a1..b1, attrib11, attrib12, ...),
  plot::Function2d(f2, x2 = a2..b2, attrib21, attrib22, ...),
  ...
):
```

In this call, each call of `plot::Function2d` creates a separate object that represents the graph of the function passed as the first argument over the plotting range passed as the second argument. An arbitrary number of plot attributes can be associated with each function graph. The objects themselves are not displayed directly. The `plot` command triggers the evaluation of the functions on some suitable numerical mesh and calls the renderer to display these numerical data in the form specified by the given attributes.

In fact, `plotfunc2d` and `plotfunc3d` do precisely the same: Internally, they generate plot objects of type `plot::Function2d` or `plot::Function3d`, respectively, and call the renderer via `plot`.

### General Principles

In general, graphical scenes are collections of “graphical primitives.” There are simple primitives such as points, line segments, polygons, rectangles

and boxes, circles and spheres, histogram plots, pie charts etc. An example of a more advanced primitive is `plot::VectorField2d` that represents a collection of arrows attached to a regular mesh visualizing a vector field over a rectangular region in  $^2$ . Yet more advanced primitives are function graphs and parametrized curves that come equipped with some internal intelligence, e.g., knowing how to evaluate themselves numerically on adaptive meshes and how to clip themselves in case of singularities. The most advanced primitives in the present `plot` library are `plot::Ode2d`, `plot::Ode3d` and `plot::Implicit2d`, `plot::Implicit3d`. The first automatically solve systems of ordinary differential equations numerically and display the solutions graphically. The latter display the solution curves or surfaces of algebraic equations  $f(x, y) = 0$  or  $f(x, y, z) = 0$ , respectively, by solving these equation numerically.

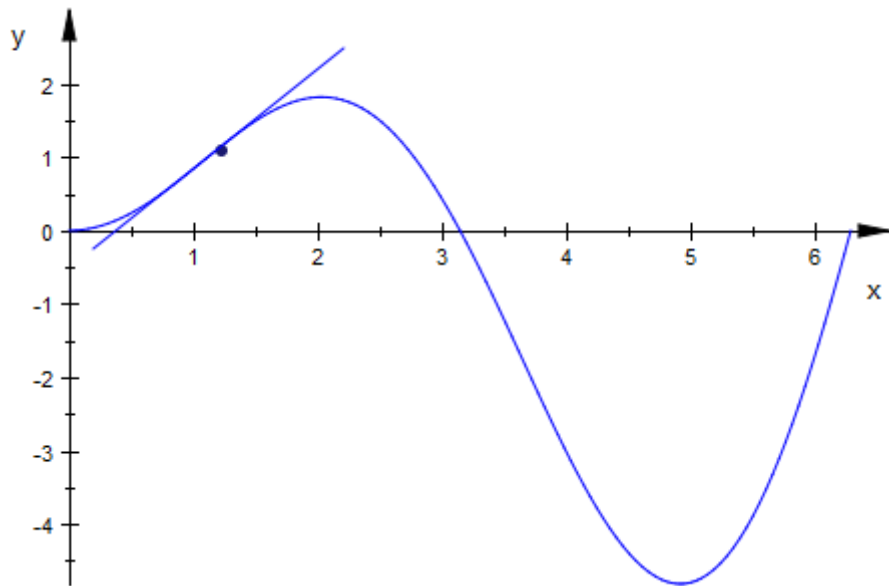
All these primitives are just “objects” representing some graphical entities. They are not rendered directly when they are created, but just serve as data structures encoding the graphical meaning, collecting attributes defining the presentation style and providing numerical routines to convert the input parameters to some numerical data that are to be sent to the renderer.

An arbitrary number of such primitives can be collected to form a graphical scene. Finally, a call to `plot` passing a sequence of all primitives in the scene invokes the renderer to draw the plot. The following example shows the graph of the function  $f(x) = x\sin(x)$ . At the point  $(x_0, f(x_0))$ , a graphical point is inserted and the tangent to the graph through this point is added:

```
f := x -> x*sin(x):
x0 := 1.2: dx := 1:
g := plot::Function2d(f(x), x = 0..2*PI):
p := plot::Point2d(x0, f(x0)):
t := plot::Line2d([x0 - dx, f(x0) - f'(x0)*dx],
                 [x0 + dx, f(x0) + f'(x0)*dx]):
```

The picture is drawn by calling `plot`:

```
plot(g, p, t):
```



Each primitive accepts a variety of plot attributes that may be passed as a sequence of equations `AttributeName = AttributeValue` to the generating call. Most prominently, each primitive allows to set the color explicitly:

```
g := plot::Function2d(f(x), x = 0..2*PI, Color = RGB::Blue):
```

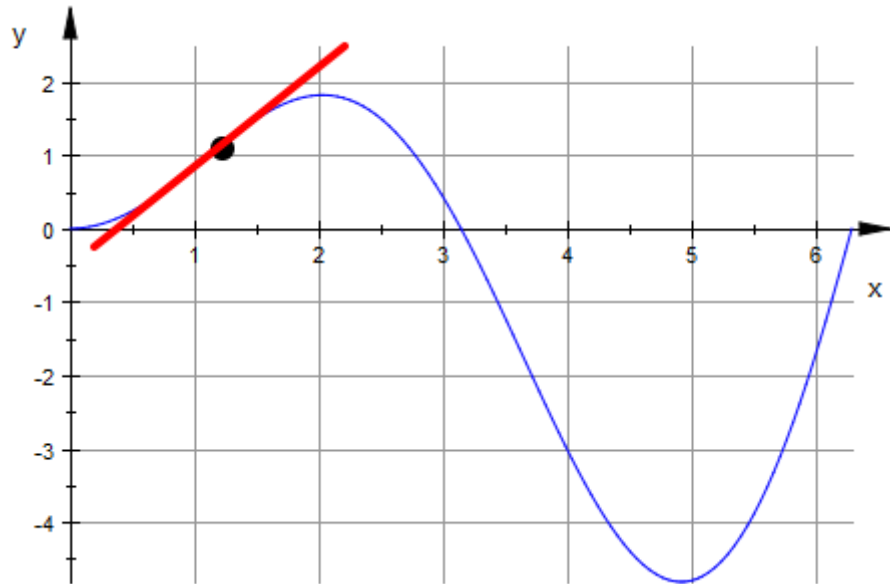
Alternatively, the generated objects allow to set attributes via slot assignments of the form `primitive::AttributeName := AttributeValue` as in

```
p::Color := RGB::Black:
p::PointSize := 3.0*unit::mm:
t::Color := RGB::Red:
t::LineWidth := 1.0*unit::mm:
```

The help page of each primitive provides a list of all attributes the primitive is reacting to.

Certain attributes such as axes style, the visibility of grid lines in the background etc. are associated with the whole scene rather than with the individual primitives. These attributes may be included in the `plot` call:

```
plot(g, p, t, GridVisible = TRUE):
```



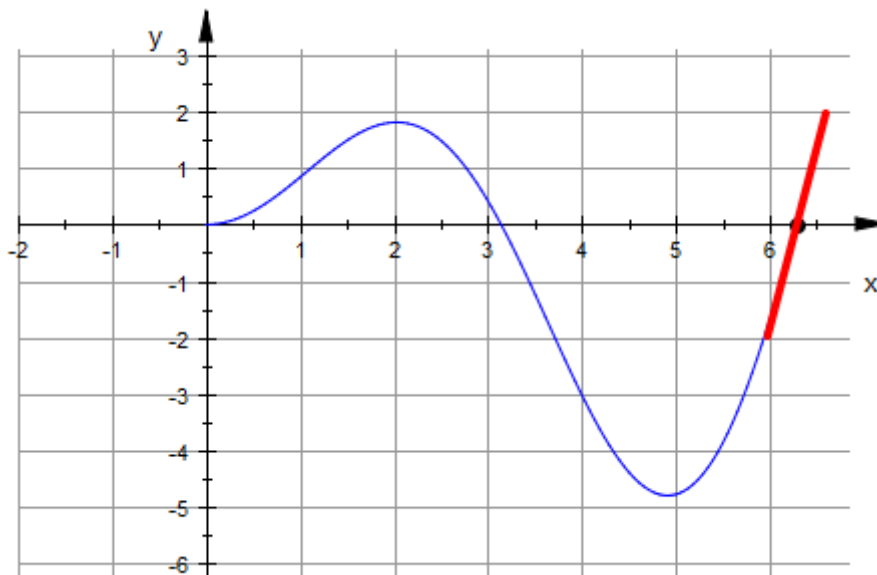
As explained in detail in section *The Full Picture: Graphical Trees*, the `plot` command automatically embeds the graphical primitives in a coordinate system, which in turn is embedded in a graphical scene, which is drawn inside a canvas. The various attributes associated with the general appearance of the whole picture are associated with these “grouping structures”: a concise list of all such attributes is provided on the help pages of `plot::Canvas`, `plot::Scene2d`, `plot::Scene3d`, `plot::CoordinateSystem2d`, and `plot::CoordinateSystem3d`, respectively.

The object browser provided by the MuPAD graphics tool allows to select each primitive in the plot. After selection, the attributes of the primitive can be changed interactively in the property inspector (see section *Viewer, Browser, and Inspector: Interactive Manipulation*).

Next, we wish to demonstrate an animation. It is remarkably simple to generate an animated picture. We want to let the point  $x_0$  at which the tangent is added move along the graph of the function. In MuPAD, you do not need to create an animation frame by frame. Instead, each primitive can be told to animate itself by simply defining it with a symbolic animation parameter and adding an animation range for this parameter. Static and animated objects can be mixed and rendered together. The static function graph of  $f(x)$  used in the previous plot is recycled for the animation. The graphical point at  $(x_0, f(x_0))$  and the tangent through this point shall be animated using the coordinate  $x_0$  as the animation parameter. Deleting its value set above, we can use the same definitions as before, now with a symbolic  $x_0$ . We just have to add the range specification  $x_0 = 0..2*PI$  for this parameter:

```
delete x0:
dx := 2/sqrt(1 + f'(x0)^2):
p := plot::Point2d(x0, f(x0), x0 = 0..2*PI,
                  Color = RGB::Black,
                  PointSize = 2.0*unit::mm):
t := plot::Line2d([x0 - dx, f(x0) - f'(x0)*dx],
                 [x0 + dx, f(x0) + f'(x0)*dx],
                 x0 = 0..2*PI, Color = RGB::Red,
                 LineWidth = 1.0*unit::mm):
plot(g, p, t, GridVisible = TRUE):
```





Details on animations and further examples are provided in section Graphics and Animations.

We summarize the construction principles for graphics with the MuPAD `plot` library:

---

**Note** Graphical scenes are built from graphical primitives. Section Graphics and Animations provides a survey of the primitives that are available in the `plot` library.

---

---

**Note** Primitives generated by the `plot` library are symbolic objects that are not rendered directly. The call `plot(Primitive1, Primitive2, ...)` generates the pictures.

---

---

**Note** Graphical attributes are specified as equations `AttributeName = AttributeValue`. Attributes for a graphical primitive may be passed in the call that generates the primitive. The help page of each primitive provides a complete list of all attributes the primitive reacts to.

---

---

**Note** Attributes determining the general appearance of the picture may be passed in the `plot` call. The help pages of `plot::Canvas`, `plot::Scene2d`, `plot::Scene3d`, `plot::CoordinateSystem2d`, and `plot::CoordinateSystem3d`, respectively, provide a complete list of all attributes determining the general appearance.

---

---

**Note** All attributes can be changed interactively in the viewer.

---

---

**Note** Presently, 2D and 3D plots are strictly separated. Objects of different dimension cannot be rendered in the same plot.

---

---

**Note** Animations are not created frame by frame but objectwise (also see section `Frame by Frame Animations`). An object is animated by generating it with a symbolic animation parameter and providing a range for this parameter in the generating call. Section `Graphics and Animations` provides for further details on animations.

---

Presently, it is not possible to add objects to an existing plot. However, using animations, it is possible to let primitives appear one after another in the animated picture. See Graphics and Animations.

## Some Examples

### Example 1

We wish to visualize the interpolation of a discrete data sample by cubic splines. First, we define the data sample, consisting of a list of points  $[[x_1, y_1], [x_2, y_2], \dots]$ . Suppose they are equidistant sample points from the graph of

the function  $f(x) = \frac{x \sin(5x)}{e^x}$ :

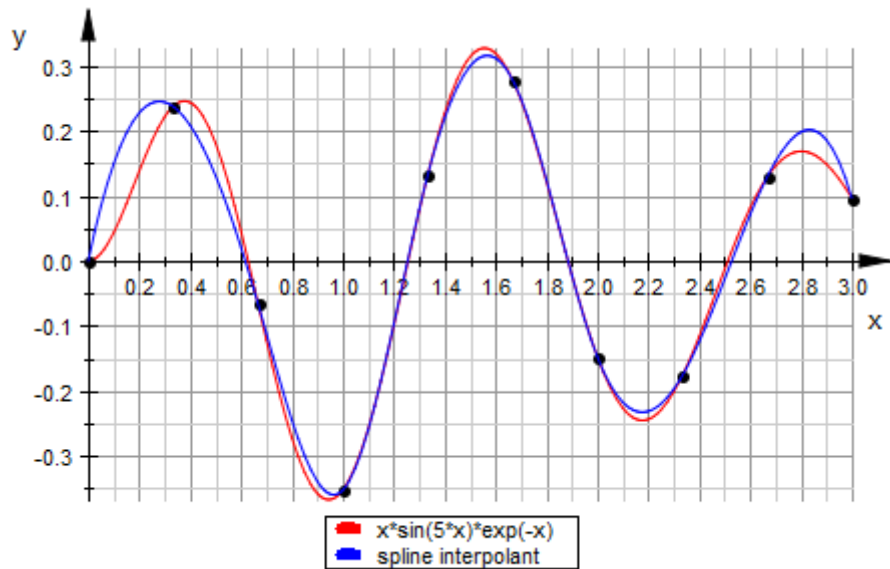
```
f := x -> x*exp(-x)*sin(5*x):
data := [[i/3, f(i/3)] $ i = 0..9]:
```

We use `numeric::cubicSpline` to define the cubic spline interpolant through these data:

```
S := numeric::cubicSpline(op(data)):
```

The plot shall consist of the function  $f(x)$  that provides the data of the sample points and of the spline interpolant  $S(x)$ . The graphs of  $f(x)$  and  $S(x)$  are generated via `plot::Function2d`. The data points are plotted as a `plot::PointList2d`:

```
plot(
  plot::Function2d(f(x), x = 0..3, Color = RGB::Red,
    LegendText = expr2text(f(x))),
  plot::PointList2d(data, Color = RGB::Black),
  plot::Function2d(S(x), x = 0..3, Color = RGB::Blue,
    LegendText = "spline interpolant"),
  GridVisible = TRUE, SubgridVisible = TRUE,
  LegendVisible = TRUE
):
```



### Example 2

A cycloid is the curve that you get when following a point fixed to a wheel rolling along a straight line. We visualize this construction by an animation in which we use the  $x$  coordinate of the hub as the animation parameter. The wheel is realized as a circle. There are 3 points fixed to the wheel: a green point on the rim, a red point inside the wheel and a blue point outside the wheel:

```

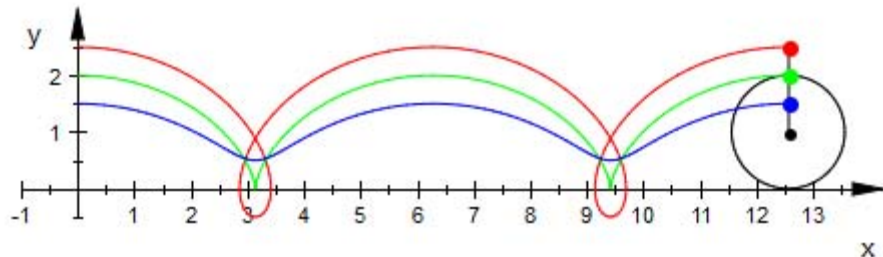
WheelRadius := 1:
WheelCenter := [x, WheelRadius]:
WheelRim := plot::Circle2d(WheelRadius, WheelCenter,
                           x = 0..4*PI,
                           LineColor = RGB::Black):
WheelHub := plot::Point2d(WheelCenter, x = 0..4*PI,
                          PointColor = RGB::Black):
WheelSpoke := plot::Line2d(WheelCenter,
                           [WheelCenter[1] + 1.5*WheelRadius*sin(x),
                            WheelCenter[2] + 1.5*WheelRadius*cos(x)],
                           x = 0..4*PI, LineColor = RGB::Black):

```

```

color:= [RGB::Red, RGB::Green, RGB::Blue]:
r := [1.5*WheelRadius, 1.0*WheelRadius, 0.5*WheelRadius]:
for i from 1 to 3 do
  Point[i] := plot::Point2d([WheelCenter[1] + r[i]*sin(x),
    WheelCenter[2] + r[i]*cos(x)],
    x = 0..4*PI,
    PointColor = color[i],
    PointSize = 2.0*unit::mm):
  Cycloid[i] := plot::Curve2d([y + r[i]*sin(y),
    WheelRadius + r[i]*cos(y)],
    y = 0..x, x = 0..4*PI,
    LineColor = color[i]):
end_for:
plot(WheelRim, WheelHub, WheelSpoke,
  Point[i] $ i = 1..3,
  Cycloid[i] $ i = 1..3,
  Scaling = Constrained,
  Width = 120*unit::mm, Height = 60*unit::mm):

```

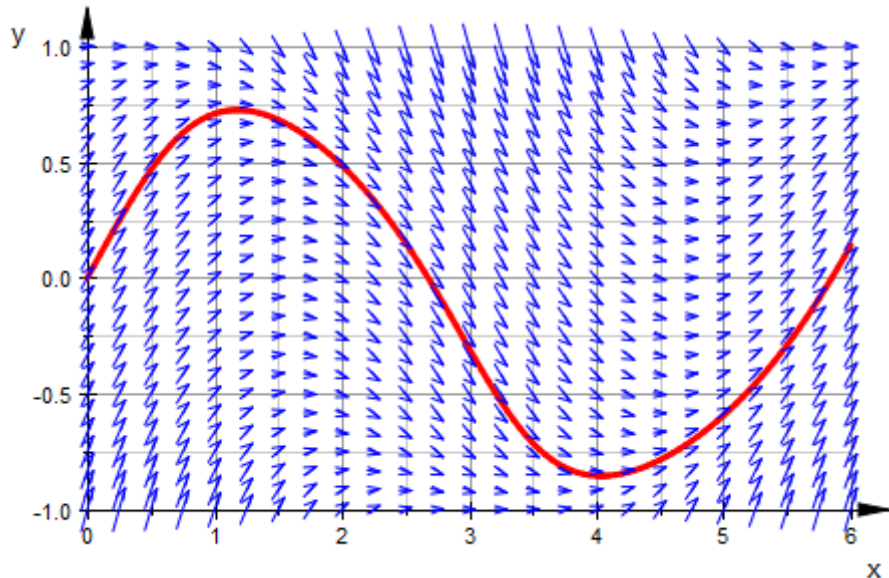


### Example 3

We wish to visualize the solution of the ordinary differential equation (ODE)  $y'(x) = -y(x)^3 + \cos(x)$  with the initial condition  $y(0) = 0$ . The solution shall be drawn together with the vector field  $\vec{v}(x, y) = (1, -y^3 + \cos(x))$  associated

with this ODE (along the solution curve, the vectors of this field are tangents of the curve). We use `numeric::odesolve2` to generate the solution as a function plot. Since the numerical integrator returns the result as a list of one floating-point value, one has to pass the single list entry as `Y(x)[1]` to `plot::Function2d`. The vector field is generated via `plot::VectorField2d`:

```
f := (x, y) -> -y^3 + cos(x):
Y := numeric::odesolve2(
    numeric::ode2vectorfield({y'(x) = f(x, y), y(0) = 0},
        [y(x)])):
plot(
    plot::Function2d(Y(x)[1], x = 0..6, Color = RGB::Red,
        LineWidth = 0.7*unit::mm),
    plot::VectorField2d([1, f(x, y)], x = 0..6, y = -1..1,
        Color = RGB::Blue, Mesh = [25, 25]),
    GridVisible = TRUE, SubgridVisible = TRUE, Axes = Frame
):
```



**Example 4**

The radius  $r$  of an object with rotational symmetry around the  $x$ -axis is measured at various  $x$  positions:

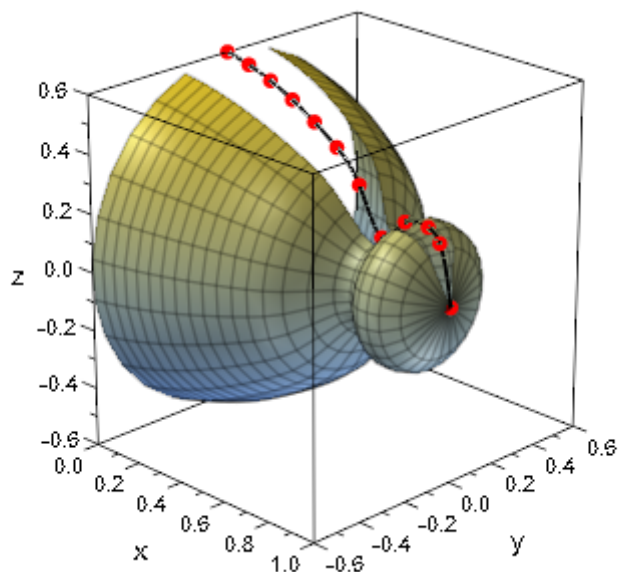
$x$	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	1.00
$r(x)$	0.60	0.58	0.55	0.51	0.46	0.40	0.30	0.15	0.23	0.24	0.20	0.00

A spline interpolation is used to define a smooth function  $r(x)$  from the measurements:

```
samplepoints :=
  [0.00, 0.60], [0.10, 0.58], [0.20, 0.55], [0.30, 0.51],
  [0.40, 0.46], [0.50, 0.40], [0.60, 0.30], [0.70, 0.15],
  [0.80, 0.23], [0.90, 0.24], [0.95, 0.20], [1.00, 0.00]:
r := numeric::cubicSpline(samplepoints):
```

We reconstruct the object as a surface of revolution via `plot::XRotate`. The rotation angle is restricted to a range that leaves a gap in the surface. The spline curve and the sample points are added as a `plot::Curve3d` and a `plot::PointList3d`, respectively, and displayed in this gap:

```
plot(
  plot::XRotate(r(x), x = 0..1, AngleRange = 0.6*PI..2.4*PI,
    Color = RGB::MuPADGold),
  plot::Curve3d([x, 0, r(x)], x = 0..1,
    LineWidth = 0.5*unit::mm, Color = RGB::Black),
  plot::PointList3d([[p[1], 0, p[2]] $ p in samplepoints],
    PointSize = 2.0*unit::mm,
    Color = RGB::Red),
  CameraDirection = [70, -70, 40]
):
```



### Example 5

The following sum is a Fourier representation of a periodic step function:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}$$

We wish to show the convergence of the partial sums

$$f_n(x) = \sum_{k=1}^n \frac{\sin((2k-1)x)}{2k-1}$$

for some small values of  $n$ . To this end, we assign the value of  $f_n(x)$  to the MuPAD identifier  $f_n$ . For our second and third example, we will need to accept fractional values of  $n$ , so the code uses `floor` to get a “proper” integer value for the sum:

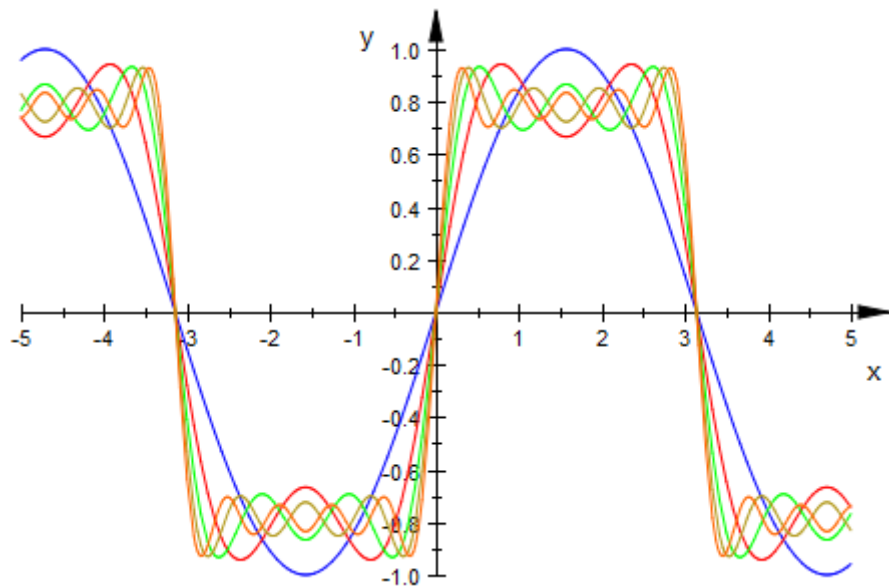
```
f_n := sum(sin((2*k-1)*x)/(2*k-1), k = 1..floor(n))
```



$$\sum_{k=1}^{|n|} \frac{\sin(x(2k-1))}{2k-1}$$

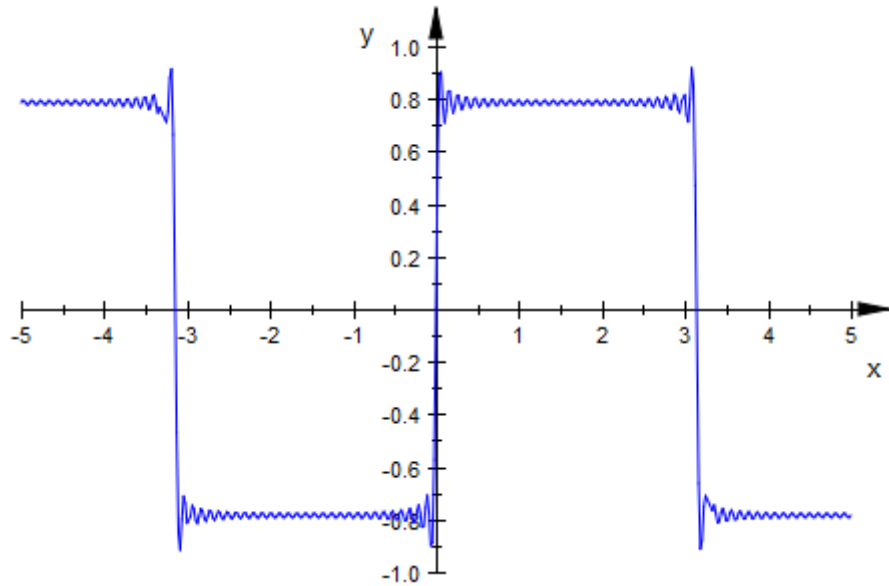
First, we use `plotfunc2d` and the sequence operator `$` to plot the first 5 partial sums into the same coordinate system (and switch off the legend, which is not useful for this application).

```
plotfunc2d(f_n $ n = 1..5, LegendVisible = FALSE)
```



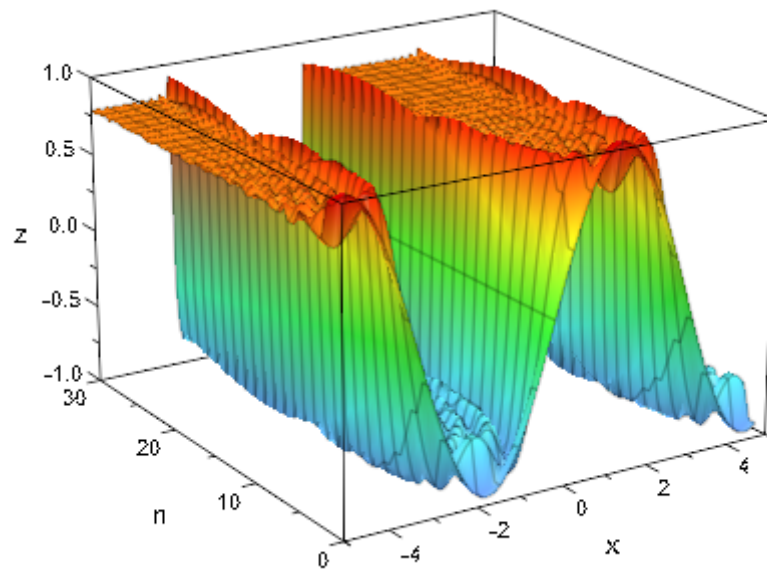
This plot clearly shows what is known as Gibbs' phenomenon: At the discontinuities of the step function, the approximation "overshoots." Plotting more approximations simultaneously is going to create a too crowded plot to be of use, so to show that using more terms in the sum does not help against Gibbs' phenomenon, we revert to animations to show the first 30 partial sums – this is one of the reasons we used `floor(n)` above:

```
plotfunc2d(f_n, x = -5..5, n = 1..30, Frames = 15)
```



Another possibility of showing the convergence behavior is to create a 3D plot with the second ordinate being the number of terms used in the approximation:

```
plotfunc3d(f_n, x = -5..5, n = 1..30,  
           Submesh = [5,1], FillColorType = Rainbow)
```



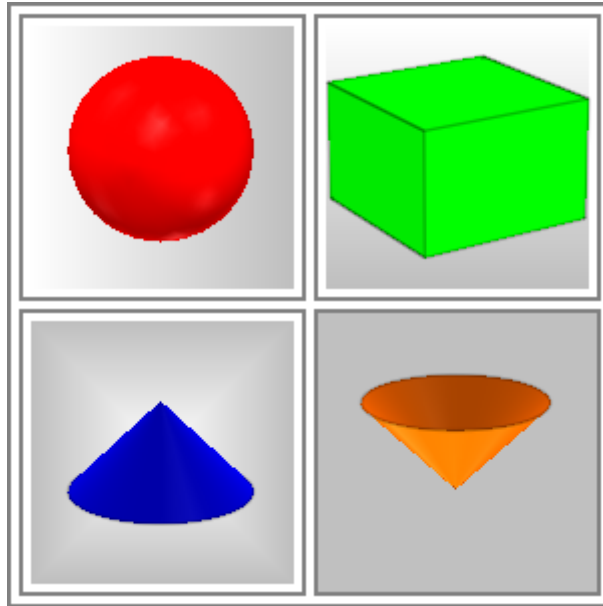
## The Full Picture: Graphical Trees

For a full understanding of the interactive features of the viewer to be discussed in the next section, we need to know the structure of a MuPAD plot as a “graphical tree.”

The root is the “canvas”; this is the drawing area into which all parts of the plot are rendered. The type of the canvas object is `plot::Canvas`. Its physical size may be specified via the attributes `Width` and `Height`.

Inside the canvas, one or more “graphical scenes” can be displayed. All of them must be of the same dimension, i.e., objects of type `plot::Scene2d` or `plot::Scene3d`, respectively. The following command displays four different 3D scenes. We set the `BorderWidth` for all objects of type `plot::Scene3d` to some positive value so that the drawing areas of the scenes become visible more clearly:

```
plot(plot::Canvas(  
  plot::Scene3d(plot::Sphere(1, [0, 0, 0], Color = RGB::Red),  
    BackgroundStyle = LeftRight),  
  plot::Scene3d(plot::Box(-1..1, -1..1, -1..1,  
    Color = RGB::Green),  
    BackgroundStyle = TopBottom),  
  plot::Scene3d(plot::Cone(1, [0, 0, 0], [0, 0, 1],  
    Color = RGB::Blue),  
    BackgroundStyle = Pyramid),  
  plot::Scene3d(plot::Cone(1, [0, 0, 1], [0, 0, 0],  
    Color = RGB::Orange),  
    BackgroundStyle = Flat,  
    BackgroundColor = RGB::Grey),  
  Width = 80*unit::mm, Height = 80*unit::mm,  
  Axes = None, BorderWidth = 0.5*unit::mm,  
  plot::Scene3d::BorderWidth = 0.5*unit::mm  
)):
```



See section [Layout of Canvas and Scenes](#) for details on how the layout of a canvas containing several scenes is set.

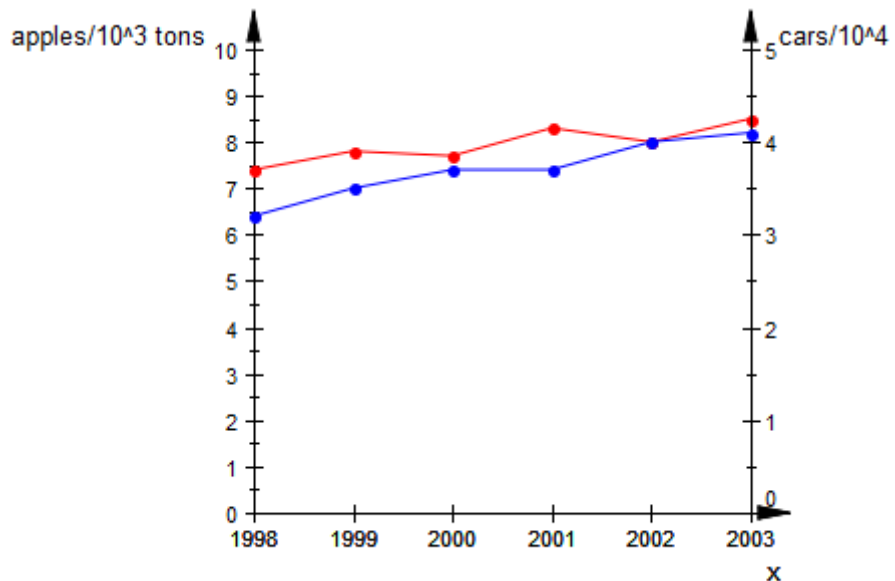
Coordinate systems of type `plot::CoordinateSystem2d` or `plot::CoordinateSystem3d` exist inside a 2D scene or a 3D scene, respectively. There may be one or more coordinate systems in a scene. Each coordinate system has its own axes. In the following example, we place two coordinate systems in one scene. The first is used to display the sales of apples in the years 1998 through 2003 in units of 1000 tons, the second is used to display the sale of cars in units of 10.000 cars. The  $y$ -axis for the apples is “flushed left” by setting the  $x$  component of its origin to the first year 1998, whilst the  $y$ -axis for the cars is “flushed right” by setting the  $x$  component of its origin to the last year 2003:

```
apples := plot::Polygon2d(
    [[1998, 7.4], [1999, 7.8], [2000, 7.7],
     [2001, 8.3], [2002, 8.0], [2003, 8.5]],
    Color = RGB::Red, PointsVisible = TRUE):
cars := plot::Polygon2d(
    [[1998, 3.2], [1999, 3.5], [2000, 3.7],
```

```

        [2001, 3.7], [2002, 4.0], [2003, 4.1]],
        Color = RGB::Blue,
        PointsVisible = TRUE):
plot(plot::Scene2d(
  plot::CoordinateSystem2d(apples, ViewingBoxYRange = 0..10,
    YAxisTitle = "apples/10^3 tons",
    Axes = Origin, AxesOriginX = 1998,
    XTicksNumber = None,
    XTicksAt = [$ 1998..2003]),
  plot::CoordinateSystem2d(cars, ViewingBoxYRange = 0..5,
    YAxisTitle = "cars/10^4",
    Axes = Origin, AxesOriginX = 2003,
    XTicksNumber = None,
    XTicksAt = [$ 1998..2003])
)):

```



Inside the coordinate systems, an arbitrary number of primitives (of the appropriate dimension) can be displayed. Thus, we always have a canvas, containing one or more scenes, with each scene containing one or more coordinate systems. The graphical primitives (or groups of such primitives)

are contained in the coordinate systems. Hence, any MuPAD plot has the following general structure:

```

Canvas
|
+-- Scene 1
|
|   +-- CoordinateSystem 1
|   |
|   |   +-- Primitive 1
|   |   +-- Primitive 2
|   |   ...
|   |
|   +-- CoordinateSystem 2
|   |
|   |   +-- Primitive
|   |   ...
|   ...
+-- Scene 2
|
|   +-- CoordinateSystem
|   |
|   |   +-- Primitive
|   |   ...
|   ...
...

```

This is the “graphical tree” that is displayed by the “object browser” of the MuPAD graphics tool (see section Viewer, Browser, and Inspector: Interactive Manipulation).

*Shortcuts:* For simple plots containing primitives inside one coordinate system only that resides in a single scene of the canvas, it would be rather cumbersome to generate the picture by a command such as

```

plot(
  plot::Canvas(
    plot::Scene2d(
      plot::CoordinateSystem2d(Primitive1, Primitive2, ...)
    )
  )
)

```

):

In fact, the command

```
plot(Primitive1, Primitive2, ...):
```

suffices: It is a shortcut of the above command. It generates automatically a coordinate system that contains the primitives, embedded it in a scene which is automatically placed inside a canvas object. Thus, this command implicitly creates the graphical tree

```
Canvas
|
+-- Scene 1
|
+-- CoordinateSystem 1
|
+-- Primitive 1
+-- Primitive 2
...

```

that becomes visible in the object browser.



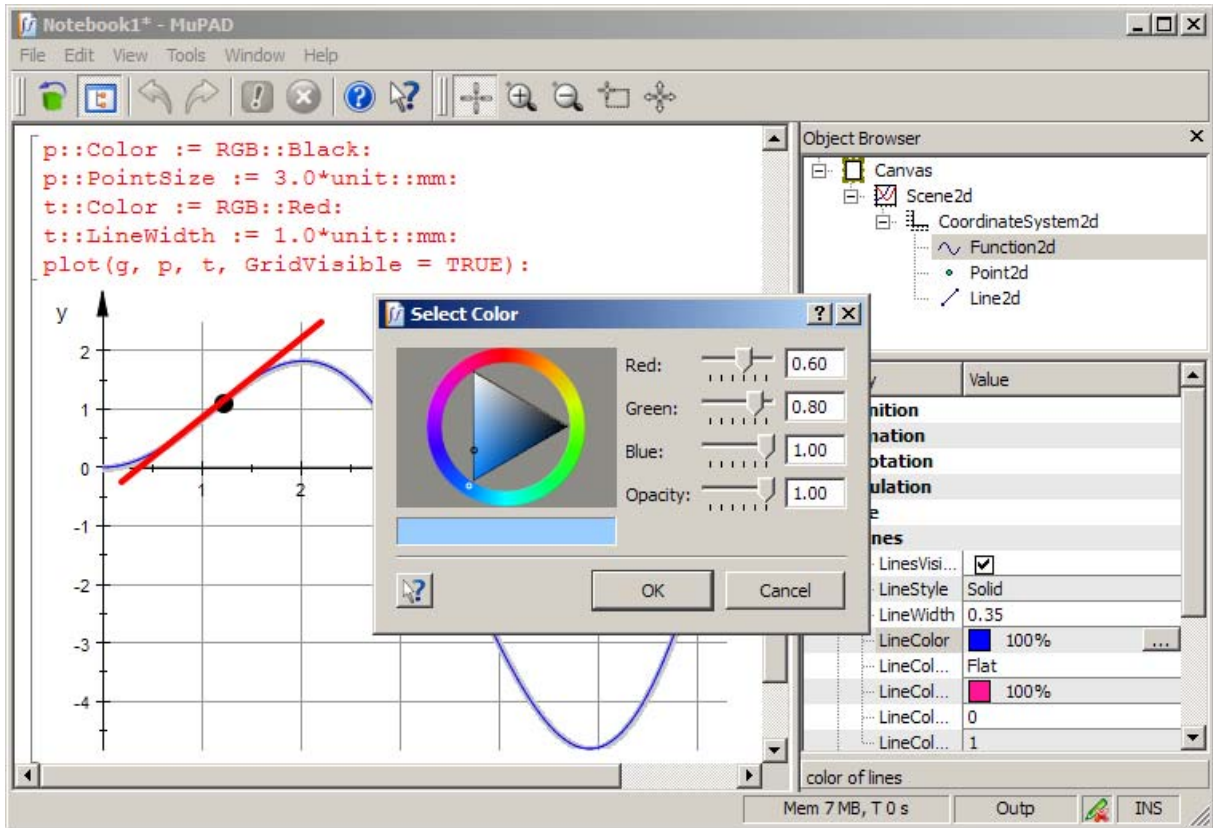
## Viewer, Browser, and Inspector: Interactive Manipulation

After a plot command is executed in a MuPAD notebook, the plot will typically appear below the input region, embedded in the notebook. To “activate” the graphic, click it once with the mouse cursor. The “Tool Bar” at the top of the window will change and the “Command Bar” at the right side of the window will be replaced by two sub-windows labelled “Object Browser” and “Properties”, which we will refer to as the “object browser” and the “property inspector.” If they are not visible, you may need to activate them with the

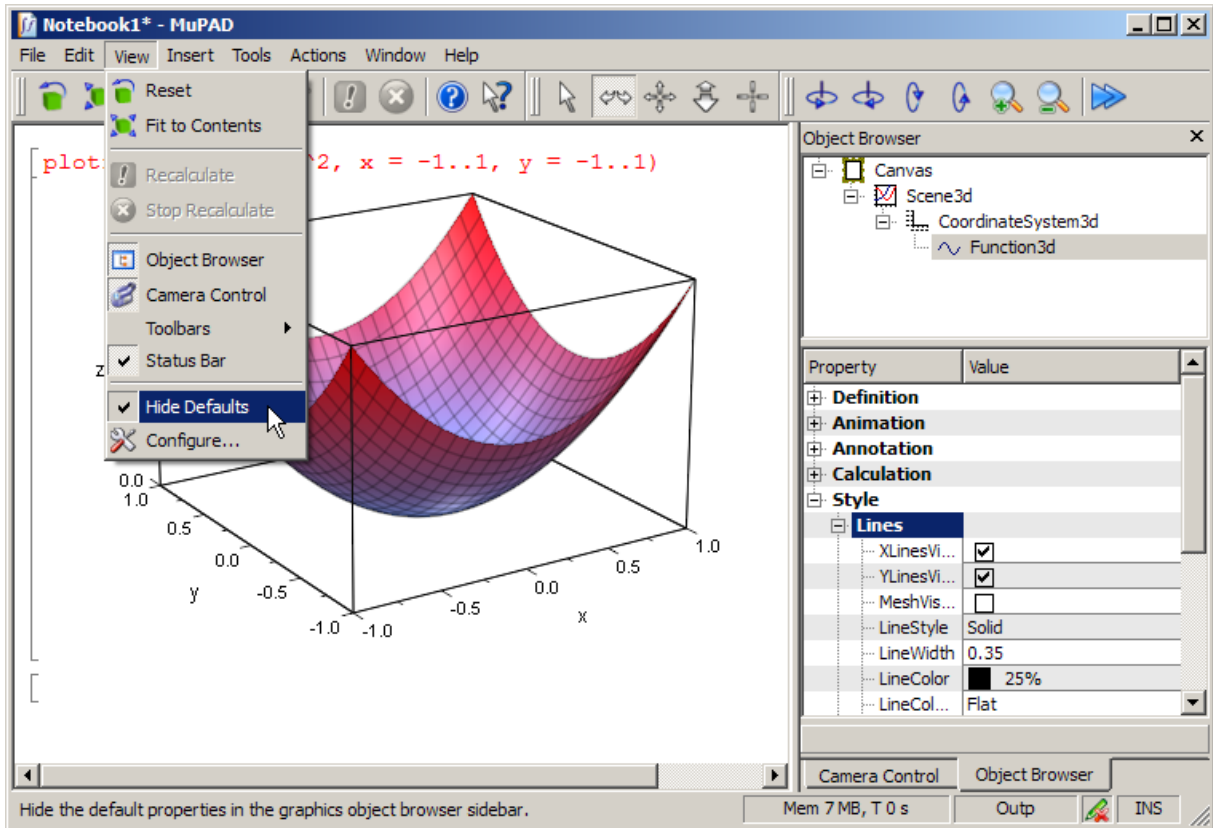
icon .

In the object browser, the graphical tree of the plot as introduced in the preceding section is visible. It allows to select any node of the graphical tree by a mouse click.

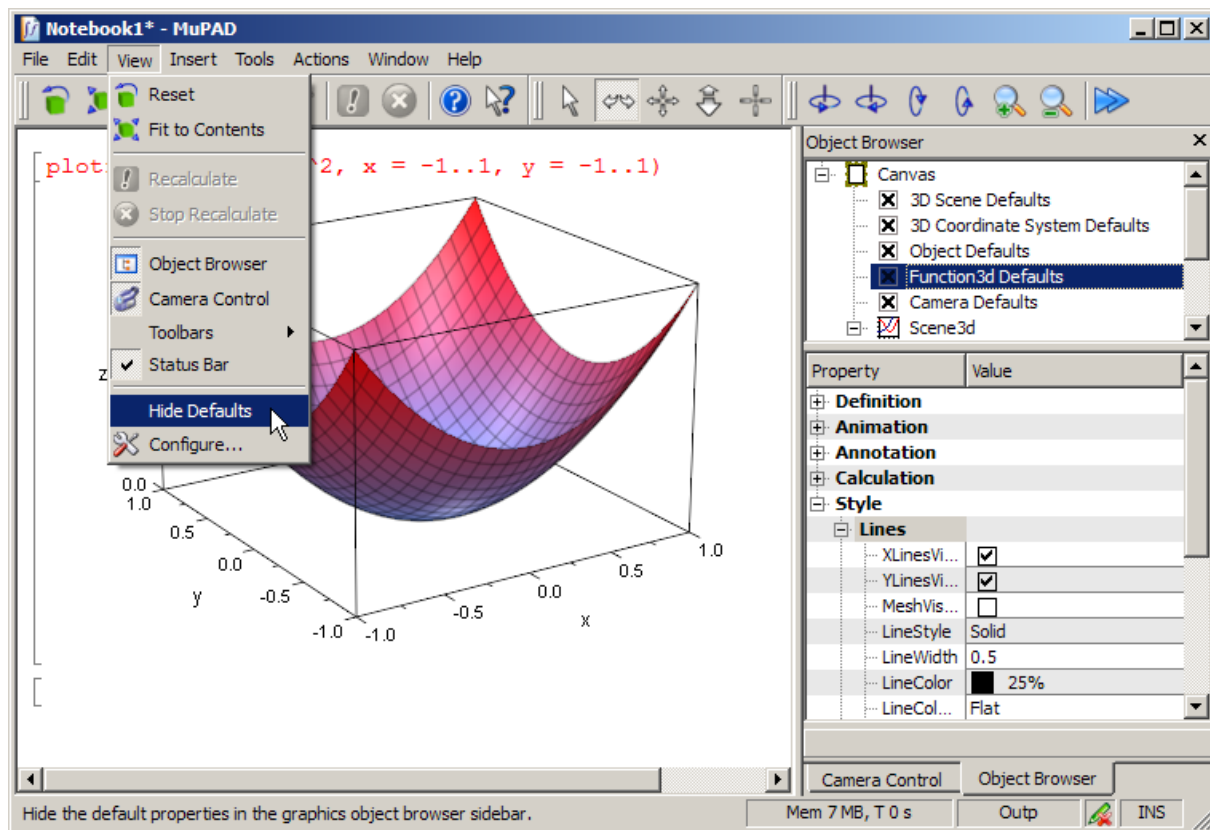
After selecting an object in the object browser, the corresponding part of the plot is highlighted in some way allowing to identify the selected object visually. The property inspector now displays all attributes the selected object reacts to. For example, for an object of type `Function2d`, the attributes visible in the property inspector are categorized as ‘Definition,’ ‘Animation,’ ‘Annotation,’ ‘Calculation,’ and ‘Style’ with the latter split into the sub-categories (Style of) ‘Lines,’ (Style of) ‘Points,’ (Style of) ‘Asymptotes.’ Opening one of these categories, one finds the attributes and their current values that apply to the currently selected object. After selection of an attribute with the mouse, its value can be changed:



There is a sophisticated way of setting defaults for the attributes via the object browser and the property inspector. The 'View' menu provides an item 'Hide Defaults.' Disabling 'Hide Defaults,' the object browser changes from



to:

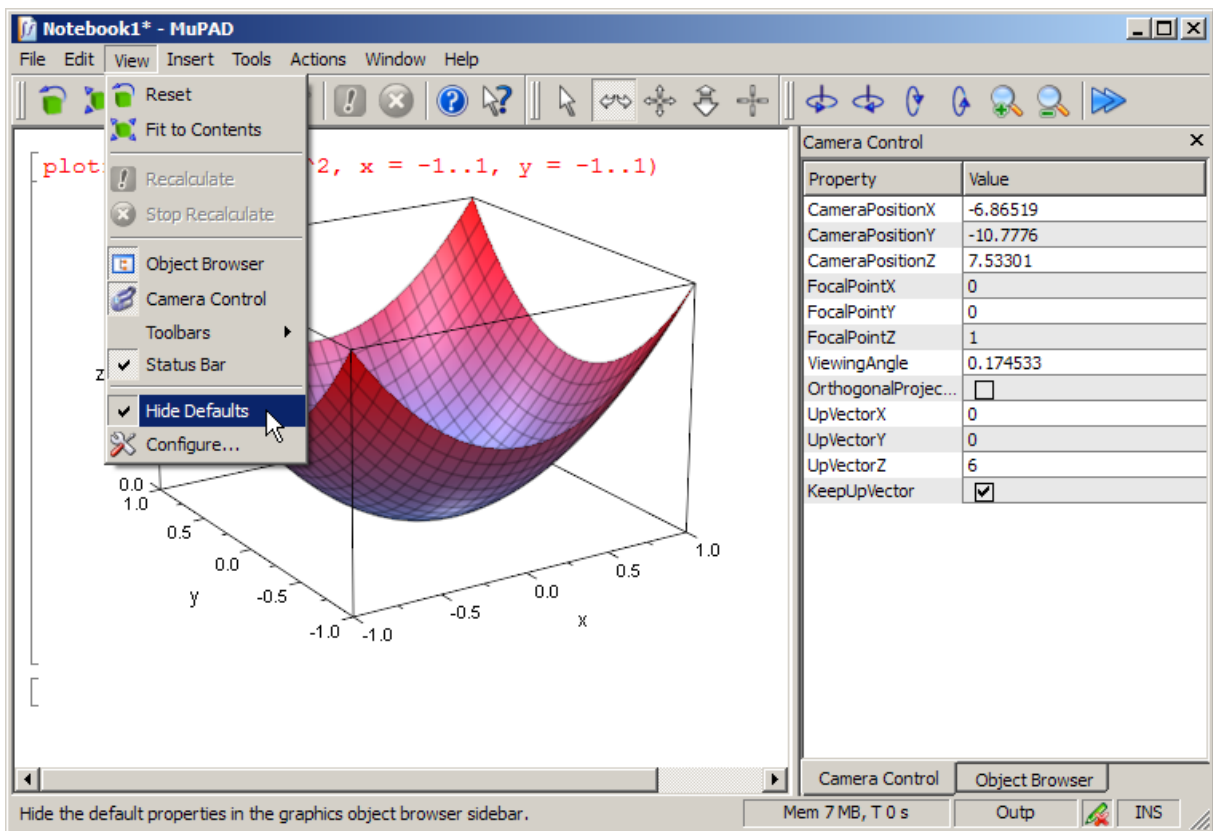


At each node of the graphical tree, default values for the attributes can be set via the property inspector. These defaults are valid for all primitives that exist below this node, unless these defaults are redefined at some other node further down in tree hierarchy.

This mechanism is particularly useful when there are many primitives of the same kind in the plot. Imagine a picture consisting of 1000 points. If you wish to change the color of all points to green, it would be quite cumbersome to set `PointColor = RGB::Green` in all 1000 points. Instead, you can set `PointColor = RGB::Green` in the `PointColor` defaults entry at some tree node that contains all the points (e.g., the canvas). Similarly, if there are 1000 points in one scene and another 1000 points in a second scene, you can change the color of all points in the first scene by an appropriate default entry in

the first scene, whilst the default entry for the second scene can be set to a different value.

A 3D plot can be rotated and shifted by the mouse. Also zooming in and out is possible. In fact, these operations are realized by moving the camera around, closer to, or farther away from the scene, respectively. There is a camera control that may be switched on and off via the 'Camera Control' item of the 'View' menu. It provides the current viewing parameters such as camera position, focal point and the angle of the camera lens:



Section Attributes for `plotfunc2d` and `plotfunc3d` provides more information on cameras.

## Primitives

In this section, we give a brief survey of the graphical primitives, grouping constructs, transformation routines etc. provided by the `plot` library.

The following table lists the 'low-level' primitives.

<code>plot::Arc2d</code>	circular arc in 2D
<code>plot::Arrow2d</code>	arrow in 2D
<code>plot::Arrow3d</code>	arrow in 3D
<code>plot::Box</code>	rectangular box in 3D
<code>plot::Circle2d</code>	circle in 2D
<code>plot::Circle3d</code>	circle in 3D
<code>plot::Cone</code>	cone/conical frustum in 3D
<code>plot::Cylinder</code>	cylinder in 3D
<code>plot::Ellipse2d</code>	ellipse in 2D
<code>plot::Ellipsoid</code>	ellipsoid in 3D
<code>plot::Line2d</code>	graphical line segment in 2D
<code>plot::Line3d</code>	graphical line segment in 3D
<code>plot::Parallelogram2d</code>	parallelogram in 2D
<code>plot::Parallelogram3d</code>	parallelogram in 3D
<code>plot::Point2d</code>	graphical point in 2D
<code>plot::Point3d</code>	graphical point in 3D
<code>plot::PointList2d</code>	collection of graphical points in 2D
<code>plot::PointList3d</code>	collection of graphical points in 3D
<code>plot::Polygon2d</code>	line segments forming a polygon in 2D
<code>plot::Polygon3d</code>	line segments forming a polygon in 3D

<code>plot::Rectangle</code>	rectangle in 2D
<code>plot::Sphere</code>	sphere in 3D
<code>plot::SurfaceSet</code>	surfaces in 3D (as a collection of 3D triangles)
<code>plot::SurfaceSTL</code>	import of 3D stl surfaces
<code>plot::Text2d</code>	text object in 2D
<code>plot::Text3d</code>	text object in 3D

In addition, there are primitives for Plato's regular polyhedra.

<code>plot::Tetrahedron</code>
<code>plot::Hexahedron</code>
<code>plot::Octahedron</code>
<code>plot::Dodecahedron</code>
<code>plot::Icosahedron</code>

The following table lists the 'high-level' primitives and 'special purpose' primitives.

<code>plot::Bars2d</code>	(statistical) data plot in 2D
<code>plot::Bars3d</code>	(statistical) data plot in 3D
<code>plot::Boxplot</code>	(statistical) box plot
<code>plot::Conformal</code>	conformal plot of complex functions
<code>plot::Curve2d</code>	parametrized curve in 2D
<code>plot::Curve3d</code>	parametrized curve in 3D
<code>plot::Density</code>	density plot in 2D
<code>plot::Function2d</code>	function graph in 2D
<code>plot::Function3d</code>	function graph in 3D
<code>plot::Hatch</code>	hatched region in 2D
<code>plot::Histogram2d</code>	(statistical) histogram plot in 2D

<code>plot::Implicit2d</code>	plot of implicitly defined curves in 2D
<code>plot::Implicit3d</code>	plot of implicitly defined surfaces in 3D
<code>plot::Inequality</code>	visualization of inequalities in 2D
<code>plot::Integral</code>	visualization of integration
<code>plot::Iteration</code>	visualization of iterations in 2D
<code>plot::Listplot</code>	lists of points in 2D
<code>plot::Lsys</code>	Lindenmayer system in 2D
<code>plot::Matrixplot</code>	visualization of matrix data as a surface in 3D
<code>plot::MuPADCube</code>	the MuPAD logo
<code>plot::Ode2d</code>	graphical solution of an ODE in 2D
<code>plot::Ode3d</code>	graphical solution of an ODE in 3D
<code>plot::Piechart2d</code>	(statistical) pie chart in 2D
<code>plot::Piechart3d</code>	(statistical) pie chart in 3D
<code>plot::Plane</code>	infinite plane in 3D
<code>plot::Raster</code>	raster and bitmap plots in 2D
<code>plot::Scatterplot</code>	(statistical) scatter plot in 2D
<code>plot::Sequence</code>	visualization of a sequence of numbers
<code>plot::SparseMatrixplot</code>	sparsity pattern of a matrix
<code>plot::Sum</code>	visualization of a sum of numbers
<code>plot::Surface</code>	parametrized surface in 3D
<code>plot::Sweep</code>	sweep surface in 3D
<code>plot::Turtle</code>	turtle plot in 2D
<code>plot::VectorField2d</code>	vector field plot in 2D
<code>plot::VectorField3d</code>	vector field plot in 3D



<code>plot::XRotate</code>	surface of revolution in 3D
<code>plot::ZRotate</code>	surface of revolution in 3D

The following table lists the various light sources available to illuminate 3D plots.

<code>plot::AmbientLight</code>	ambient (undirected) light
<code>plot::DistantLight</code>	directed light
<code>plot::PointLight</code>	(undirected) point light
<code>plot::SpotLight</code>	(directed) spot light

The following table lists various grouping constructs.

<code>plot::Canvas</code>	drawing area, container for 2D or 3D scenes
<code>plot::Scene2d</code>	container for 2D coordinate systems
<code>plot::Scene3d</code>	container for 3D coordinate systems
<code>plot::CoordinateSystem2d</code>	container for 2D primitives and <code>plot::Group2d</code>
<code>plot::CoordinateSystem3d</code>	container for 3D primitives and <code>plot::Group3d</code>
<code>plot::Group2d</code>	group of primitives in 2D
<code>plot::Group3d</code>	group of primitives in 3D

Primitives or groups of primitives can be transformed by the following routines.

<code>plot::Scale2d</code>	scaling transformation in 2D
<code>plot::Scale3d</code>	scaling transformation in 3D
<code>plot::Reflect2d</code>	reflection in 2D
<code>plot::Reflect3d</code>	reflection in 3D
<code>plot::Rotate2d</code>	rotation in 2D

<code>plot::Rotate3d</code>	rotation in 3D
<code>plot::Translate2d</code>	translation in 2D
<code>plot::Translate3d</code>	translation in 3D
<code>plot::Transform2d</code>	general linear transformation in 2D
<code>plot::Transform3d</code>	general linear transformation in 3D

The following special plot routines are provided.

<code>plot::Cylindrical</code>	cylindrical plot in 3D
<code>plot::Polar</code>	polar plot in 2D
<code>plot::Spherical</code>	polar plot in 3D
<code>plot::Tube</code>	tube plot in 3D

Additionally, there are.

<code>plot::Camera</code>	camera in 3D
<code>plot::ClippingBox</code>	clipping box in 3D

# Attributes

## In this section...

“Default Values” on page 5-116

“Inheritance of Attributes” on page 5-117

“Primitives Requesting Special Scene Attributes: “Hints”” on page 5-125

“The Help Pages of Attributes” on page 5-127

The `plot` library provides for more than 400 attributes for fine-tuning of the graphical output. Because of this large number, the attributes are grouped into various categories in the object browser (see section Viewer, Browser, and Inspector: Interactive Manipulation) and the documentation:

category	meaning
Animation	parameters relevant for animations
Annotation	footer, header, titles, and legends
Axes	axes style and axes titles
Cameras	cameras in 3D
Lights	lights in 3D
Calculation	numerical evaluation (mesh parameters)
Definition	parameters that change the object itself
Grid Lines	grid lines in the background (rulings)
Layout	layout parameters for canvas and scenes
Style	parameters that do not change the objects but their presentation (visibility, color, line width, point size etc.)
Arrow Style	style parameters for arrows
Line Style	style parameters for line objects

Point Style	style parameters for point objects
Surface Style	style parameters for surface objects in 3D and filled areas in 2D
Tick Marks	axes tick marks: style and labels

On the help page for each primitive, there is a complete list of all attributes the primitive reacts to. Clicking on an attribute, you are lead to the help page for this attribute which provides for all the necessary information concerning its semantics and admissible values. The examples on the help page demonstrate the typical use of the attribute.

## Default Values

Most attributes have a default value that is used if no value is specified explicitly. As an example, we consider the attribute `LinesVisible` that is used by several primitives such as `plot::Box`, `plot::Circle2d`, `plot::Cone`, `plot::Curve2d`, `plot::Raster` etc. Although they all use the same attribute named `LinesVisible`, its default value differs among the different primitive types. The specific defaults are accessible by the slots `plot::Box::LinesVisible`, `plot::Circle2d::LinesVisible` etc.:

```
plot::getDefault(plot::Box::LinesVisible),  
plot::getDefault(plot::Circle2d::LinesVisible),  
plot::getDefault(plot::Cone::LinesVisible),  
plot::getDefault(plot::Raster::LinesVisible)
```

`TRUE, TRUE, TRUE, FALSE`

If any of the default values provided by the MuPAD system do not seem appropriate for your applications, change these defaults via `plot::setDefault`:

```
plot::setDefault(plot::Box::LinesVisible = FALSE)
```

`TRUE`

(The return value is the previously valid default value.) Several defaults can be changed simultaneously:

```
plot::setDefault(plot::Box::LinesVisible = FALSE,  
                 plot::Circle2d::LinesVisible = FALSE,  
                 plot::Circle2d::Filled = TRUE)
```

```
FALSE, TRUE, FALSE
```

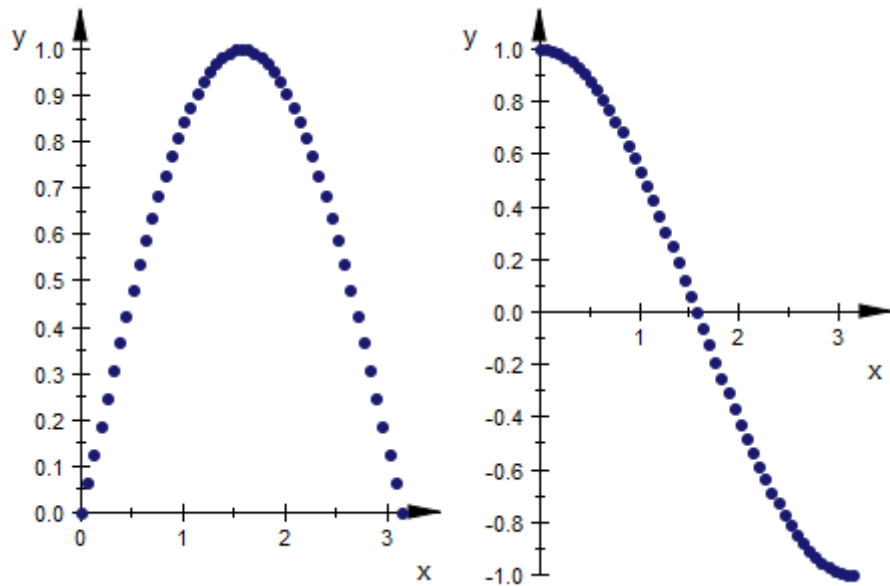
```
plot::getDefault(plot::Box::LinesVisible)
```

```
FALSE
```

## Inheritance of Attributes

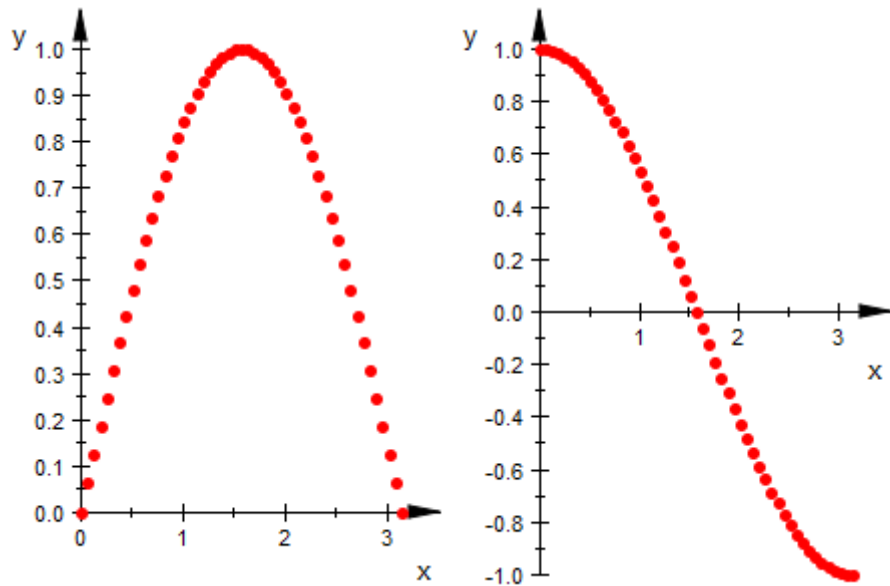
The setting of default values for attributes is quite sophisticated. Assume that you have two scenes that are to be displayed in one single canvas. Both scenes consist of 51 graphical points, each:

```
points1 := plot::Point2d(i/50*PI, sin(i/50*PI)) $ i = 0..50:  
points2 := plot::Point2d(i/50*PI, cos(i/50*PI)) $ i = 0..50:  
S1 := plot::Scene2d(points1):  
S2 := plot::Scene2d(points2):  
plot(S1, S2):
```



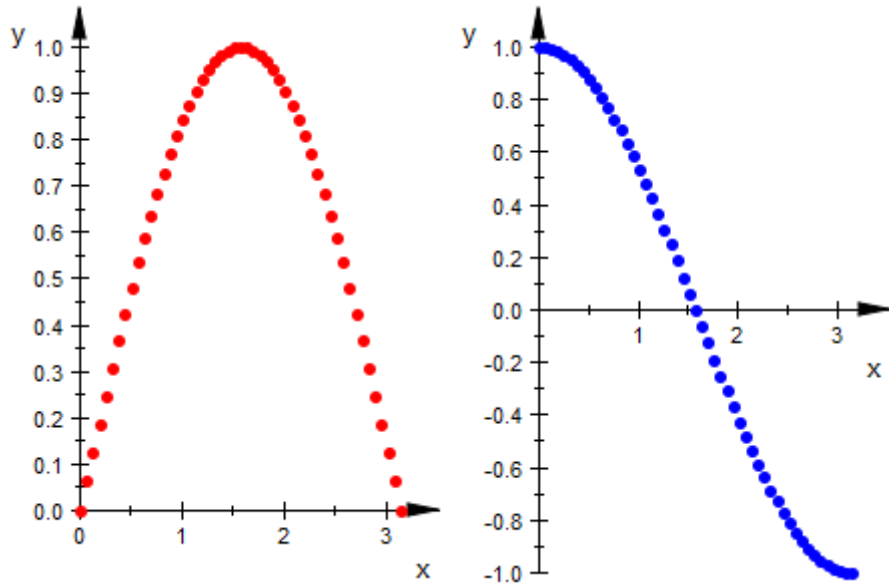
If we wish to color all points in both scenes red, we can set a default for the point color in the plot command:

```
plot(S1, S2, PointColor = RGB::Red):
```



If we wish to color all points in the first scene red and all points in the second scene blue, we can give each of the points the desired color in the generating call to `plot::Point2d`. Alternatively, we can set separate defaults for the point color inside the scenes:

```
S1 := plot::Scene2d(points1, PointColor = RGB::Red):  
S2 := plot::Scene2d(points2, PointColor = RGB::Blue):  
plot(S1, S2):
```



Here is the general rule for setting defaults inside a graphical tree (see The Full Picture: Graphical Trees):

---

**Note** When an attribute is specified in a node of the graphical tree, the specified value serves as the default value for the attribute for all primitives that exist in the sub-tree starting from that node.

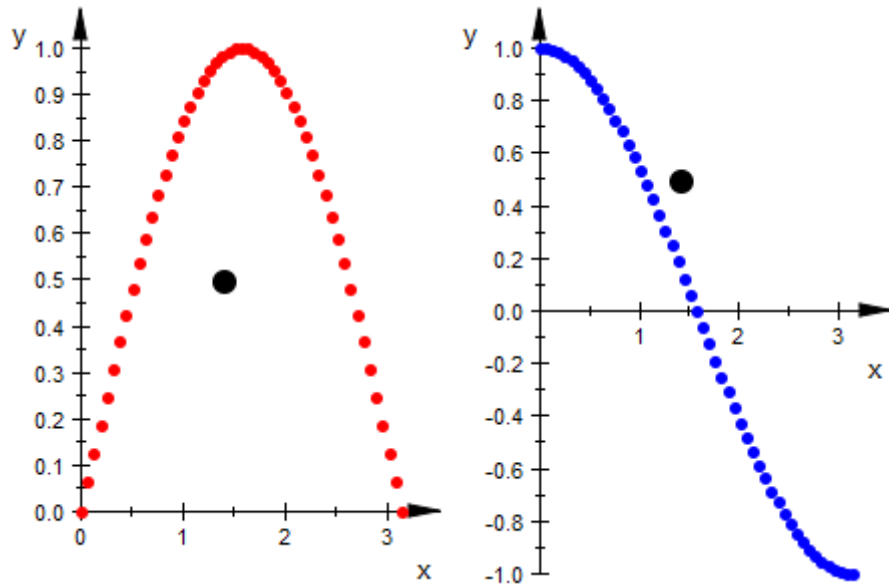
---

The default value can be overwritten by another value at each node further down in the sub-tree (for example, finally, by a specific value in the primitives). In the following call, the default color 'red' is set in the canvas. This value is accepted and used in the first scene. The second scene sets the default color 'blue' overriding the default value 'red' set in the canvas. Additionally, there is an extra point with a color set explicitly to 'black.' This point ignores the defaults set further up in the tree hierarchy:

```
extrapoint := plot::Point2d(1.4, 0.5, PointSize = 3*unit::mm,
                           PointColor = RGB::Black):
S1 := plot::Scene2d(points1, extrapoint):
```

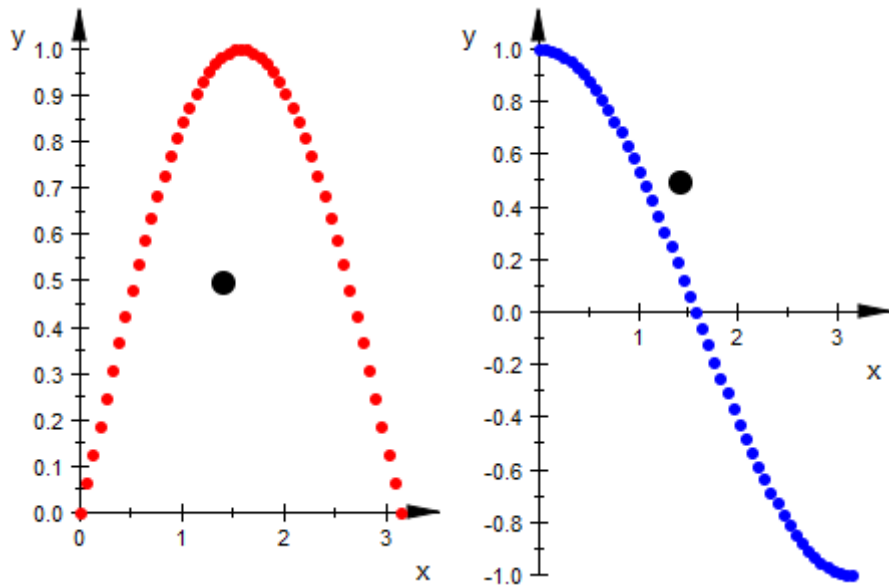


```
S2 := plot::Scene2d(points2, extrapoint,  
    PointColor = RGB::Blue):  
plot(plot::Canvas(S1, S2, PointColor = RGB::Red)):
```



The following call generates the same result. Note that the `plot` command automatically creates a canvas object and passes the attribute `PointColor = RGB::Red` to the canvas:

```
plot(S1, S2, PointColor = RGB::Red):
```



We note that there are different primitives that react to the same attribute. We used `LinesVisible` in the previous section to demonstrate this fact. One of the rules for inheriting attributes in a graphical tree is:

---

**Note** If an attribute such as `LinesVisible = TRUE` is specified in some node of the graphical tree, *all* primitives below this node that react to this attribute use the specified value as the default value.

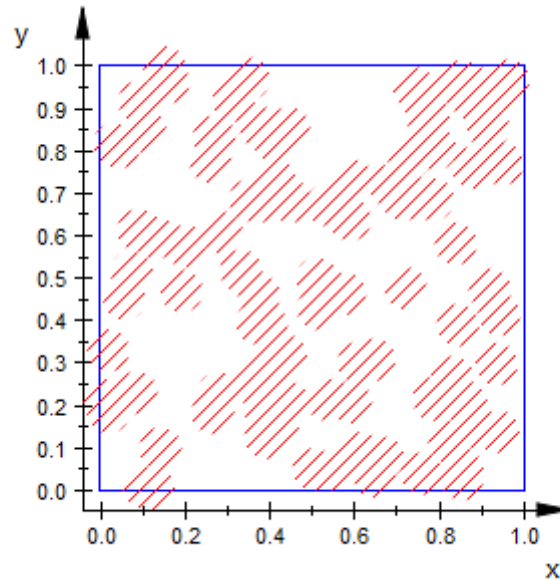
If a type specific attribute such as `plot::Circle2d::LinesVisible = TRUE` is specified, the new default value is valid only for primitives of that specific type.

---

In the following example, we consider 100 randomly placed circles with a rectangle indicating the area into which all circle centers are placed:

```
rectangle := plot::Rectangle(0..1, 0 ..1):
circles := plot::Circle2d(0.05, [frandom(), frandom()])
           $ i = 1..100:
```

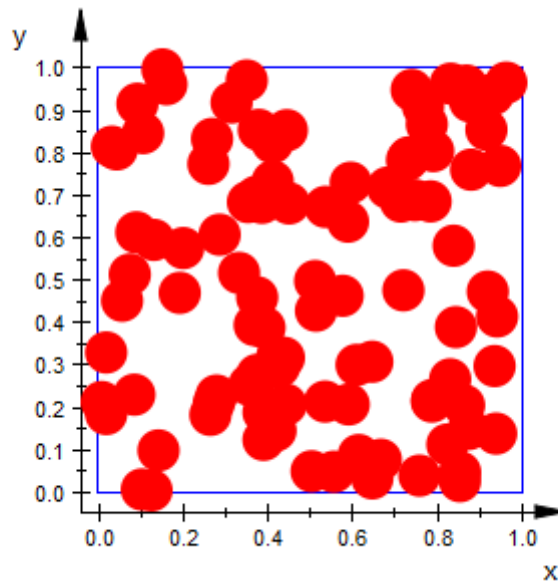
```
plot(rectangle, circles, Axes = Frame):
```



We wish to turn the circles into filled circles by `Filled = TRUE`, `LinesVisible = FALSE`:

```
plot(rectangle, circles,  
      Filled = TRUE, FillPattern = Solid,  
      LinesVisible = FALSE, Axes = Frame):
```





## Primitives Requesting Special Scene Attributes: “Hints”

The default values for the attributes are chosen in such a way that they produce reasonable pictures in “typical” plot commands. For example, the default axes type in 3D scenes is `Axes = Boxed` because this is the most appropriate axes type in the majority of 3D plots:

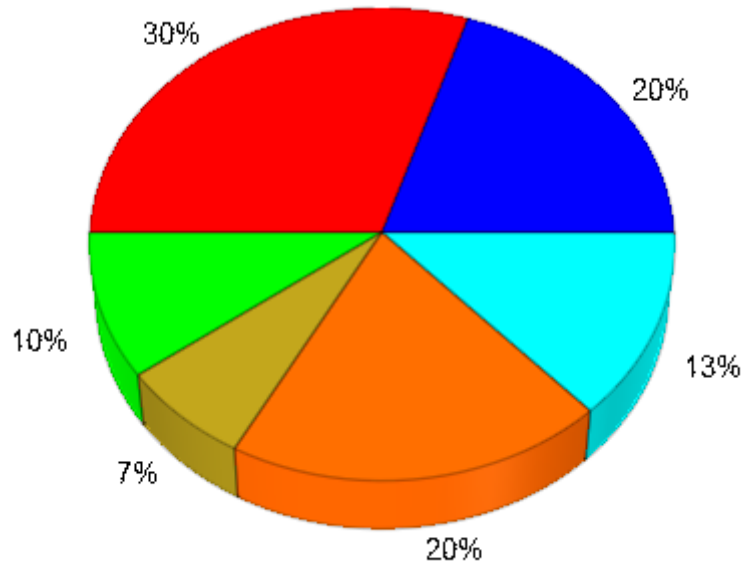
```
plot::getDefault(plot::CoordinateSystem3d::Axes)
```

### Boxed

However, there are exceptions. E.g., a plot containing a 3D pie chart should probably have no axes at all. Since it is not desirable to use `Axes = None` as the default setting for all plots, exceptional primitives such as `plot::Piechart3d` are given a chance to override the default axes automatically. In a pie chart plot, no axes are used by default:

```
plot(plot::Piechart3d([20, 30, 10, 7, 20, 13],
```

```
Titles = [1 = "20%", 2 = "30%", 3 = "10%",
          4 = "7%", 5 = "20%", 6 = "13%"]):
```



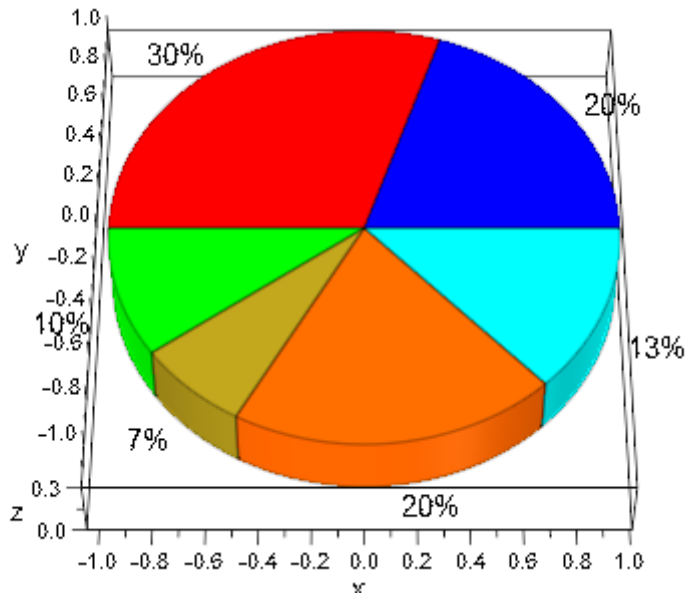
Note that `Axes` is a scene attribute that cannot be processed by pie chart objects directly. Hence, a separate mechanism for requesting special scene attributes by primitives is implemented: so-called “hints.”

A “hint” is an attribute of one of the superordinate nodes in the graphical tree, for example, an attribute of a coordinate system, a scene or the canvas. The help pages of the primitives give information on what “hints” are sent by the primitive. If several primitives send conflicting hints, the first “hint” is used.

“Hints” are implemented internally and cannot be switched off by the user. Note, however, that the “hints” concept only concerns *default values* for attributes. You can always specify the attribute explicitly if you think that a default value or a “hint” is not appropriate. E.g., we explicitly request `Axes = Boxed` in the following call:

```
plot(plot::Piechart3d([20, 30, 10, 7, 20, 13],
                      Titles = [1 = "20%", 2 = "30%", 3 = "10%",
```

4 = "7%", 5 = "20%", 6 = "13%"]),  
 Axes = Boxed):



## The Help Pages of Attributes

We have a brief look at a typical help page for a plot attribute to explain the information provided there:

The item “*Acceptable Values*” states the type of the number  $n$  that is admissible when passing the attributes  $UMesh = n$ ,  $VMesh = n$  etc.

The item “*Attribute type: inherited*” states that these attributes may not only be specified in the generating call of graphical primitives. They can also be specified at higher nodes of a graphical tree to be inherited to the primitives in the corresponding sub-tree (see section Inheritance of Attributes).

The sections “*Object types reacting to UMesh*” etc. provide complete listings of all primitives reacting to the attribute(s) together with the specific default values.

The “*Details*” section gives further information on the semantics of the attribute(s). Finally, there are “*Examples*” of plot commands using the described attribute(s).



## Layout of Canvas and Scenes

In this section...
“Layout of the Canvas” on page 5-129
“Layout of Scenes” on page 5-135

### Layout of the Canvas

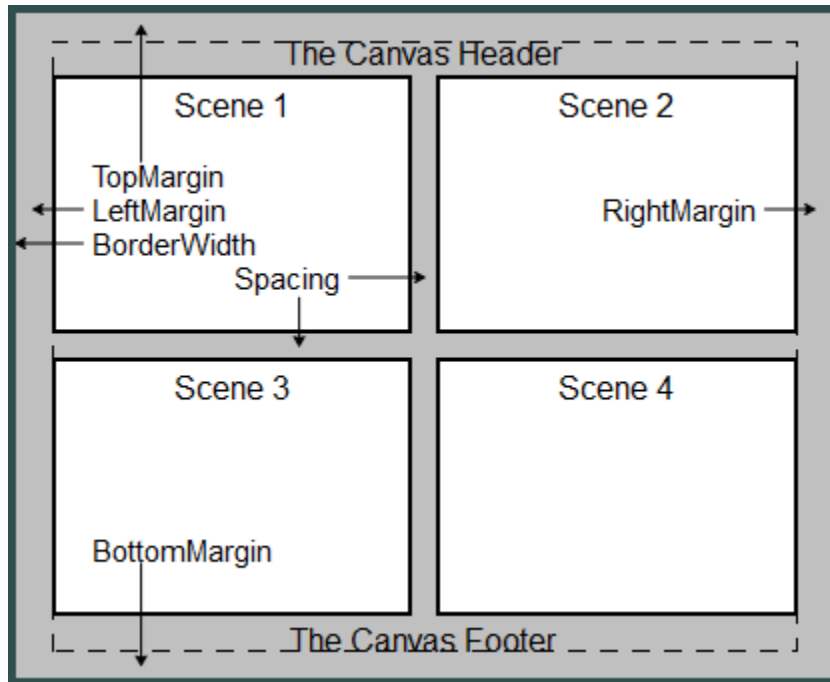
The following canvas attributes are relevant for its layout and its style.

attribute name	possible values/example	meaning	default
Width	12*unit::cm	physical width of the canvas	120*unit::mm
Height	8*unit::cm	physical height of the canvas	80*unit::mm
BackgroundColor	RGB color	color of the background	RGB::White
BorderColor	RGB color	color of the border	RGB::Grey50
BorderWidth	1*unit::mm	width of the border	0
Margin	1*unit::mm	common width for all margins: BottomMargin, LeftMargin, etc.	1*unit::mm
BottomMargin	1*unit::mm	width of the bottom margin	1*unit::mm
LeftMargin	1*unit::mm	width of the left margin	1*unit::mm
RightMargin	1*unit::mm	width of the right margin	1*unit::mm

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
TopMargin	1*unit::mm	width of the top margin	1*unit::mm
Footer	string	footer text	" " (no footer)
Header	string	header text	" " (no header)
FooterAlignment	Left, Center, Right	horizontal footer alignment	Center
HeaderAlignment	Left, Center, Right	horizontal header alignment	Center
FooterFont	see section Fonts	font for the footer	sans-serif 12
HeaderFont	see section Fonts	font for the header	sans-serif 12
Layout	Tabular, Horizontal, Vertical, Absolute, Relative	automatic or user-defined layout?	Tabular
Rows	integer > 0	number of rows in automatic tabular layout mode	
Columns	integer > 0	number of columns in automatic tabular layout mode	
Spacing	1.0*unit::mm	space between scenes	1.0*unit::mm

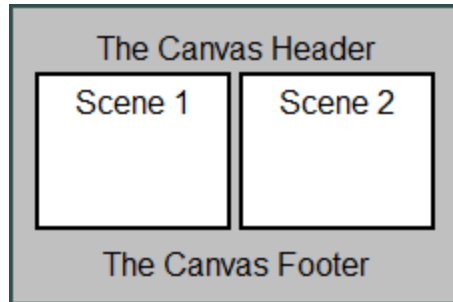
A canvas may contain one or more scenes. The following picture shows a canvas with four scenes to illustrate the meaning of the layout attributes

BorderWidth, BottomMargin, LeftMargin, RightMargin, TopMargin, and Spacing:

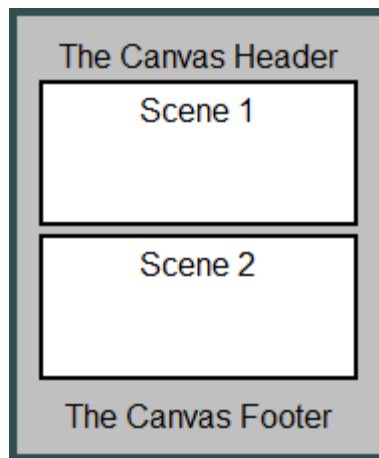


The basic attribute that switches between “automatic” and “user-defined” layout is `Layout`:

- With the default setting `Layout = Tabular`, a sequence of scenes in a canvas is displayed as a graphical array. The number of columns or rows of this array may be chosen via the attributes `Columns` or `Rows`, respectively. If none of these attributes is specified, the tabular layout scheme chooses some suitable values automatically.
- The setting `Layout = Horizontal` places the scenes side by side in a single row. It is a shortcut for the setting `Layout = Tabular, Rows = 1`:



- The setting `Layout = Vertical` places the scenes one below the other in a single column. It is a shortcut for the setting `Layout = Tabular, Columns = 1`:



The settings `Layout = Absolute` and `Layout = Relative` switch the automatic layout mode off and allow to position each scene via the scene attributes `Left` and `Bottom`. These attributes determine the position of the lower left corner of the scene and can be set separately for each scene.

- With `Layout = Absolute`, the values for the lower left corner of the scene as well as its width and height may be specified as absolute physical lengths such as `Left = 3.0*unit::mm`, `Bottom = 4.5*unit::mm`, `Width = 10*unit::cm`, `Height = 4*unit::inch`.

- With `Layout = Relative`, these values may be specified as fractions of the canvas height and width. E.g.,

```
Layout = Relative,
Left = 0.3, Bottom = 0.2,
Width = 0.5, Height = 0.5
```

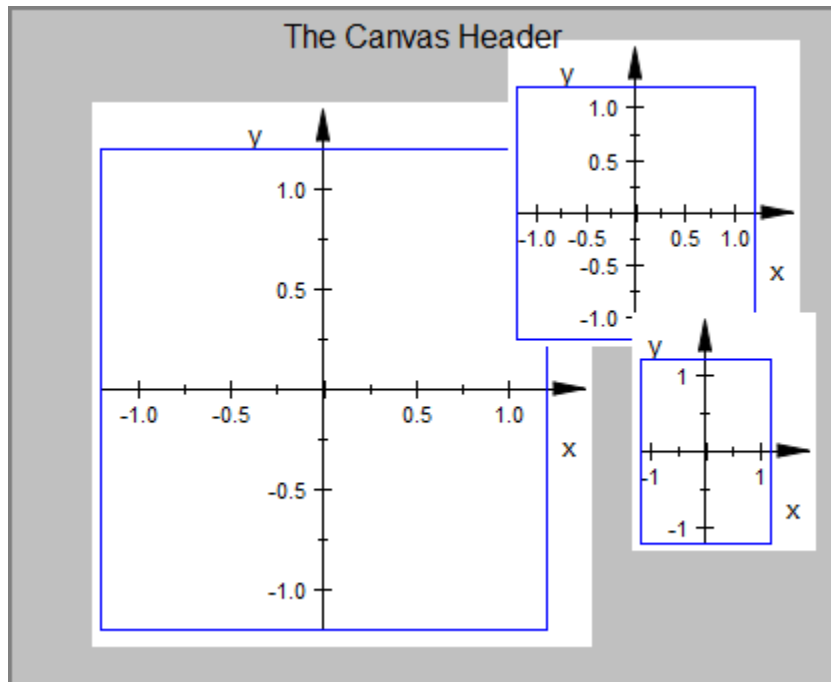
is equivalent to

```
Layout = Absolute,
Left = 0.3*canvaswidth, Bottom = 0.2*canvasheight,
Width = 0.5*canvaswidth, Height = 0.5*canvasheight,
```

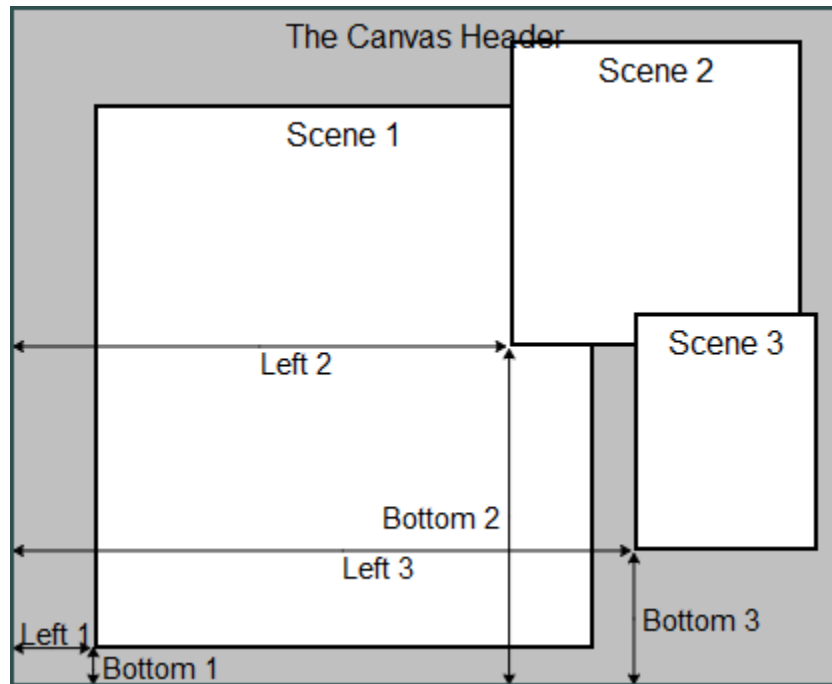
where `canvaswidth` and `canvasheight` are the physical width and height of the canvas.

The following command uses `Layout = Relative` to position 3 scenes by specifying the lower left corner together with their widths and heights as fractions of the canvas dimensions:

```
Left1 := 0.1: Bottom1 := 0.06: Width1 := 0.60: Height1 := 0.8:
Left2 := 0.6: Bottom2 := 0.5: Width2 := 0.35: Height2 := 0.45:
Left3 := 0.75: Bottom3 := 0.2: Width3 := 0.22: Height3 := 0.35:
plot(plot::Canvas(
  BorderWidth = 0.5*unit::mm,
  BackgroundColor = RGB::Grey,
  plot::Scene2d(plot::Rectangle(-1.2..1.2, -1.2..1.2),
    Left = Left1, Bottom = Bottom1,
    Width = Width1, Height = Height1),
  plot::Scene2d(plot::Rectangle(-1.2..1.2, -1.2..1.2),
    Left = Left2, Bottom = Bottom2,
    Width = Width2, Height = Height2),
  plot::Scene2d(plot::Rectangle(-1.2..1.2, -1.2..1.2),
    Left = Left3, Bottom = Bottom3,
    Width = Width3, Height = Height3),
  Header = "The Canvas Header", Layout = Relative,
  Width = 110*unit::mm, Height = 90*unit::mm)):
```



In detail:



## Layout of Scenes

The following scene attributes are relevant for the layout and the style of a scene:

### Layout and style parameters for scenes

attribute name	possible values/example	meaning	default
Width	0.8, 12*unit::cm	width of the scene (relative or absolute value)	automatic
Height	0.8, 8*unit::cm	height of the scene (relative or absolute value)	automatic

**Layout and style parameters for scenes (Continued)**

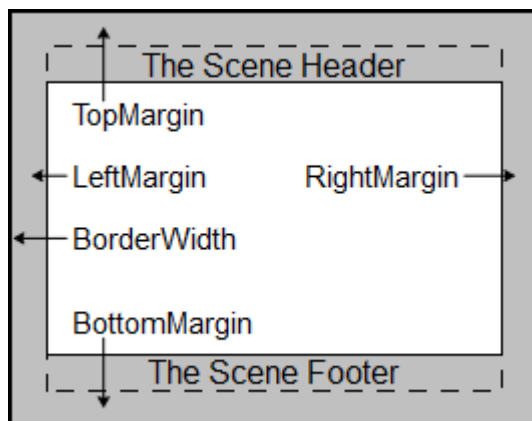
<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
BackgroundColor	RGB color	color of the background	RGB::White
BackgroundColor2	RGB color	secondary color for color blend (3D only)	RGB::Grey75
BackgroundStyle	Flat, LeftRight, Pyramid, TopBottom	style of color blend (3D only)	FLAT
BackgroundTransparent	TRUE, FALSE	background transparent or opaque?	FALSE
BorderColor	RGB color	color of the border	RGB::SlateGrey
BorderWidth	1*unit::mm	width of the border	0
Margin	1*unit::mm	common width for all margins: BottomMargin, LeftMargin, etc.	1*unit::mm
BottomMargin	1*unit::mm	width of the bottom margin	1*unit::mm
LeftMargin	1*unit::mm	width of the left margin	1*unit::mm
RightMargin	1*unit::mm	width of the right margin	1*unit::mm
TopMargin	1*unit::mm	width of the top margin	1*unit::mm
Footer	string	footer text	" " (no footer)
Header	string	header text	" " (no header)



**Layout and style parameters for scenes (Continued)**

<b>attribute name</b>	<b>possible values/example</b>	<b>meaning</b>	<b>default</b>
FooterAlignment	Left, Center, Right	horizontal footer alignment	Center
HeaderAlignment	Left, Center, Right	horizontal header alignment	Center
FooterFont	see section Fonts	font for the footer	sans-serif 12
HeaderFont	see section Fonts	font for the header	sans-serif 12
LegendVisible	TRUE, FALSE	legend on/off	FALSE
LegendAlignment	Center, Left, Right	horizontal legend alignment	Center
LegendPlacement	Bottom, Top	vertical legend placement	Bottom
Left	0.1, 1.0*unit::mm	distance of the left side of the scene to the left side of the canvas (relative or absolute value)	0
Bottom	0.1, 1.0*unit::mm	distance of the bottom side of the scene to the bottom side of the canvas (relative or absolute value)	0

Similar to the canvas, scenes can have a border (set via `BorderWidth` and `BorderColor`). There is a (small) margin into which no graphical content is rendered. Further, a header and a footer can be specified:



# Animations

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## Generate Simple Animations

Each primitive of the `plot` library knows how many specifications of type “range” it has to expect. For example, a univariate function graph in 2D such as

```
plot::Function2d(sin(x), x = 0..2*PI):
```

expects one plot range for the  $x$  coordinate, whereas a bivariate function graph in 3D expects two plot ranges for the  $x$  and  $y$  coordinate:

```
plot::Function3d(sin(x^2 + y^2), x = 0..2, y = 0..2):
```

A contour plot in 2D expects 2 ranges for the  $x$  and  $y$  coordinate:

```
plot::Implicit2d(x^2 + y^2 - 1, x = -2..2, y = - 2..2):
```

A contour plot in 3D expects 3 ranges for the  $x$ ,  $y$ , and  $z$  coordinate:

```
plot::Implicit3d(x^2 + y^2 + z^2 - 1, x = -2..2,
                y = - 2..2, z = - 2..2):
```

A line in 2D does not expect any range specification:

```
plot::Line2d([0, 0], [1, 1]):
```

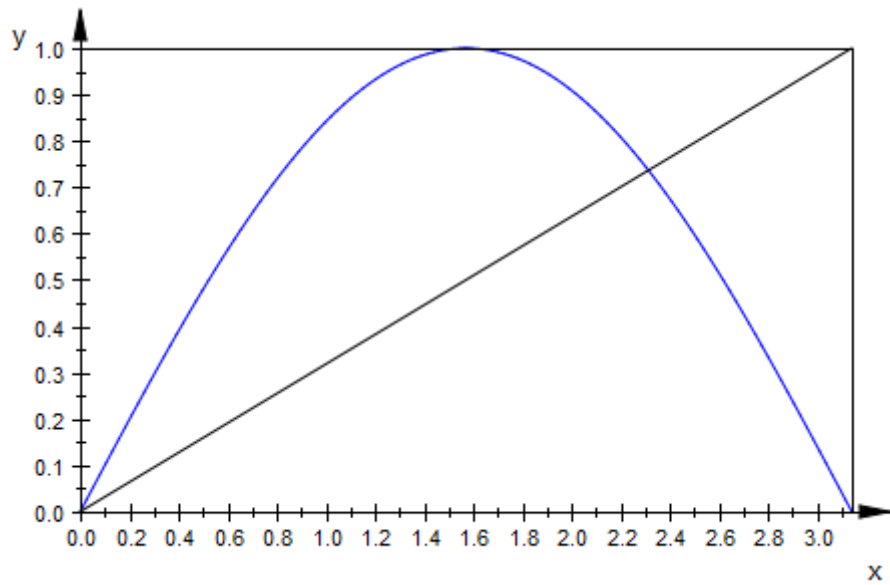
---

**Note** Whenever a graphical primitive receives a “surplus” range specification by an equation such as  $a = \text{amin}.. \text{amax}$ , the parameter  $a$  is interpreted as an “animation parameter” assuming values from  $\text{amin}$  to  $\text{amax}$ .

---

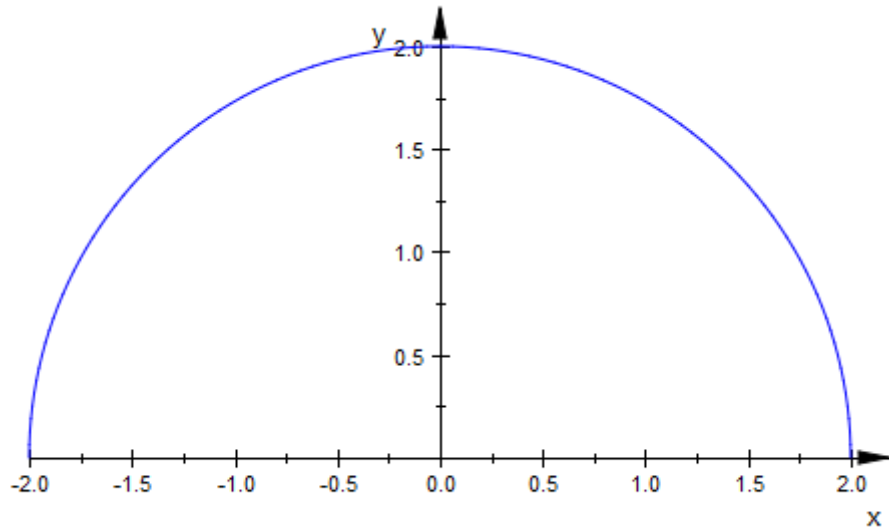
Thus, it is very easy indeed to create animated objects: Just pass a “surplus” range equation  $a = \text{amin}.. \text{amax}$  to the generating call of the primitive. All other entries and attributes of the primitive that are symbolic expressions of the animation parameter will be animated. In the following call, both the function expression as well as the  $x$  range of the function graph depend on the animation parameter. Also, the ranges defining the width and the height of the rectangle as well as the end point of the line depend on it:

```
plot(  
  plot::Function2d(a*sin(x), x = 0..a*PI, a = 0.5..1),  
  plot::Rectangle(0..a*PI, 0..a, a = 0.5..1,  
    LineColor = RGB::Black),  
  plot::Line2d([0, 0], [PI*a, a], a = 0.5 ..1,  
    LineColor = RGB::Black)  
)
```



Additional range specifications may enter via the graphical attributes. Here is an animated arc whose radius and “angle range” depend on the animation parameter:

```
plot(plot::Arc2d(1 + a, [0, 0], AngleRange = 0..a*PI, a = 0..1)):
```



Here, the attribute `AngleRange` is identified by its attribute name and thus not assumed to be the specification of an animation parameter with animation range.

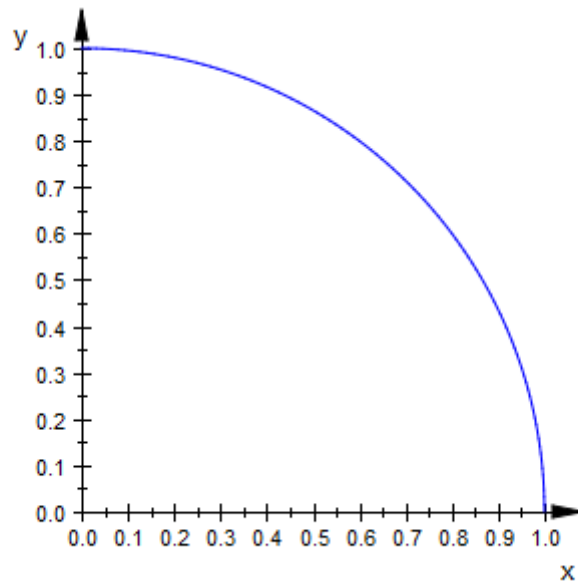
---

**Note** Do make sure that attributes are specified by their correct names. If an incorrect attribute name is used, it may be mistaken for an animation parameter!

---

In the following examples, we wish to define a static semicircle, using `plot::Arc2d` with `AngleRange = 0..PI`. However, `AngleRange` is spelled incorrectly. A plot is created. It is an animated full circle with the animation parameter `AngelRange`!

```
plot(plot::Arc2d(1, [0, 0], AngelRange = 0..PI)):
```



The animation parameter may be any symbolic parameter (`identifier` or `indexed identifier`) that is different from the symbols used for the mandatory range specifications (such as the names of the independent variables in function graphs). The parameter must also be different from any of the protected names of the plot attributes.

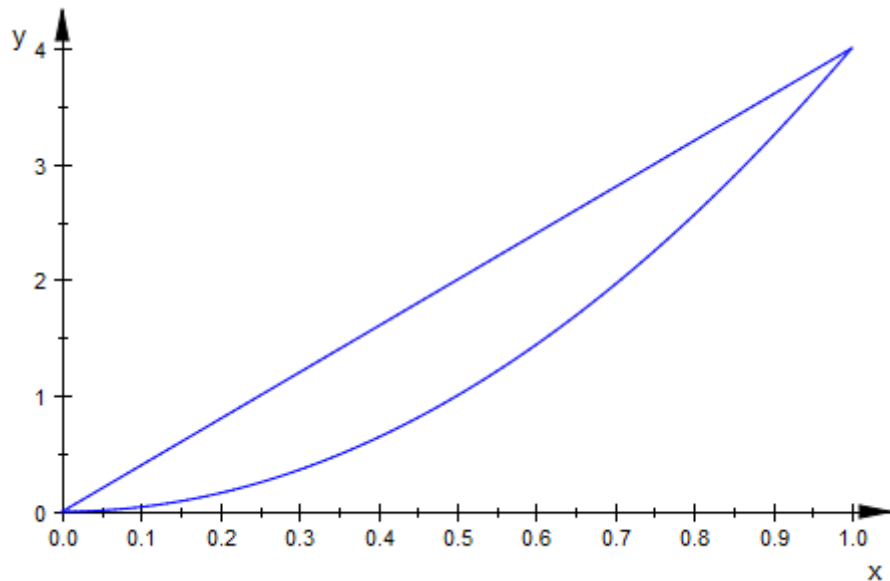
---

**Note** Animations are created object by object. The names of the animation parameters in different objects need not coincide.

---

In the following example, different names `a`, `b` are used for the animation parameters of the two functions:

```
plot(plot::Function2d(4*a*x, x = 0..1, a = 0..1),  
      plot::Function2d(b*x^2, x = 0..1, b = 1..4)):
```

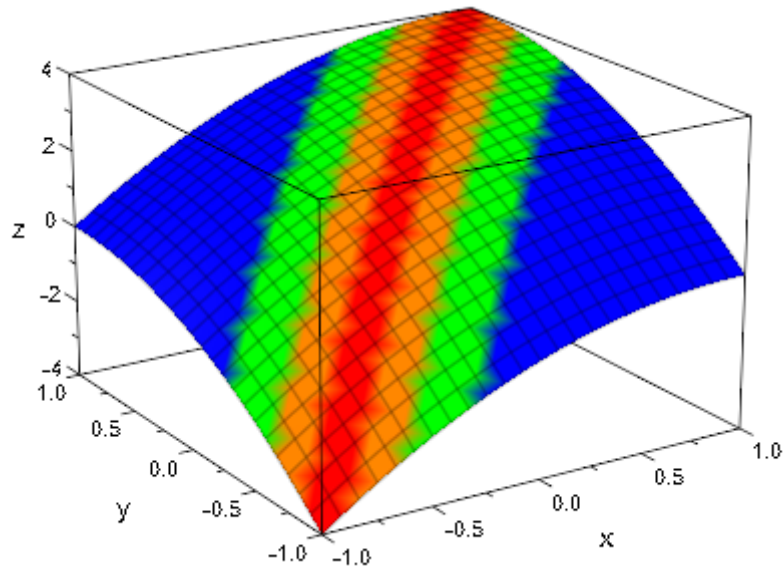


An animation parameter is a global symbolic name. It can be used as a global variable in procedures defining the graphical object. The following example features the 3D graph of a bivariate function that is defined by a procedure using the globally defined animation parameter. Further, a `fill color` function `mycolor` is defined that changes the color in the course of the animation. It could use the animation parameter as a global parameter, just as the function `f` does. Alternatively, the animation parameter may be declared as an additional input parameter. Refer to the help page of `FillColorFunction` to find out, how many input parameters the fill color function expects and which of the input parameters is fed with the animation parameter. One finds that for `plot::Function3d`, the fill color function is called with the coordinates  $x, y, z$  of the points on the graph. The next input parameter (the 4th argument of `mycolor`) is the animation parameter:

```
f := (x, y) -> 4 - (x - a)^2 - (y - a)^2:
mycolor := proc(x, y, z, a)
  local t;
  begin
    t := sqrt((x - a)^2 + (y - a)^2):
    if t < 0.1 then
```

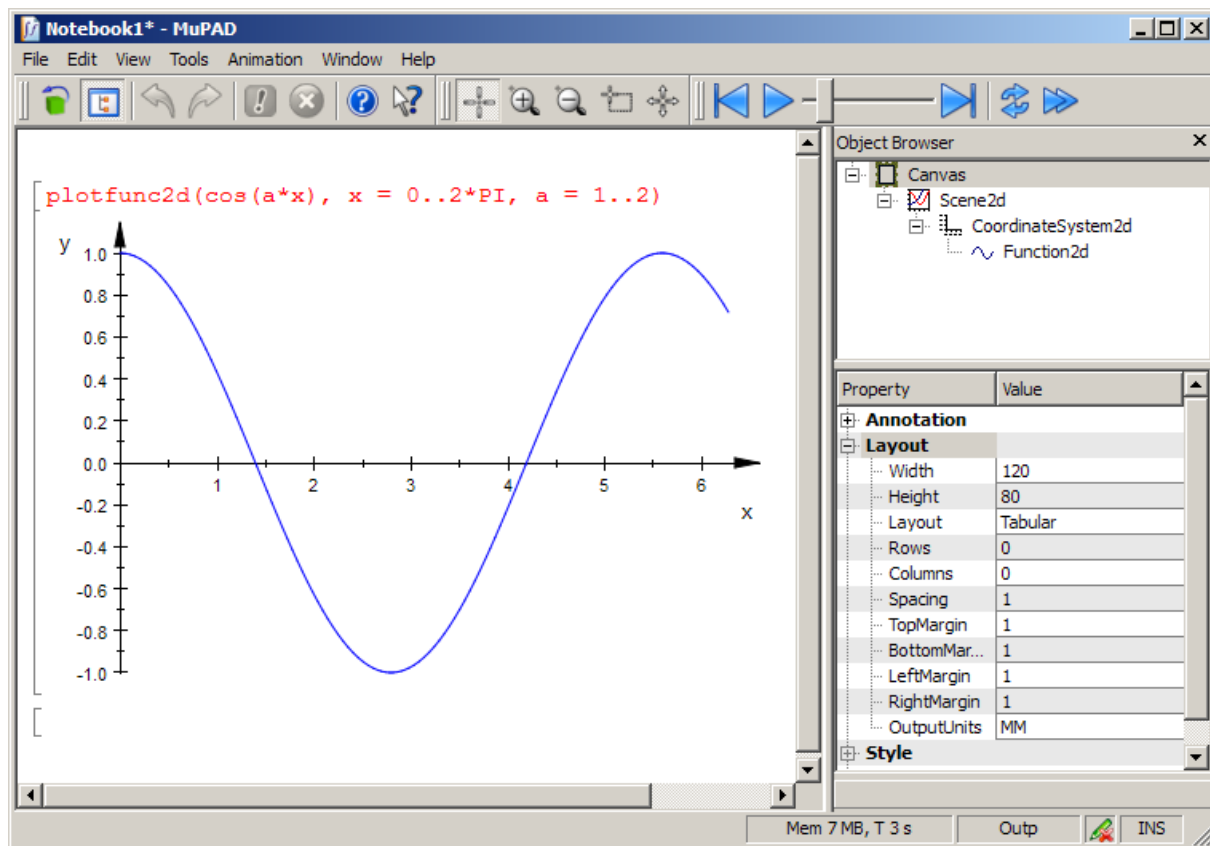


```
        return(RGB::Red)
    elif t < 0.4 then
        return(RGB::Orange)
    elif t < 0.7 then
        return(RGB::Green)
    else return(RGB::Blue)
    end_if;
end:
plot(plot::Function3d(f, x = -1..1, y = -1..1, a = -1..1,
                    FillColorFunction = mycolor)):
```



## Play Animations

When an animated plot is created in a MuPAD notebook, the first frame of the animation appears as a static picture below the input region. To start the animation, double click on the plot. An icon for starting the animation will appear (make sure the item 'Animation Bar' of the 'View' menu is enabled):



One can also use the slider to animate the picture “by hand.” Alternatively, the ‘Animation’ menu provides an item for starting the animation.

## The Number of Frames and the Time Range

By default, an animation consists of 50 different frames. The number of frames can be set to be any positive number  $n$  by specifying the attribute `Frames = n`. This attribute can be set in the generating call of the animated primitives, or at some higher node of the graphical tree. In the latter case, this attribute is inherited to all primitives that exist below the node. With `a = amin..amax`, `Frames = n`, the  $i$ -th frame consists of a snapshot of the primitive with

$$a = a_{\min} + \frac{i-1}{n-1} (a_{\max} - a_{\min}), i = 1, \dots, n$$

Increasing the number of frames does not mean that the animation runs longer; the renderer does not work with a fixed number of frames per second but processes all frames within a fixed time interval.

In the background, there is a “real time clock” used to synchronize the animation of different animated objects. An animation has a time range measured by this clock. The time range is set by the attributes `TimeBegin = t0`, `TimeEnd = t1` or, equivalently, `TimeRange = t0..t1`, where `t0`, `t1` are real numerical values representing physical times in seconds. These attribute can be set in the generating call of the animated primitives, or at some higher node of the graphical tree. In the latter case, these attributes are inherited by all primitives that exist below the node.

The absolute value of `t0` is irrelevant if all animated objects share the same time range. Only the time difference `t1 - t0` matters. It is (an approximation of) the physical time in seconds that the animation will last.

---

**Note** The parameter range `amin..amax` in the specification of the animation parameter `a = amin..amax` together with `Frames = n` defines an equidistant time mesh in the time interval set by `TimeBegin = t0` and `TimeEnd = t1`. The frame with `a = amin` is visible at the time `t0`, the frame with `a = amax` is visible at the time `t1`.

---

---

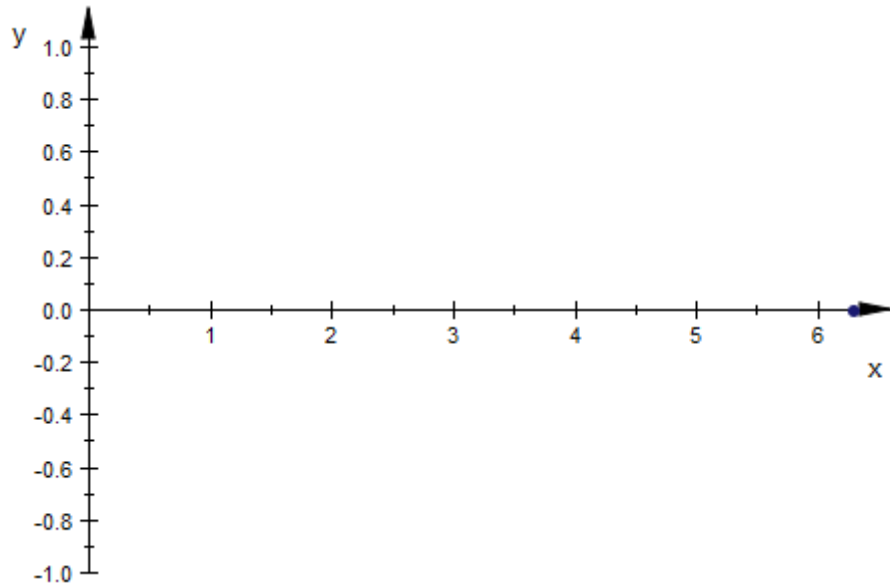
**Note** With the default `TimeBegin = 0`, the value of the attribute `TimeEnd` gives the physical time of the animation in seconds. The default value is `TimeEnd = 10`, i.e., an animation using the default values will last about 10 seconds. The number of frames set by `Frames = n` does not influence the time interval, but changes the number of frames displayed in this time interval.

---

Here is a simple example:

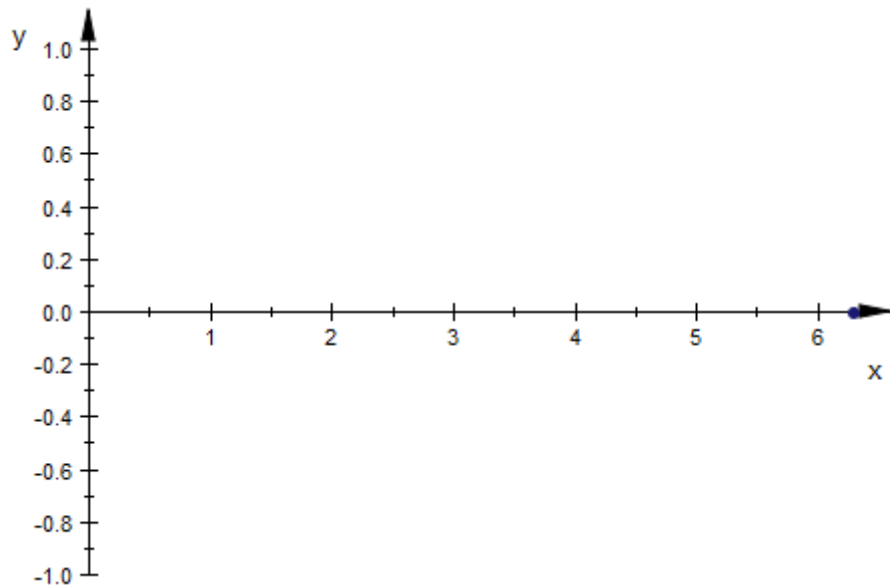
```
plot(plot::Point2d([a, sin(a)], a = 0..2*PI,
```

```
Frames = 100, TimeRange = 0..5)):
```



The point will be animated for about 5 physical seconds in which it moves along one period of the sine graph. Each frame is displayed for about 0.05 seconds. After increasing the number of frames by a factor of 2, each frame is displayed for about 0.025 seconds, making the animation somewhat smoother:

```
plot(plot::Point2d([a, sin(a)], a = 0..2*PI,  
    Frames = 200, TimeRange = 0..5)):
```



Note that the human eye cannot distinguish between different frames if they change with a rate of more than 25 frames per second. Thus, the number of frames  $n$  set for the animation should satisfy

$$n < 25 (t_1 - t_0)$$

Hence, with the default time range  $\text{TimeBegin} = t_0 = 0, \text{TimeEnd} = t_1 = 10$  (seconds), it does not make sense to specify  $\text{Frames} = n$  with  $n > 250$ . If a higher frame number is required to obtain a sufficient resolution of the animated object, one should increase the time for the animation by a sufficiently high value of  $\text{TimeEnd}$ .

## What Can Be Animated?

We may regard a graphical primitive as a collection of plot attributes. (Indeed, also the function expression  $\sin(x)$  in `plot::Function2d(sin(x), x = 0..2*PI)` is internally realized at the attribute `Function = sin(x)`.) So, the question is:

*“Which attributes can be animated?”*

The answer is: *“Almost any attribute can be animated!”* Instead of listing the attributes that allow animation, it is much easier to characterize the attributes that cannot be animated:

- None of the `canvas` attributes can be animated. This includes layout parameters such as the physical size of the picture. See the help page of `plot::Canvas` for a complete list of all canvas attributes.
- None of the attributes of `2D scenes` and `3D scenes` can be animated. This includes layout parameters, background color and style, camera positioning in 3D etc. See the help pages of `plot::Scene2d` and `plot::Scene3d` for a complete list of all scene attributes.

Note that there are camera objects of type `plot::Camera` that can be placed in a 3D scene. These camera objects can be animated and allow to realize a “flight” through a 3D scene. See section *Cameras in 3D* for details.

- None of the attributes of `2D coordinate systems` and `3D coordinate systems` can be animated. This includes viewing boxes, axes, axes ticks, and grid lines (rulings) in the background. See the help pages of `plot::CoordinateSystem2d` and `plot::CoordinateSystem3d` for a complete list of all attributes for coordinate systems.

Although the `ViewingBox` attribute of a coordinate system cannot be animated, the user can still achieve animated visibility effects in 3D by clipping box objects of type `plot::ClippingBox`.

- None of the attributes that are declared as *“Attribute Type: inherited”* on their help page can be animated. This includes size specifications such as `PointSize`, `LineWidth` etc.
- RGB and RGBA values cannot be animated. However, it is possible to animate the coloring of lines and surfaces via user defined procedures. See the help pages `LineColorFunction` and `FillColorFunction` for details.
- The texts of annotations such as `Footer`, `Header`, `Title`, `legend entries`, etc. cannot be animated. The position of titles, however, can be animated.

There are special text objects `plot::Text2d` and `plot::Text3d` that allow to animate the text as well as their position.

- Fonts cannot be animated.

- Attributes such as `DiscontinuitySearch = TRUE` or `FillPattern = Solid` that can assume only finitely many values from a fixed discrete set cannot be animated.

Nearly all attributes not falling into one of these categories can be animated. You will find detailed information on this issue on the corresponding help pages of primitives and attributes.

## Advanced Animations: The Synchronization Model

As already explained in section The Number of Frames and the Time Range, there is a “real time clock” running in the background that synchronizes the animation of different animated objects.

Each animated object has its own separate “real time life span” set by the attributes `TimeBegin = t0`, `TimeEnd = t1` or, equivalently, `TimeRange = t0..t1`. The values `t0`, `t1` represent seconds measured by the “real time clock.”

In most cases, there is no need to bother about specifying the life span. If `TimeBegin` and `TimeEnd` are not specified, the default values `TimeBegin = 0` and `TimeEnd = 10` are used, i.e., the animation will last about 10 seconds. These values only need to be modified

- if a shorter or longer real time period for the animation is desired, or
- if the animation contains several animated objects, where some of the animated objects are to remain static while others change.

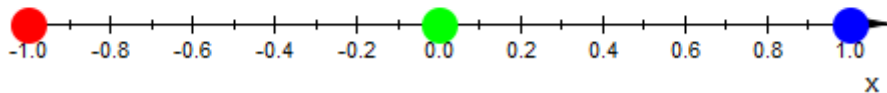
Here is an example for the second situation. The plot consists of 3 jumping points. For the first 5 seconds, the left point jumps up and down, while the other points remain at their initial position. Then, all points stay static for 1 second. After a total of 6 seconds, the middle point starts its animation by jumping up and down, while the left point remains static in its final position and the right points stays static in its initial position. After 9 seconds, the right point begins to move as well. The overall time span for the animation is the hull of the time ranges of all animated objects, i.e., 15 seconds in this example:

```
p1 := plot::Point2d(-1, sin(a), a = 0..PI, Color = RGB::Red,
```

```

        PointSize = 5*unit::mm,
        TimeBegin = 0, TimeEnd = 5):
p2 := plot::Point2d(0, sin(a), a = 0..PI, Color = RGB::Green,
        PointSize = 5*unit::mm,
        TimeBegin = 6, TimeEnd = 12):
p3 := plot::Point2d(1, sin(a), a = 0..PI, Color = RGB::Blue,
        PointSize = 5*unit::mm,
        TimeBegin = 9, TimeEnd = 15):
plot(p1, p2, p3, PointSize = 3.0*unit::mm,
     YAxisVisible = FALSE):

```



Here, all points use the default settings `VisibleBeforeBegin = TRUE` and `VisibleAfterEnd = TRUE` which make them visible as static objects outside the time range of their animation. We set `VisibleAfterEnd = FALSE` for the middle point, so that it disappears after the end of its animation. With `VisibleBeforeBegin = FALSE`, the right point is not visible until its animation starts:

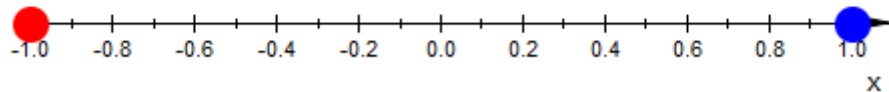
```

p2::VisibleAfterEnd := FALSE:
p3::VisibleBeforeBegin := FALSE:

```



```
plot(p1, p2, p3, PointSize = 3.0*unit::mm,  
     YAxisVisible = FALSE):
```



We summarize the synchronization model of animations:

---

**Note** The total real time span of an animated plot is the physical real time given by the minimum of the `TimeBegin` values of all animated objects in the plot to the maximum of the `TimeEnd` values of all the animated objects.

---

- When a plot containing animated objects is created, the real time clock is set to the minimum of the `TimeBegin` values of all animated objects in the plot. The real time clock is started when pushing the 'play' button for animations in the graphical user interface.
- Before the real time reaches the `TimeBegin` value `t0` of an animated object, this object is static in the state corresponding to the begin of its animation. Depending on the attribute `VisibleBeforeBegin`, it may be visible or invisible before `t0`.

- During the time from `t0` to `t1`, the object changes from its original to its final state.
- After the real time reaches the `TimeEnd` value `t1`, the object stays static in the state corresponding to the end of its animation. Depending on the value of the attribute `VisibleAfterEnd`, it may stay visible or become invisible after `t1`.
- The animation of the entire plot ends with the physical time given by the maximum of the `TimeEnd` values of all animated objects in the plot.

### Frame by Frame Animations

There are some special attributes such as `VisibleAfter` that are very useful to build animations from purely static objects:

---

**Note** With `VisibleAfter = t0`, an object is invisible from the start of the animation until time `t0`. Then it will appear and remain visible for the rest of the animation.

---

---

**Note** With `VisibleBefore = t1`, an object is visible from the start of the animation until time `t1`. Then it will disappear and remain invisible for the rest of the animation.

---

These attributes should not be combined to define a “visibility range” from `t0` to `t1`. Use the attribute `VisibleFromTo` instead:

---

**Note** With `VisibleFromTo = t0..t1`, an object is invisible from the start of the animation until time `t0`. Then it will appear and remain visible until time `t1`, when it will disappear and remain invisible for the rest of the animation.

---

We continue the example of the previous section in which we defined the following animated points:

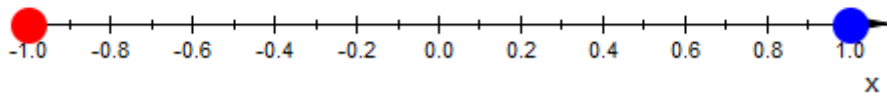
```
p1 := plot::Point2d(-1, sin(a), a = 0..PI, Color = RGB::Red,
    PointSize = 5*unit::mm,
    TimeBegin = 0, TimeEnd = 5):
p2 := plot::Point2d(0, sin(a), a = 0..PI, Color = RGB::Green,
    PointSize = 5*unit::mm,
    TimeBegin = 6, TimeEnd = 12):
p3 := plot::Point2d(1, sin(a), a = 0..PI, Color = RGB::Blue,
    PointSize = 5*unit::mm,
    TimeBegin = 9, TimeEnd = 15):
p2::VisibleAfterEnd := FALSE:
p3::VisibleBeforeBegin := FALSE:
```

We add a further point `p4` that is not animated. We make it invisible at the start of the animation via the attribute `VisibleFromTo`. It is made visible after 7 seconds to disappear again after 13 seconds:

```
p4 := plot::Point2d(0.5, 0.5, Color = RGB::Black,
    PointSize = 5*unit::mm,
    VisibleFromTo = 7..13):
```

The start of the animation is determined by `p1` which bears the attribute `TimeBegin = 0`, the end of the animation is determined by `p3` which has set `TimeEnd = 15`:

```
plot(p1, p2, p3, p4, PointSize = 3.0*unit::mm,
    YAxisVisible = FALSE):
```



Although a typical MuPAD animation is generated object by object, each animated object taking care of its own animation, we can also use the attributes `VisibleAfter`, `VisibleBefore`, `VisibleFromTo` to build up an animation frame by frame:

---

**Note** “*Frame by frame animations*”: Choose a collection of (typically static) graphical primitives that are to be visible in the  $i$ -th frame of the animation. Set `VisibleFromTo = t[i]..t[i+1]` for these primitives, where `t[i]..t[i+1]` is the real time life span of the  $i$ -th frame (in seconds). Finally, plot all frames in a single `plot` command.

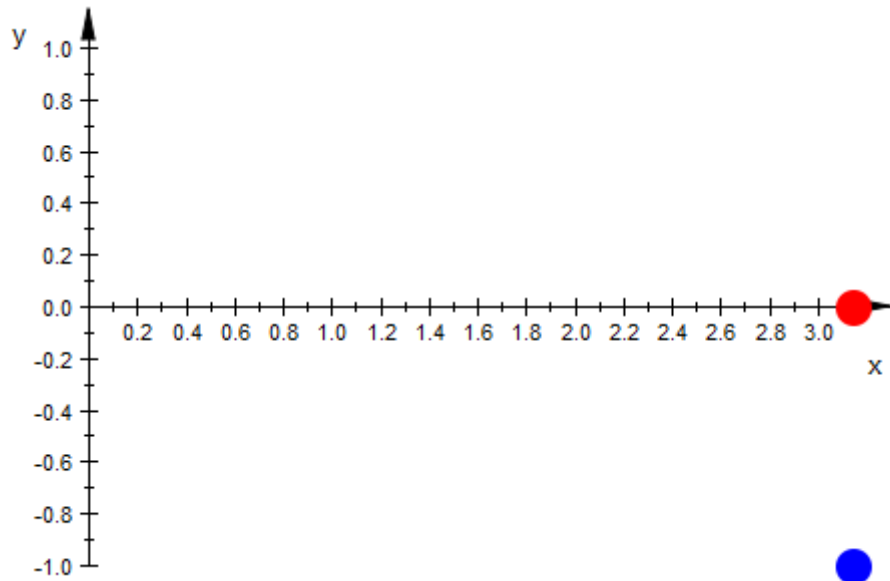
---

Here is an example. We let two points wander along the graphs of the sine and the cosine function, respectively. Each frame is to consist of a picture of two points. We use `plot::Group2d` to define the frame; the group forwards the attribute `VisibleFromTo` to all its elements:

```
for i from 0 to 101 do
  t[i] := i/10;
```

```

end_for:
for i from 0 to 100 do
  x := i/100*PI;
  myframe[i] := plot::Group2d(
    plot::Point2d([x, sin(x)], Color = RGB::Red),
    plot::Point2d([x, cos(x)], Color = RGB::Blue),
    VisibleFromTo = t[i]..t[i + 1]);
end_for:
plot(myframe[i] $ i = 0..100, PointSize = 5.0*unit::mm):
    
```

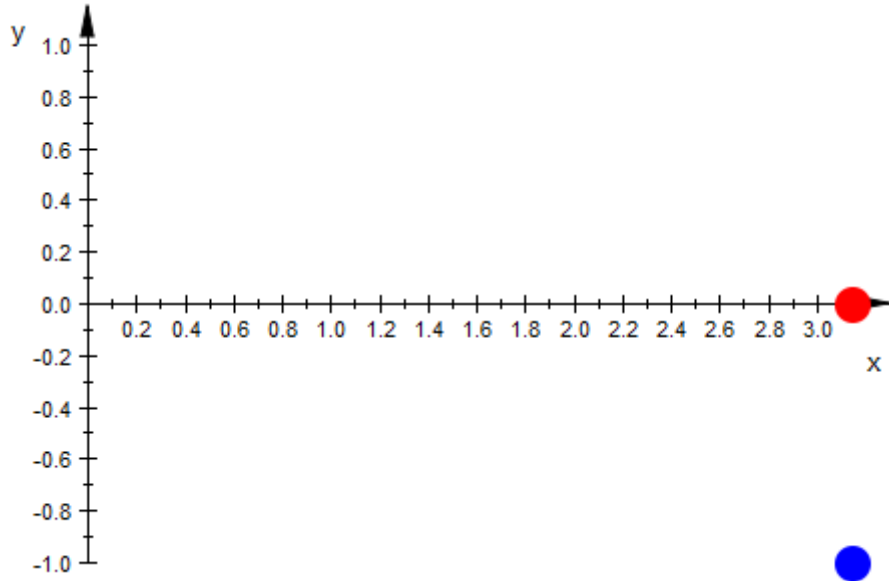


This “frame by frame” animation certainly needs a little bit more coding effort than the equivalent objectwise animation, where each of the points is animated:

```

delete i:
plot(plot::Point2d([i/100*PI, sin(i/100*PI)], i = 0..100,
  Color = RGB::Red),
  plot::Point2d([i/100*PI, cos(i/100*PI)], i = 0..100,
  Color = RGB::Blue),
  Frames = 101, TimeRange = 0..10,
    
```

PointSize = 5.0\*unit::mm):



There is, however, a special kind of plot where “frame by frame” animations are very useful. Note that in the present version of the graphics, new plot objects cannot be added to a scene that is already rendered. With the special “visibility” animations for static objects, however, one can easily simulate a plot that gradually builds up: Fill the frames of the animation with static objects that are visible for a limited time only. The visibility can be chosen very flexibly by the user. For example, the static objects can be made visible only for one frame (`VisibleFromTo`) so that the objects seem to move.

In the following example, we use `VisibleAfter` to fill up the plot gradually. We demonstrate the caustics generated by sunlight in a tea cup. The rim of

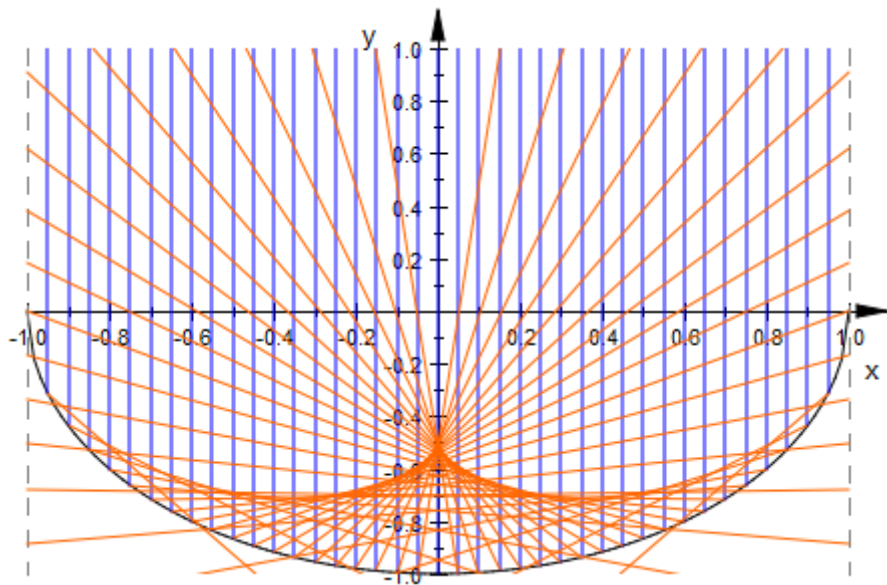
the cup, regarded as a mirror, is given by the function  $f(x) = -\sqrt{1-x^2}$ ,  $x \in [-1, 1]$  (a semicircle). Sun rays parallel to the  $y$ -axis are reflected by the rim. After reflection at the point  $(x, f(x))$  of the rim, a ray heads into the

direction  $\left(-1, -\frac{f'(x) - \frac{1}{2}}{f'(x)}\right)$  if  $x$  is positive. It heads into the direction

$\left(1, \frac{f'(x) - 1}{f'(x)}\right)$  if  $x$  is negative. Sweeping through the mirror from left to right, the incoming rays as well as the reflected rays are visualized as lines. In the animation, they become visible after the time  $5x$ , where  $x$  is the coordinate of the rim point at which the ray is reflected:

```

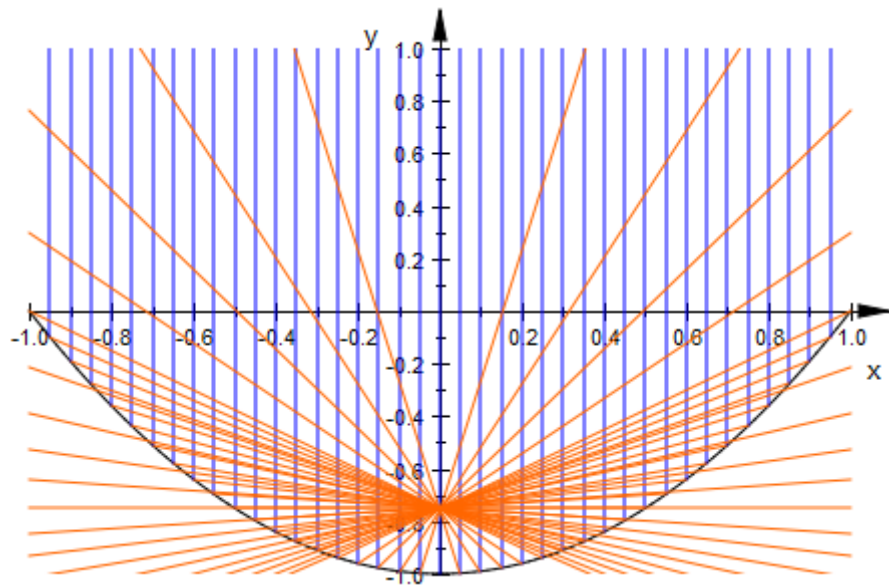
f := x -> -sqrt(1 - x^2):
plot(// The static rim:
      plot::Function2d(f(x), x = -1..1, Color = RGB::Black),
      // The incoming rays:
      plot::Line2d([x, 2], [x, f(x)], VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 1..39],
      // The reflected rays leaving to the right:
      plot::Line2d([x, f(x)],
                   [1, f(x) + (1-x)*(f'(x) - 1/f'(x))/2],
                   Color = RGB::Orange, VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 1..19],
      // The reflected rays leaving to the left:
      plot::Line2d([x, f(x)],
                   [-1, f(x) - (x+1)*(f'(x) - 1/f'(x))/2],
                   Color = RGB::Orange, VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 21..39],
      ViewingBox = [-1..1, -1..1]):
    
```



Compare the spherical mirror with a parabolic mirror that has a true focal point:

```
f := x -> -1 + x^2:
plot(// The static rim:
      plot::Function2d(f(x), x = -1..1, Color = RGB::Black),
      // The incoming rays:
      plot::Line2d([x, 2], [x, f(x)], VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 1..39],
      // The reflected rays leaving to the right:
      plot::Line2d([x, f(x)],
                   [1, f(x) + (1-x)*(f'(x) - 1/f'(x))/2],
                   Color = RGB::Orange, VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 1..19],
      // The reflected rays leaving to the left:
      plot::Line2d([x, f(x)],
                   [-1, f(x) - (x+1)*(f'(x) - 1/f'(x))/2],
                   Color = RGB::Orange, VisibleAfter = 5*x
                   ) $ x in [-1 + i/20 $ i = 21..39],
      ViewingBox = [-1..1, -1..1]):
```





## Examples

### Example 1

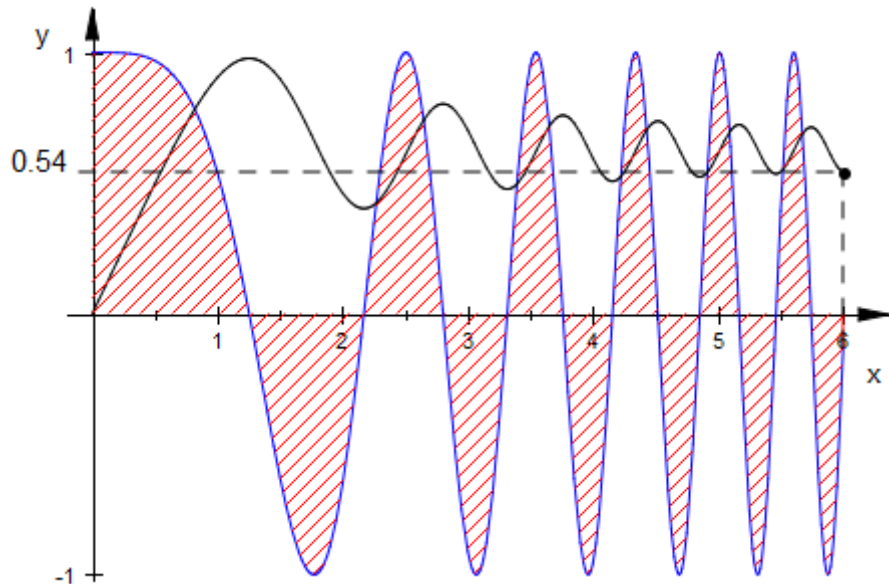
We build a 2D animation that displays a function  $f(x)$  together with the integral  $F(x) = \int^x f(y) \, dy$ . The area between the graph of  $f$  and the  $x$ -axis is displayed as an animated hatch object. The current value of  $F(x)$  is displayed by an animated text:

```
DIGITS := 2:
// the function:
f := x -> cos(x^2):
// the anti-derivative:
F := x -> numeric::int(f(y), y = 0..x):
// the graph of f(x):
g := plot::Function2d(f(x), x = 0..6, Color = RGB::Blue):
// the graph of F(x):
G := plot::Function2d(F(x), x = 0..6, Color = RGB::Black):
// a point moving along the graph of F(x):
```

```

p := plot::Point2d([a, F(a)], a = 0..6, Color = RGB::Black):
// hatched region between the origin and the moving point p:
h := plot::Hatch(g, 0, 0 ..a, a = 0..6, Color = RGB::Red):
// the right border line of the hatched region:
l := plot::Line2d([a, 0], [a, f(a)], a = 0..6,
                  Color = RGB::Red):
// a dashed vertical line from f to F:
L1 := plot::Line2d([a, f(a)], [a, F(a)], a = 0..6,
                  Color = RGB::Black, LineStyle = Dashed):
// a dashed horizontal line from the y axis to F:
L2 := plot::Line2d([-0.1, F(a)], [a, F(a)], a = 0..6,
                  Color = RGB::Black, LineStyle = Dashed):
// the current value of F at the moving point p:
t := plot::Text2d(a -> F(a), [-0.2, F(a)], a = 0..6,
                  HorizontalAlignment = Right):
plot(g, G, p, h, l, L1, L2, t,
     YTicksNumber = None, YTicksAt = [-1, 1]):
delete DIGITS:

```



## Example 2

We build two 3D animations. The first starts with a rectangular strip that is deformed to an annulus in the  $x, y$  plane:

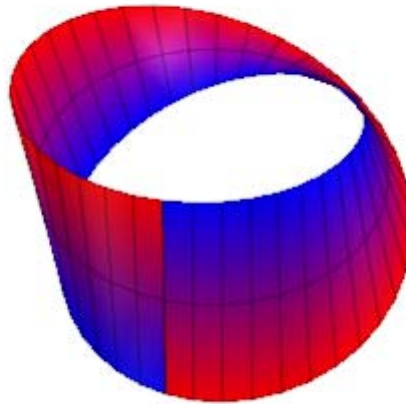
```
c := a -> 1/2 *(1 - 1/sin(PI/2*a)):
mycolor := (u, v, x, y, z) -> [(u - 0.8)/0.4, 0, (1.2 - u)/0.4]:
rectangle2annulus := plot::Surface(
    [c(a) + (u - c(a))*cos(PI*v), (u - c(a))*sin(PI*v), 0],
    u = 0.8..1.2, v = -a..a, a = 1/10^10..1,
    FillColorFunction = mycolor, Mesh = [3, 40], Frames = 40):
plot(rectangle2annulus, Axes = None,
    CameraDirection = [-11, -3, 3]):
```



The second animation twists the annulus to become a Moebius strip:

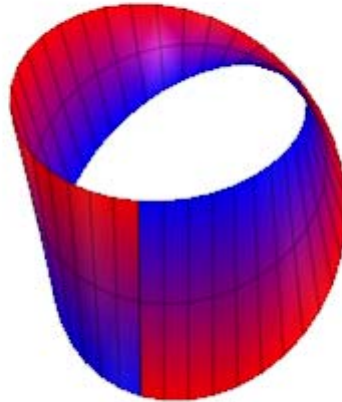
```
annulus2moebius := plot::Surface(
    [((u - 1)*cos(a*v*PI/2) + 1)*cos(PI*v),
     ((u - 1)*cos(a*v*PI/2) + 1)*sin(PI*v),
     (u - 1)*sin(a*v*PI/2)],
    u = 0.8..1.2, v = -1..1, a = 0..1,
```

```
FillColorFunction = mycolor, Mesh = [3, 40], Frames = 20):  
plot(annulus2moebius, Axes = None,  
      CameraDirection = [-11, -3, 3]):
```



Note that the final frame of the first animation coincides with the first frame of the second animation. To join the two separate animations, we can set appropriate visibility ranges and plot them together. After 5 seconds, the first animation object vanishes and the second takes over:

```
rectangle2annulus::VisibleFromTo := 0..5:  
annulus2moebius::VisibleFromTo := 5..7:  
plot(rectangle2annulus, annulus2moebius, Axes = None,  
      CameraDirection = [-11, -3, 3]):
```



### Example 3

In this example, we consider the planar celestial 3 body problem. We solve the system of differential equations

$$m_s x_s'' = - \frac{m_s m_1 (x_s - x_1)}{\sqrt{(x_s - x_1)^2 + (y_s - y_1)^2}^3} - \frac{m_s m_2 (x_s - x_2)}{\sqrt{(x_s - x_2)^2 + (y_s - y_2)^2}^3},$$

$$m_s y_s'' = - \frac{m_s m_1 (y_s - y_1)}{\sqrt{(x_s - x_1)^2 + (y_s - y_1)^2}^3} - \frac{m_s m_2 (y_s - y_2)}{\sqrt{(x_s - x_2)^2 + (y_s - y_2)^2}^3},$$

$$m_1 x_1'' = - \frac{m_1 m_s (x_1 - x_s)}{\sqrt{(x_1 - x_s)^2 + (y_1 - y_s)^2}^3} - \frac{m_1 m_2 (x_1 - x_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}^3},$$

$$m_1 y_1'' = - \frac{m_1 m_s (y_1 - y_s)}{\sqrt{(x_1 - x_s)^2 + (y_1 - y_s)^2}^3} - \frac{m_1 m_2 (y_1 - y_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}^3},$$

$$m_2 x_2'' = - \frac{m_2 m_s (x_2 - x_s)}{\sqrt{(x_2 - x_s)^2 + (y_2 - y_s)^2}^3} - \frac{m_2 m_1 (x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3},$$

$$m_2 y_2'' = - \frac{m_2 m_s (y_2 - y_s)}{\sqrt{(x_2 - x_s)^2 + (y_2 - y_s)^2}^3} - \frac{m_2 m_1 (y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3},$$

which is nothing but the equations of motions for two planets with masses  $m_1, m_2$  at positions  $(x_1, y_1), (x_2, y_2)$  revolving in the  $x, y$  plane around a sun of mass  $m_s$  positioned at  $(x_s, y_s)$ . We specify the mass ratios: The first planet is a giant with a mass  $m_1$  that is 4% of the sun's mass. The second planet is much smaller:

`ms := 1: m1 := 0.04: m2 := 0.0001:`

As we will see, the motion of the giant is nearly undisturbed by the small planet. The small one, however, is heavily disturbed by the giant and, finally, kicked out of the system after a near collision.

We solve the ODEs via the MuPAD numerical ODE solve `numeric::odesolve2` that provides a solution vector

$$Y(t) = [x_s(t), x_s'(t), y_s(t), y_s'(t), x_1(t), x_1'(t), y_1(t), y_1'(t), x_2(t), x_2'(t), y_2(t), y_2'(t)]$$

The initial conditions are chosen such that the total momentum vanishes, i.e., the total center of mass stays put (at the origin):

```
Y := numeric::odesolve2(numeric::ode2vectorfield(
  {xs''(t) =
    -m1*(xs(t)-x1(t))/sqrt((xs(t)-x1(t))^2 + (ys(t)-y1(t))^2)^3
    -m2*(xs(t)-x2(t))/sqrt((xs(t)-x2(t))^2 + (ys(t)-y2(t))^2)^3,
  ys''(t) =
```

```

        -m1*(ys(t)-y1(t))/sqrt((xs(t)-x1(t))^2 + (ys(t)-y1(t))^2)^3
        -m2*(ys(t)-y2(t))/sqrt((xs(t)-x2(t))^2 + (ys(t)-y2(t))^2)^3,
x1''(t) =
        -ms*(x1(t)-xs(t))/sqrt((x1(t)-xs(t))^2 + (y1(t)-ys(t))^2)^3
        -m2*(x1(t)-x2(t))/sqrt((x1(t)-x2(t))^2 + (y1(t)-y2(t))^2)^3,
y1''(t) =
        -ms*(y1(t)-ys(t))/sqrt((x1(t)-xs(t))^2 + (y1(t)-ys(t))^2)^3
        -m2*(y1(t)-y2(t))/sqrt((x1(t)-x2(t))^2 + (y1(t)-y2(t))^2)^3,
x2''(t) =
        -ms*(x2(t)-xs(t))/sqrt((x2(t)-xs(t))^2 + (y2(t)-ys(t))^2)^3
        -m1*(x2(t)-x1(t))/sqrt((x2(t)-x1(t))^2 + (y2(t)-y1(t))^2)^3,
y2''(t) =
        -ms*(y2(t)-ys(t))/sqrt((x2(t)-xs(t))^2 + (y2(t)-ys(t))^2)^3
        -m1*(y2(t)-y1(t))/sqrt((x2(t)-x1(t))^2 + (y2(t)-y1(t))^2)^3,
xs(0) = -m1,    x1(0) = ms,    x2(0) = 0,
ys(0) = 0.7*m2, y1(0) = 0,    y2(0) = -0.7*ms,
xs'(0) = -1.01*m2, x1'(0) = 0,    x2'(0) = 1.01*ms,
ys'(0) = -0.9*m1, y1'(0) = 0.9*ms, y2'(0) = 0},
[xs(t), xs'(t), ys(t), ys'(t),
 x1(t), x1'(t), y1(t), y1'(t),
 x2(t), x2'(t), y2(t), y2'(t)]
)):

```

The positions  $[x_s(t), y_s(t)] = [Y(t)[1], Y(t)[3]]$ ,  $[x_1(t), y_1(t)] = [Y(t)[5], Y(t)[7]]$ ,  $[x_2(t), y_2(t)] = [Y(t)[9], Y(t)[11]]$  are computed on an equidistant time mesh with  $dt = 0.05$ . The animation is built up “frame by frame” by defining static points with suitable values of `VisibleFromTo` and static line segments with suitable values of `VisibleAfter`.

Setting `VisibleFromTo = t..t + 0.99*dt`, each solution point is visible only for a short time (the factor `0.99` makes sure that not two points can be visible simultaneously on each orbit). The orbits of the points are realized as line segments from the positions at time  $t - dt$  to the positions at time  $t$ . The line segments become visible at time  $t$  and stay visible for the rest of the animation (`VisibleAfter = t`), thus leaving a “trail” of the moving points. We obtain the following graphical solution (the computation takes about two minutes on a 1 GHz computer):

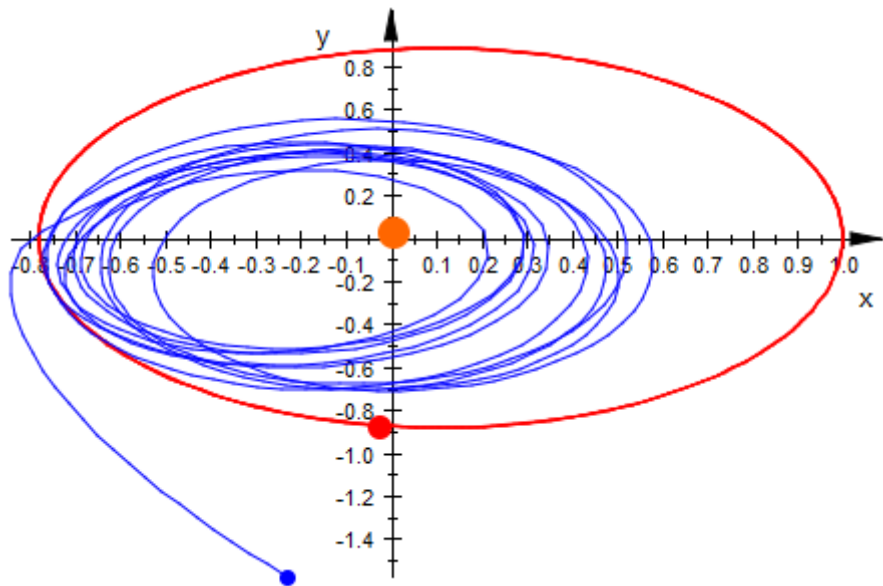
```

dt := 0.05: imax := 516:
plot(// The sun:

```

```
plot::Point2d(Y(t)[1], Y(t)[3], Color = RGB::Orange,
              VisibleFromTo = t..t + 0.99*dt,
              PointSize = 4*unit::mm
              ) $ t in [i*dt $ i = 0..imax],
// The giant planet:
plot::Point2d(Y(t)[5], Y(t)[7], Color = RGB::Red,
              VisibleFromTo = t..t + 0.99*dt,
              PointSize = 3*unit::mm
              ) $ t in [i*dt $ i = 0..imax],
// The orbit of the giant planet:
plot::Line2d([Y(t - dt)[5], Y(t - dt)[7]],
             [Y(t)[5], Y(t)[7]], Color = RGB::Red,
             VisibleAfter = t
             ) $ t in [i*dt $ i = 1..imax],
// The small planet:
plot::Point2d(Y(t)[9], Y(t)[11], Color = RGB::Blue,
              VisibleFromTo = t..t + 0.99*dt,
              PointSize = 2*unit::mm
              ) $ t in [i*dt $ i = 0..imax],
// The orbit of the small planet:
plot::Line2d([Y(t - dt)[9], Y(t - dt)[11]],
             [Y(t)[9], Y(t)[11]], Color = RGB::Blue,
             VisibleAfter = t
             ) $ t in [i*dt $ i = 1..imax]
):
```



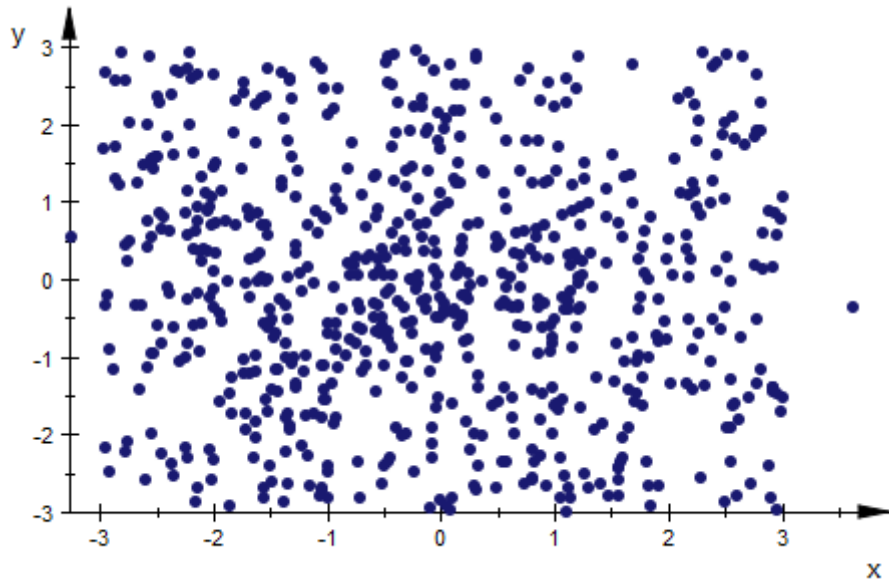


## Groups of Primitives

An arbitrary number of graphical primitives in 2D or 3D can be collected in groups of type `plot::Group2d` or `plot::Group3d`, respectively. This is useful for inheriting attribute values to all elements in a group.

In the following example, we visualize random generators with different distributions by using them to position random points:

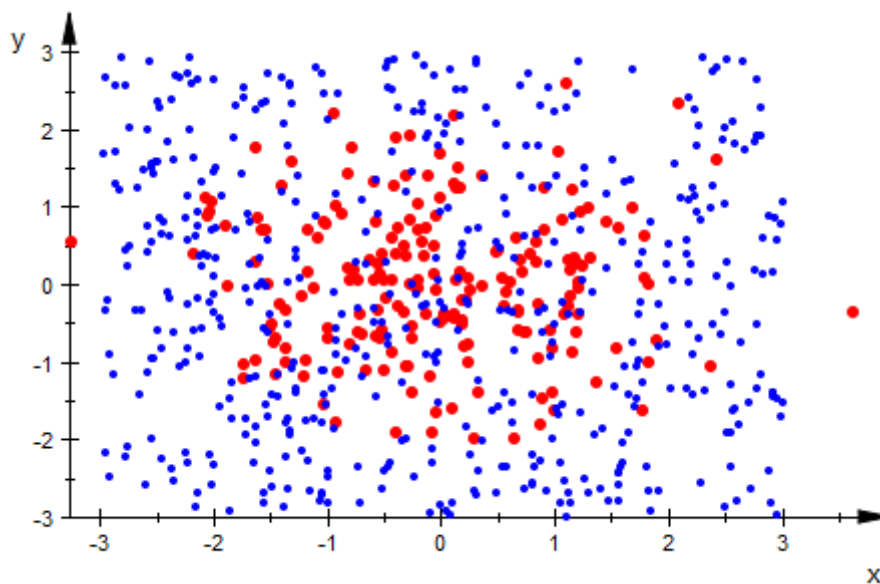
```
r1 := stats::normalRandom(0, 1):
group1 := plot::Group2d(plot::Point2d(r1(), r1()) $ i = 1..200):
r2 := stats::uniformRandom(-3, 3):
group2 := plot::Group2d(plot::Point2d(r2(), r2()) $ i = 1..500):
plot(group1, group2, Axes = Frame):
```



We cannot distinguish between the two kinds of points. Due to the grouping, it is very easy to change their color and size by setting the appropriate attributes in the groups. Now, the two kinds of points can be distinguished easily:

```
group1::PointColor := RGB::Red:
```

```
group1::PointSize := 1.5*unit::mm:  
group2::PointColor := RGB::Blue:  
group2::PointSize := 1.3*unit::mm:  
plot(group1, group2, Axes = Frame):
```



## Transformations

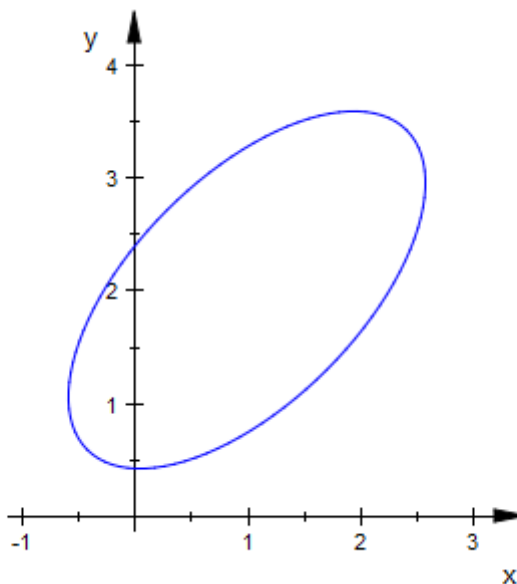
Affine linear transformations  $x \rightarrow Ax + b$  with a vector  $b$  and a matrix  $A$  can be applied to graphical objects via transformation objects. There are special transformations such as translations, scaling, and rotations as well as general affine linear transformations:

- `plot::Translate2d([b1, b2], Primitive1, Primitive2, ...)` applies the translation  $x \rightarrow x + b$  by the vector  $b = [b1, b2]$  to all points of 2D primitives.
- `plot::Translate3d([b1, b2, b3], Primitive1, ...)` applies the translation  $x \rightarrow x + b$  by the vector  $b = [b1, b2, b3]$  to all points of 3D primitives.
- `plot::Reflect2d([x1, y1], [x2, y2], Primitive1, ...)` reflects all 2D primitives about the line through the points  $[x1, y1]$  and  $[x2, y2]$ .
- `plot::Reflect3d([x, y, z], [nx, ny, nz], Primitive1, ...)` reflects all 3D primitives about the plane through the point  $[x, y, z]$  with the normal  $[nx, ny, nz]$ .
- `plot::Rotate2d(angle, [c1, c2], Primitive1, ...)` rotates all points of 2D primitives counter clockwise by the given angle about the pivot point  $[c1, c2]$ .
- `plot::Rotate3d(angle, [c1, c2, c3], [d1, d2, d3], Primitive1, ...)` rotates all points of 3D primitives by the given angle around the rotation axis specified by the pivot point  $[c1, c2, c3]$  and the direction  $[d1, d2, d3]$ .
- `plot::Scale2d([s1, s2], Primitive1, ...)` applies the diagonal scaling matrix `diag(s1, s2)` to all points of 2D primitives.
- `plot::Scale3d([s1, s2, s3], Primitive1, ...)` applies the diagonal scaling matrix `diag(s1, s2, s3)` to all points of 3D primitives.
- `plot::Transform2d([b1, b2], A, Primitive1, ...)` applies the general affine linear transformation  $x \rightarrow Ax + b$  with a  $2 \times 2$  matrix  $A$  and a vector  $b = [b1, b2]$  to all points of 2D primitives.

- `plot::Transform3d([b1, b2, b3], A, Primitive1, ...)` applies the general affine linear transformation  $x \rightarrow Ax + b$  with a  $3 \times 3$  matrix  $A$  and a vector  $b = [b1, b2, b3]$  to all points of 3D primitives.

The ellipses `plot::Ellipse2d` provided by the `plot` library have axes parallel to the coordinate axes. We use a rotation to create an ellipse with a different orientation:

```
center := [1, 2]:
ellipse := plot::Ellipse2d(2, 1, center):
plot(plot::Rotate2d(PI/4, center, ellipse))
```



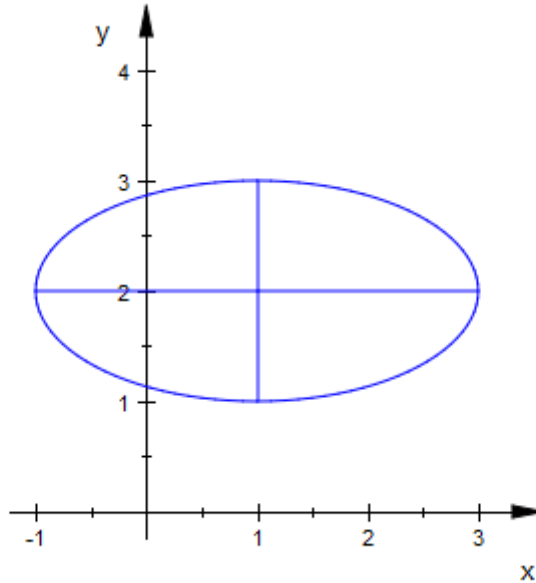
Transform objects can be animated. We build a group consisting of the ellipse and its symmetry axes. An animated rotation is applied to the group:

```
g := plot::Group2d(
    ellipse,
    plot::Line2d(center, [center[1] + 2, center[2]]),
    plot::Line2d(center, [center[1] - 2, center[2]]),
    plot::Line2d(center, [center[1], center[2] + 1]),
```

```

        plot::Line2d(center, [center[1], center[2] - 1])
    ):
    plot(plot::Rotate2d(a, center, a = 0..2*PI, g)):

```



Objects inside an animated transformation can be animated, too. The animations run independently and may be synchronized via suitable values of the `TimeRange` as described in section “Advanced Animations: The Synchronization Model” on page 5-151.

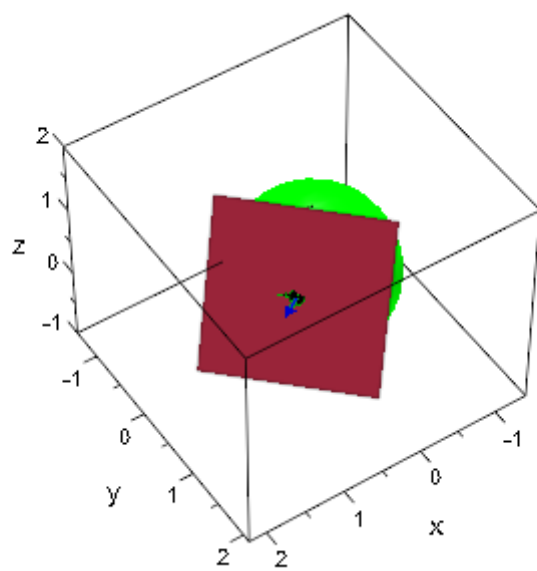
We generate a sphere  $s$  of radius  $r$  with center  $c = (c_x, c_y, c_z)$ . We wish to visualize the tangent plane at various points of the surface. We start with the tangent plane of the north pole and rotate it around the  $y$  axis (i.e., along the line with zero longitude) by the polar angle  $\theta$  for the first 3 seconds. Then it is rotated around the  $z$ -axis (i.e., along the line with constant latitude) by the azimuth angle  $\varphi$ . We end up with the tangent plane at the point  $x = c_x + \cos(\varphi)\sin(\theta)$ ,  $y = c_y + \sin(\varphi)\sin(\theta)$ ,  $z = c_z + \cos(\theta)$ . The two rotations are realized as a nested animation: By specifying disjoint time ranges, the second rotation (around the  $z$ -axis) starts when the first rotation (around the  $y$ -axis) is finished:

```

r := 1:          // the radius of the sphere
R := 1.01:      // increase the radius a little bit
c := [0, 0, 0]: // the center of the sphere
thet := PI/3:   // spherical coordinates of
phi := PI/4:    // the final point p
// the final point:
p := plot::Point3d(c[1] + R*cos(phi)*sin(thet),
                  c[2] + R*sin(phi)*sin(thet),
                  c[3] + R*cos(thet),
                  PointSize = 2*unit::mm,
                  Color = RGB::Black):

// the sphere:
s := plot::Sphere(r, c, Color = RGB::Green):
// the meridian at thet = 0
c1 := plot::Curve3d([c[1] + R*sin(t), c[2], c[3] + R*cos(t)],
                   t = 0..thet, Color = RGB::Black):
// the meridian at thet = 0
c2 := plot::Curve3d([c[1] + R*cos(t)*sin(thet),
                   c[2] + R*sin(t)*sin(thet),
                   c[3] + R*cos(thet)],
                   t = 0..phi, Color = RGB::Black):
// form a group consisting of the tangent plane and its normal:
g := plot::Group3d(
  plot::Surface([c[1] + u, c[2] + v, c[3] + R],
               u = -1..1, v = -1..1,
               Mesh = [2, 2], Color = RGB::Red.[0.3]),
  plot::Arrow3d([c[1], c[2], c[3] + R],
               [c[1], c[2], c[3] + R + 0.7])
):
// rotate the group for 3 seconds along the meridian:
g := plot::Rotate3d(a, c, [0, 1, 0], a = 0..thet,
                  g, TimeRange = 0..3):
// rotate the group for further 3 seconds along the azimuth:
g := plot::Rotate3d(a, c, [0, 0, 1], a = 0..phi,
                  g, TimeRange = 3..6):
plot(s, g, c1, c2, p, CameraDirection = [2, 3, 4]):

```

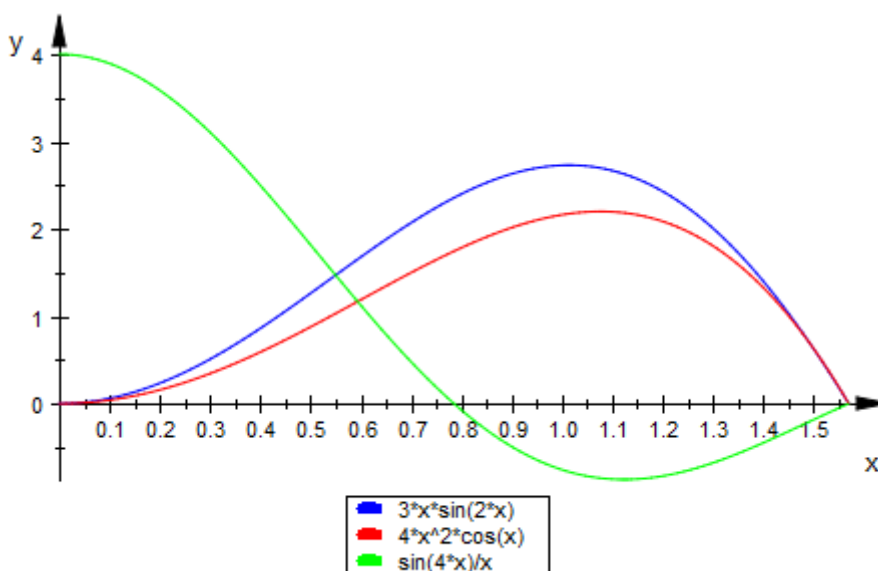




## Legends

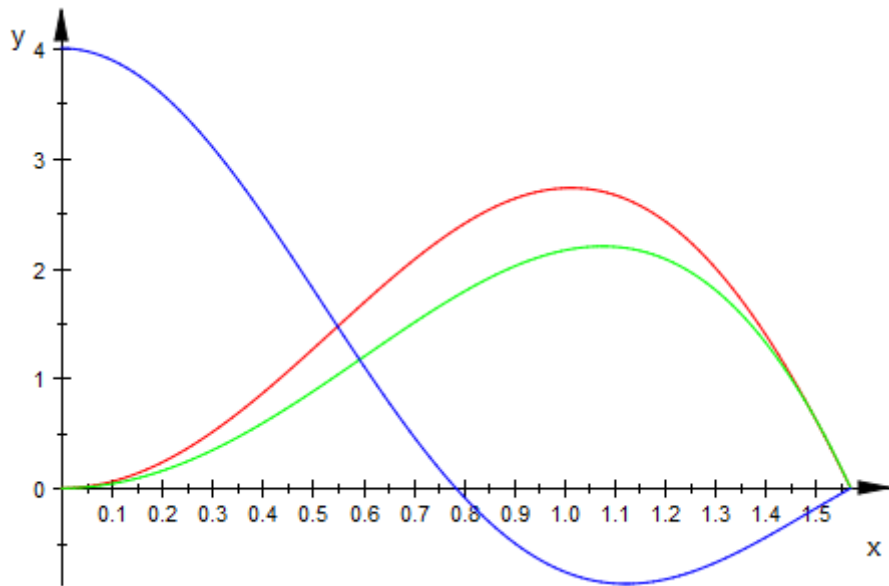
The annotations of a MuPAD plot may include a legend. A legend is a small table that relates the color of an object with some text explaining the object:

```
f := 3*x*sin(2*x):
g := 4*x^2*cos(x):
h := sin(4*x)/x:
plotfunc2d(f, g, h, x = 0..PI/2):
```



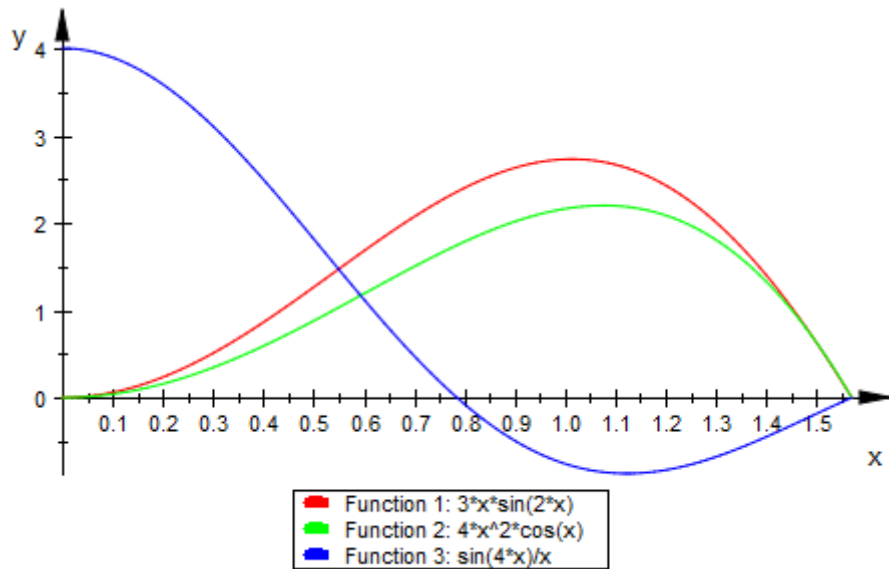
By default, legends are provided only by `plotfunc2d` and `plotfunc3d`. These routines define the legend texts as the expressions with which the functions are passed to `plotfunc2d` or `plotfunc3d`, respectively. A corresponding plot command using primitives of the plot library does not generate the legend automatically:

```
plot(plot::Function2d(f, x = 0..PI/2, Color = RGB::Red),
      plot::Function2d(g, x = 0..PI/2, Color = RGB::Green),
      plot::Function2d(h, x = 0..PI/2, Color = RGB::Blue)):
```



However, legends can be requested explicitly:

```
plot(plot::Function2d(f, x = 0..PI/2, Color = RGB::Red,  
      Legend = "Function 1: ".expr2text(f)),  
      plot::Function2d(g, x = 0..PI/2, Color = RGB::Green,  
      Legend = "Function 2: ".expr2text(g)),  
      plot::Function2d(h, x = 0..PI/2, Color = RGB::Blue,  
      Legend = "Function 3: ".expr2text(h))  
):
```



Each graphical primitive accepts the attribute `Legend`. Passing this attribute to an object triggers several actions:

- The object attribute `LegendText` is set to the given string.
- The object attribute `LegendEntry` is set to `TRUE`.
- A hint is sent to the scene containing the object advising it to use the scene attribute `LegendVisible = TRUE`.

The attributes `LegendText` and `LegendEntry` are visible in the “object inspector” of the interactive viewer (see section `Viewer, Browser, and Inspector: Interactive Manipulation`) and can be manipulated interactively for each single primitive after selection in the “object browser.” The attribute `LegendVisible` is associated with the scene object accessible via the “object browser.”

At most 20 entries can be displayed in a legend. If more entries are specified in a plot command, the surplus entries are ignored. Further, the legend may not cover more than 50% of the height of the drawing area of a scene. Only

those legend entries fitting into this space are displayed; remaining entries are ignored.

If the attribute `LegendText = TRUE` is set for a primitive, its legend entry is determined as follows:

- If the attribute `LegendText` is specified, its value is used for the legend text.
- If no `LegendText` is specified, but the `Name` attribute is set, the name is used for the legend text.
- If no `Name` attribute is specified either, the type of the object such as `Function2d`, `Curve2d` etc. is used for the legend text.

Here are all attributes relevant for legends:

attribute name	possible values/examples	default	browser entry
Legend	string	sets <code>LegendText</code> to the given string, <code>LegendEntry</code> to TRUE, and <code>LegendVisible</code> to TRUE.	
LegendEntry	TRUE, FALSE	add this object to the legend?	primitive
LegendText	string	legend text	primitive
LegendVisible	TRUE, FALSE	legend on/off	Scene2d/3d
LegendPlacement	Top, Bottom	vertical placement	Scene2d/3d

**(Continued)**

<b>attribute name</b>	<b>possible values/examples</b>	<b>default</b>	<b>browser entry</b>	
LegendAlignment	Left, Center, Right	horizontal alignment	Center	Scene2d/3d
LegendFont	see section Fonts	font for the legend text	sans-serif 8	Scene2d/3d

## Fonts

The plot attributes allow to specify the fonts `AxesTitleFont`, `FooterFont`, `HeaderFont`, `LegendFont`, `TextFont`, `TicksLabelFont`, and `TitleFont`. Each such font is specified by a MuPAD list of the following form:

```
[<family>, <size>, <Bold>, <Italic>, <color>, <alignment>].
```

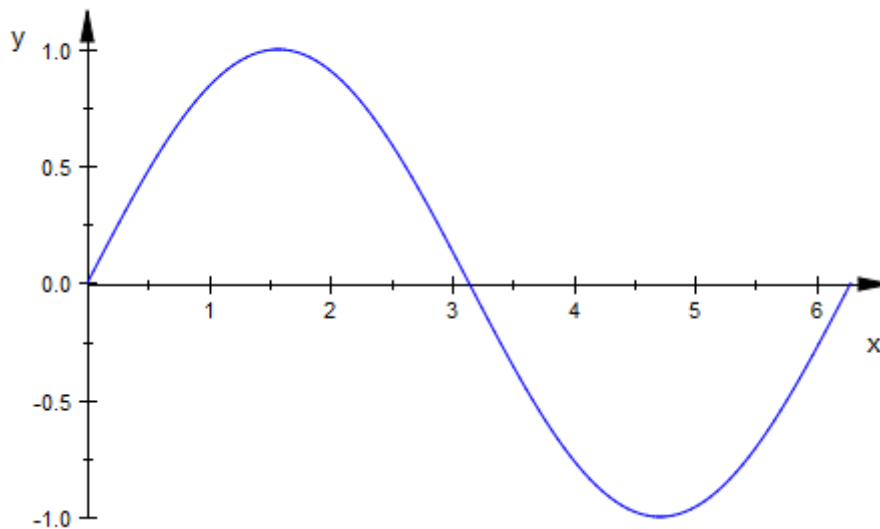
The parameters are:

<code>family</code>	–	the font family name: a string. The available font families depend on the fonts that are installed on your machine. For example, typical font families available on Microsoft Windows systems are "Times New Roman" (of type 'serif'), "Arial" (of type 'sans-serif'), or "Courier New" (of type 'monospace'). To find out which fonts are available on your machine, open the menu 'Format,' submenu 'Font' in your MuPAD notebook. The first column in the font dialog provides the names of the font families that you may specify. You may also specify one the three generic family names "serif", "sans-serif", or "monospace", and the system will automatically choose one of the available font families of the specified type for you.
<code>size</code>	–	the size of the font in integral points: a positive integer.
<code>Bold</code>	–	if specified, the font is bold
<code>Italic</code>	–	if specified, the font is italic.
<code>color</code>	–	an RGB color value: a list of 3 numerical values between 0 and 1
<code>alignment</code>		text alignment in case of new-lines: one of the flags <code>Left</code> , <code>Center</code> , or <code>Right</code> .

In the following example, we specify the font for the canvas header:

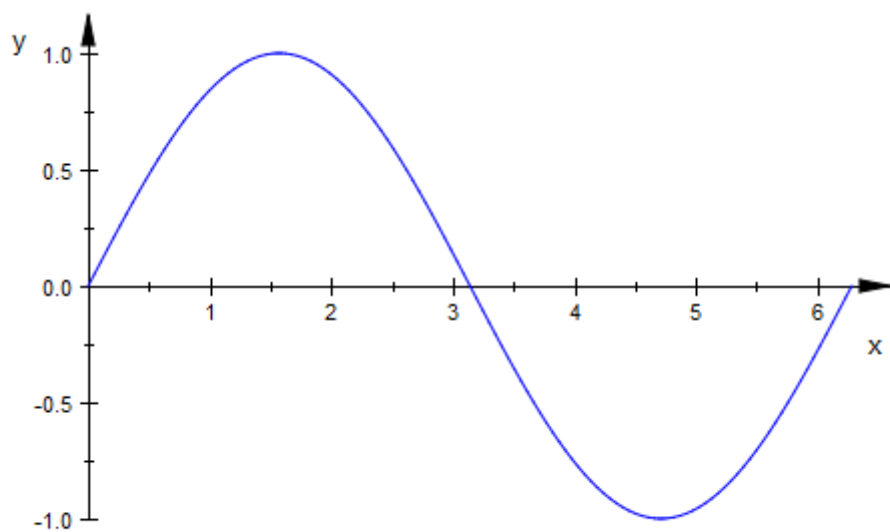
```
plot(plot::Function2d(sin(x), x = 0..2*PI),  
      Header = "The sine function",  
      HeaderFont = ["monospace", 14, Bold]):
```

### The sine function



All font parameters are optional; some default values are chosen for entries that are not specified. For example, if you do not care about the footer font family for your plot, but you insist on a specific font size, you may specify an 18 pt font as follows:

```
plot(plot::Function2d(sin(x), x = 0..2*PI),  
      Footer = "The sine function", FooterFont = [18]):
```



The sine function



# Colors

In this section...
“RGB Colors” on page 5-185
“HSV Colors” on page 5-189

The most prominent plot attribute, available for all primitives, is `Color`. The MuPAD plot library knows 3 different types of colors:

- The attribute `PointColor` refers to the color of points in 2D and 3D (of type `plot::Point2d` and `plot::Point3d`, respectively).
- The attribute `LineColor` refers to the color of line objects in 2D and 3D. This includes the color of function graphs in 2D, curves in 2D and in 3D, polygon lines in 2D and in 3D, etc. Also 3D objects such as function graphs in 3D, parametrized surfaces etc. react to the attribute `LineColor`; it defines the color of the coordinate mesh lines that are displayed on the surface.
- The attribute `FillColor` refers to the color of closed and filled polygons in 2D and 3D as well as hatched regions in 2D. Further, it sets the surface color of function graphs in 3D, parametrized surfaces etc. This includes spheres, cones etc.

The primitives also accept the attribute `Color` as a shortcut for any one of these colors. Depending on the primitive, either `PointColor`, `LineColor`, or `FillColor` is set with the `Color` attribute.

## RGB Colors

MuPAD uses the RGB color model, i.e., colors are specified by lists `[r, g, b]` of red, green, and blue values between 0 and 1. Black and white correspond to `[0, 0, 0]` and `[1, 1, 1]`, respectively. The library `RGB` contains numerous color names with corresponding RGB values:

`RGB::Black`, `RGB::White`, `RGB::Red`, `RGB::SkyBlue`

`[0.0, 0.0, 0.0]`, `[1.0, 1.0, 1.0]`, `[1.0, 0.0, 0.0]`, `[0.0, 0.8, 1.0]`

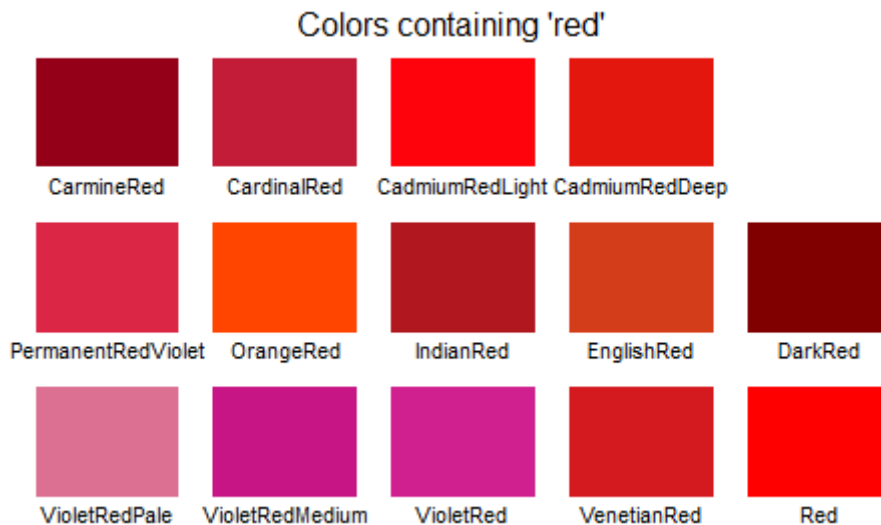
You may list all color names available in the RGB library via `info(RGB)`. Alternatively, there is the command `RGB::ColorNames()` returning a complete list of names, optionally filtered. For example, let us list all the colors whose names contain “Red”:

```
RGB::ColorNames(Red)
```

```
[CadmiumRedDeep, CadmiumRedLight, CardinalRed, CarmineRed, DarkRed, EnglishRed,  
IndianRed, OrangeRed, PermanentRedViolet, Red, VenetianRed, VioletRed, VioletRedMedium,  
VioletRedPale]
```

To get them displayed, use

```
RGB::plotColorPalette("red")
```



After loading the color library via `use(RGB)`, you can use the color names in the short form `Black`, `White`, `IndianRed` etc.

In MuPAD, the color of all graphic elements can either be specified by RGB or RGBa values.

RGBA color values consist of lists  $[r, g, b, a]$  containing a fourth entry: the “opacity” value  $a$  between 0 and 1. For  $a = 0$ , a surface patch painted with this RGBA color is fully transparent (i.e., invisible). For  $a = 1$ , the surface patch is opaque, i.e., it hides plot objects that are behind it. For  $0 < a < 1$ , the surface patch is semitransparent, i.e., plot objects behind it “shine through.”

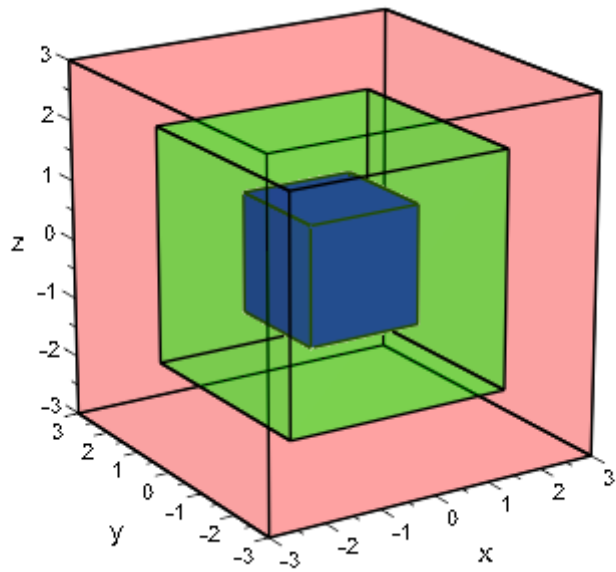
RGBA color values can be constructed easily via the RGB library. One only has to append a fourth entry to the  $[r, g, b]$  lists provided by the color names. The easiest way to do this is to append the list  $[a]$  to the RGB list via the concatenation operator `'.'`. We create a semitransparent ‘grey’:

```
RGB::Grey.[0.5]
```

```
[0.752907, 0.752907, 0.752907, 0.5]
```

The following command plots a highly transparent red box, containing a somewhat less transparent green box with an opaque blue box inside:

```
plot(  
  plot::Box(-3..3, -3..3, -3..3, FillColor = RGB::Red.[0.2]),  
  plot::Box(-2..2, -2..2, -2..2, FillColor = RGB::Green.[0.3]),  
  plot::Box(-1..1, -1..1, -1..1, FillColor = RGB::Blue),  
  LinesVisible = TRUE, LineColor = RGB::Black,  
  Scaling = Constrained  
):
```



In the following example, we plot points randomly distributed with random colors and random translucencies:

```
plot(plot::PointList2d([[frandom() $ i = 1..2,  
                        [frandom() $ i = 1..4]]  
                        $ i = 1..300],  
      PointSize=4),  
      Axes=None, Scaling=Constrained)
```



## HSV Colors

Apart from the RGB model, there are various other popular color formats used in computer graphics. One is the HSV model (Hue, Saturation, Value). The RGB library provides the routines `RGB::fromHSV` and `RGB::toHSV` to convert HSV colors to RGB colors and vice versa:

```
hsv := RGB::toHSV(RGB::Orange)
```

```
[24.0, 1.0, 1.0]
```

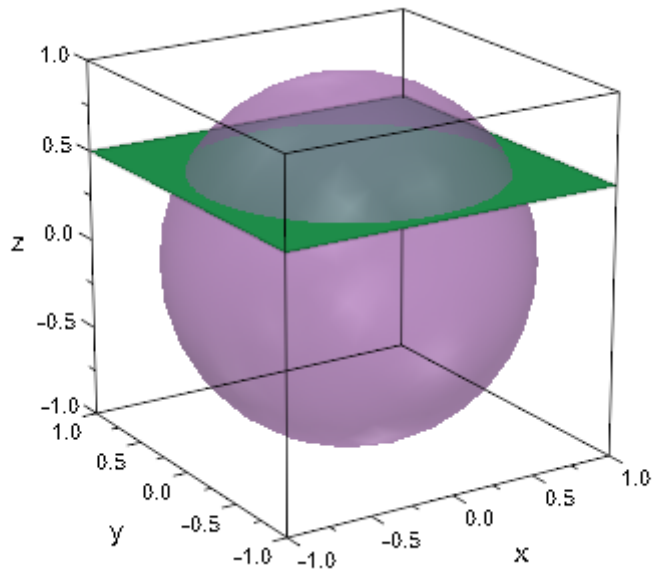
```
RGB::fromHSV(hsv) = RGB::Orange
```

```
[1.0, 0.4, 0.0] = [1.0, 0.4, 0.0]
```

With the `RGB::fromHSV` utility, all colors in a MuPAD plot can be specified easily as HSV colors. For example, the color 'violet' is given by the HSV values `[290, 0.4, 0.6]`, whereas 'dark green' is given by the HSV specification

[120, 1, 0.4]. Hence, a semitransparent violet sphere intersected by an opaque dark green plane may be specified as follows:

```
plot(plot::Sphere(1, [0, 0, 0],  
              Color = RGB::fromHSV([290, 0.4, 0.6]).[0.5]),  
      plot::Surface([x, y, 0.5], x = -1..1, y = -1..1,  
                    Mesh = [2, 2],  
                    Color = RGB::fromHSV([120, 1, 0.4]))  
):
```



## Save and Export Pictures

### In this section...

“Save and Export Interactively” on page 5-191

“Save in Batch Mode” on page 5-191

### Save and Export Interactively

The MuPAD kernel uses an xml format to communicate with the renderer. Usually, a `plot` command in MuPAD sends a stream of xml data directly to the viewer which renders the picture.

After double clicking on the picture, the viewer (see section Viewer, Browser, and Inspector: Interactive Manipulation) provides a menu item ‘Edit/Export...’ that opens a dialog allowing to save the picture in a variety of graphical formats:

- The ‘VCam Graphics’ format, indicated by the file extension ‘.xvz’. This is a compressed version of the xml ascii data used by MuPAD.
- The ‘Uncompressed VCam Graphics’ format, indicated by the file extension ‘.xvc,’ saves these xml data in an uncompressed ascii file (the resulting file can be read with any text editor).

One can use the MuPAD graphics tool ‘VCam’ to open such files and display the xml data.

- Further, there are various standard bitmap formats such as bmp, jpg, eps etc. in which the image may be stored.

### Save in Batch Mode

MuPAD plots can also be saved in “batch mode” by specifying the attribute `OutputFile = filename` in a plot call:

```
plot(primitives, OutputFile = "mypicture.xvz");
```

Here, the extension `.xvz` of the file name `"mypicture.xvz"` indicates that the MuPAD xml data are to be written and, finally, the file is to be compressed.

Alternatively, the extension `.xvc` may be used to write the xml data without final compression of the file (the resulting text file can be read with any text editor). Files in both formats can be opened by the MuPAD graphics tool 'VCam' to generate the plot encoded by the xml data.

If the MuPAD environment variable `WRITEPATH` does not have a value, the previous call creates the file in the directory where MuPAD is installed. An absolute pathname can be specified to place the file anywhere else:

```
plot(primitives, OutputFile = "C:\\Documents\\mypicture.xvz"):
```

Alternatively, the environment variable `WRITEPATH` can be set:

```
WRITEPATH := "C:\\Documents":  
plot(primitives, OutputFile = "mypicture.xvz"):
```

Now, the plot data are stored in the file `'C:\\Documents\\mypicture.xvz'`.

If a MuPAD notebook is saved to a file, its location is available inside the notebook as the environment variable `NOTEBOOKPATH`. If you wish to save your plot in the same folder as the notebook, you may call

```
plot(primitives, OutputFile = NOTEBOOKPATH."mypicture.xvz"):
```

Apart from saving files as xml data, MuPAD pictures can also be saved in a variety of standard graphical formats such as jpg, eps, svg, bmp etc. In batch mode, the export is triggered by the `OutputFile` attribute in the same way as for saving in xml format. Just use an appropriate extension of the filename indicating the format. The following commands save the plot in four different files in jpg, eps, svg, and bmp format, respectively:

```
plot(primitives, OutputFile = "mypicture.jpg"):  
plot(primitives, OutputFile = "mypicture.eps"):  
plot(primitives, OutputFile = "mypicture.svg"):  
plot(primitives, OutputFile = "mypicture.bmp"):
```

On Windows systems, an animated MuPAD plot can be exported to avi format:

```
plot(plot::Function2d(sin(x - a), x = 0..2*PI, a = 0..5)  
      OutputFile = "myanimation.avi"):
```



If no file extension is specified by the file name, the default extension `.xvc` is used, i.e., compressed xml data are written.

In addition to `OutputFile`, there is the attribute `OutputOptions` to specify parameters for some of the export formats. The admissible value for this attribute is a list of equations

```
OutputOptions = [<ReduceTo256Colors = b >, <DotsPerInch = n1>,
<Quality = n2>, <JPEGMode = n3>, <EPSMode = n4>, <AVIMode = n5>,
<WMFMode = n6>, <FramesPerSecond = n7>, <PlotAt = l1>]
```

Each entry of the list is optional. The parameters are:

- b TRUE or FALSE. Has an effect for export to some raster formats only. With TRUE, only 256 different colors are stored in the raster file. The default value is FALSE.
- n<sub>1</sub> Positive integer setting the resolution in dpi (dots per inch). Has an effect for export to raster formats only. The default value depends on the hardware.
- n<sub>2</sub> One of the integers 1, 2, ..., 100. This integer represents a percentage value determining the quality of the export. Has an effect for jpg, 3D eps, 3D wmf, and avi export only. The default value is 75.
- n<sub>3</sub> 0, 1, or 2. Has an effect for jpg export only. The flag 0 represents the jpg mode 'Baseline Sequential,' 1 represents 'Progressive,' 2 represents 'Sequential Optimized.' The default value is 0.
- n<sub>4</sub> 0 or 1. Has an effect for eps export only. The flag 0 represents the eps mode 'Painter's Algorithm,' 1 represents 'BSP Tree Algorithm.' The default value is 0.
- n<sub>5</sub> 0, 1 or 2. Has an effect for avi export only. With 0, the 'Microsoft Video 1 Codec' is used. With 1, the 'Uncompressed Single Frame Codec' is used. With 2, the 'Radius Cinepak Codec' is used. The default value is 0.
- n<sub>6</sub> 0, 1 or 2. Has an effect for wmf export only. With 0, the 'Painter's Algorithm' is used. With 1, the 'BSP Tree Algorithm' is used. With 2, a 'embedded bitmap' is created. The default value is 0.

- $n_7$  Positive integer setting the frames per second for the avi to be generated. Has an effect for avi export only. The default value is 15.
- $l_1$  List of real values between `TimeBegin` and `TimeEnd` which determines the times at which pictures should be saved from an animation.

## Import Pictures

MuPAD does not provide for many tools to import standard graphical *vector* formats, yet. Presently, the only supported vector type is the stl format, popular in Stereo Lithography, which encodes 3D surfaces. It can be imported via the routine `plot::SurfaceSTL`.

In contrast to graphical *vector* formats, there are many standard *bitmap* formats such as bmp, gif, jpg, ppm etc. that can be imported. One can read such a file via `import::readbitmap`, thus creating a MuPAD array of RGB color values that can be manipulated at will. In particular, it can be fed into the routine `plot::Raster` which creates an object that can be used in any 2D MuPAD plot. Note, however, that the import of bitmap data consumes a lot of memory, i.e., only reasonably small bitmaps (up to a few hundred pixels in each direction) should be processed.

In the following example, we plot the probability density function and the cumulative density function of the standard normal (“Gaussian”) distribution. Paying tribute to Carl Friedrich Gauss, we wish to display his picture in this plot. Assume that we have his picture as a ppm bitmap file “Gauss.ppm.” We import the file via `import::readbitmap` that provides us with the width and height in pixels and the color data:

```
[width, height, gauss] := import::readbitmap("Gauss.ppm");
```

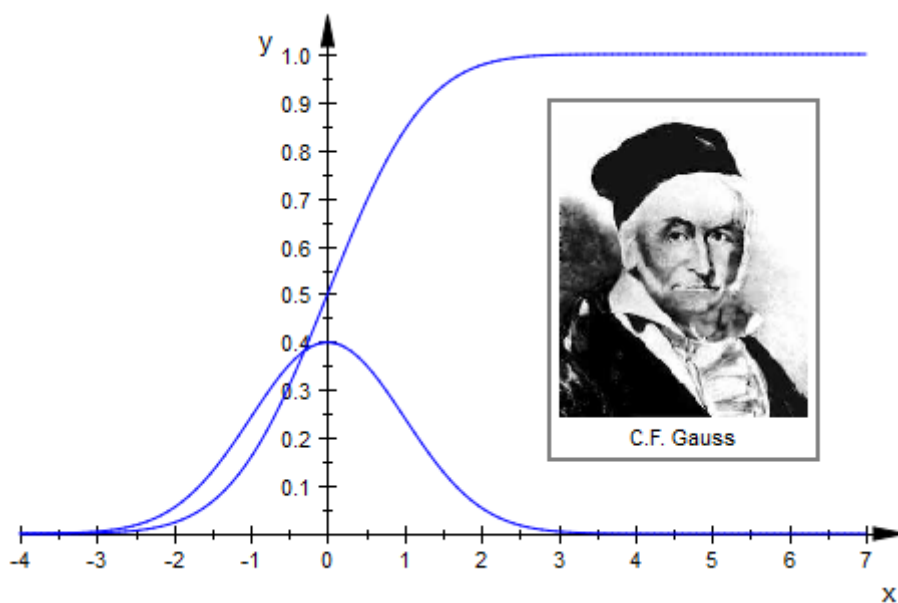
We have to use `Scaling = Constrained` to preserve the aspect ratio of the image. Unfortunately, this setting is not appropriate for the function plots. So we use two different scenes that are positioned via `Layout = Relative` in one canvas (see section `Layout of Canvas and Scenes`).

The first scene plots the two functions using the default setting `Scaling = Unconstrained` for function plots. With the sizes `Width = 1`, `Height = 1`, representing fractions of the canvas size, this scene fills the whole canvas.

The second scene is given a width and height of approximately the desired magnitude. It uses `Scaling = Constrained` to get the aspect ratio right, automatically. With the attributes `Bottom` and `Left`, the lower left corner of Gauss’ image is moved to an appropriate point of the canvas:

```
pdf := stats::normalPDF(0, 1):
```

```
cdf := stats::normalCDF(0, 1):  
plot(plot::Scene2d(plot::Function2d(pdf(x), x = -4..7),  
  plot::Function2d(cdf(x), x = -4..7),  
  Width = 1, Height = 1),  
  plot::Scene2d(plot::Raster(gauss),  
    Scaling = Constrained,  
    Width = 0.3, Height = 0.6,  
    Bottom = 0.25, Left = 0.6,  
    BorderWidth = 0.5*unit::mm,  
    Footer = "C.F. Gauss",  
    FooterFont = [8]),  
  Layout = Relative  
):
```



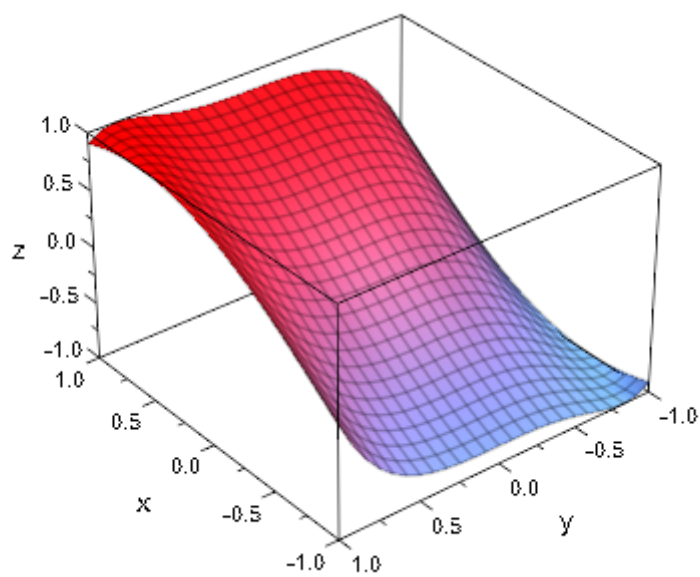
## Cameras in 3D

The MuPAD 3D graphics model includes an observer at a specific position, pointing a camera with a lens of a specific opening angle to some specific focal point. The specific parameters “position”, “angle”, and “focal point” determine the picture that the camera will take.

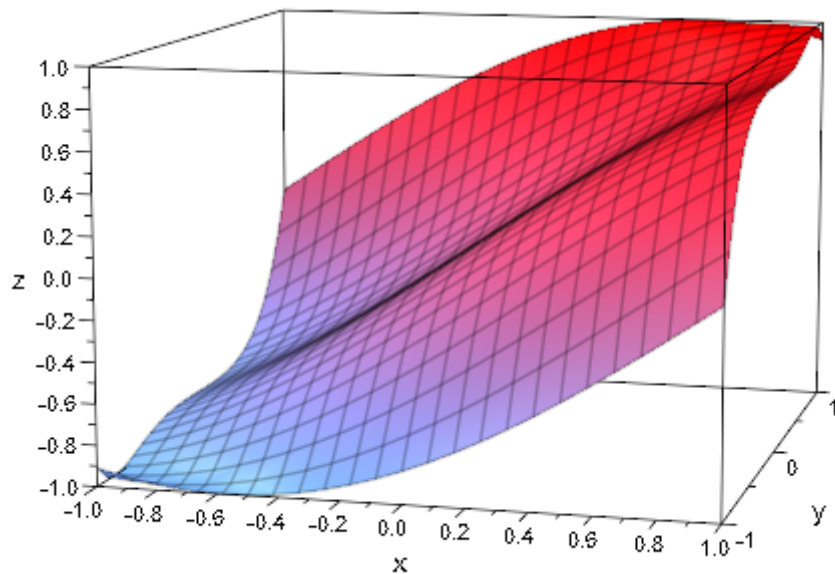
When a 3D picture is created in MuPAD, a camera with an appropriate default lens is positioned automatically. Its focal point is chosen as the center of the graphical scene. The interactive viewer allows to rotate the scene which, in fact, is implemented internally as a change of the camera position. Also interactive zooming in and zooming out is realized by moving the camera closer to or farther away from the scene.

Apart from interactive camera motions, the perspective of a 3D picture can also be set in the calls generating the plot. One way is to specify the direction from which the camera is pointing towards the scene. This is done via the attribute `CameraDirection`:

```
plot(plot::Function3d(sin(x + y^3), x = -1..1, y = -1..1),  
      CameraDirection = [-25, 20, 30]):
```



```
plot(plot::Function3d(sin(x + y^3), x = -1..1, y = -1..1),  
      CameraDirection = [10, -40, 10]):
```



In these calls, the position of the camera is not fully specified by `CameraDirection`. This attribute just requests the camera to be placed at some large distance from the scene along the ray in the direction given by the attribute. The actual distance from the scene is determined automatically to let the graphical scene fill the picture optimally.

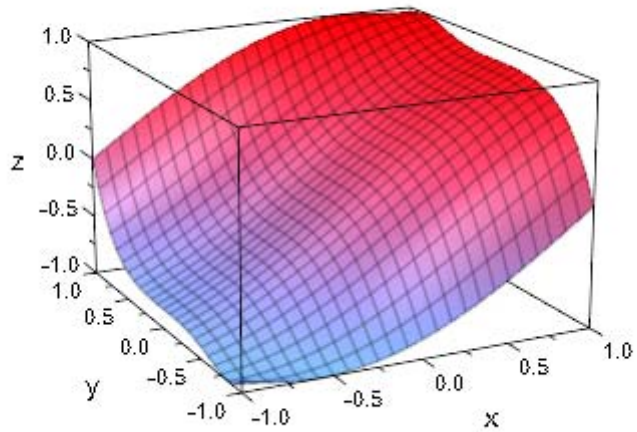
For a full specification of the perspective, there are camera objects of type `plot::Camera` that allow to specify the position of the camera, its focal point and the opening angle of its lens:

```
position := [-5, -10, 5]:
focalpoint := [0, 0, 0]:
angle := PI/12:
camera := plot::Camera(position, focalpoint, angle):
```

This camera can be passed like any graphical object to the `plot` command generating the scene. Once a camera object is specified in a graphical scene, it determines the view. No “automatic camera” is used:

```
plot(plot::Function3d(sin(x + y^3), x = -1..1, y = -1..1),
```

camera):



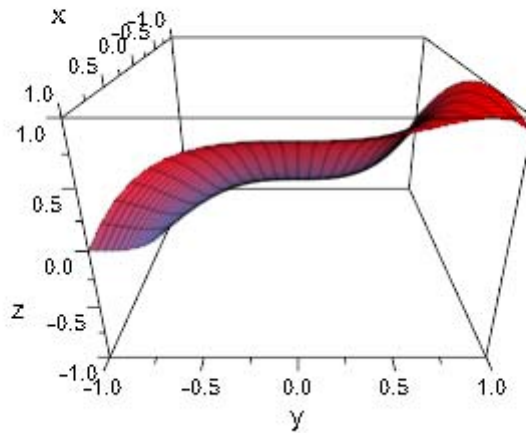
Camera objects can be animated:

```
camera := plot::Camera([3*cos(a), 3*sin(a), 1 + cos(2*a)],  
                        [0, 0, 0], PI/3, a = 0..2*PI,  
                        Frames = 100):
```

Inserting the animated camera in a graphical scene, we obtain an animated plot simulating a “flight around the scene”:

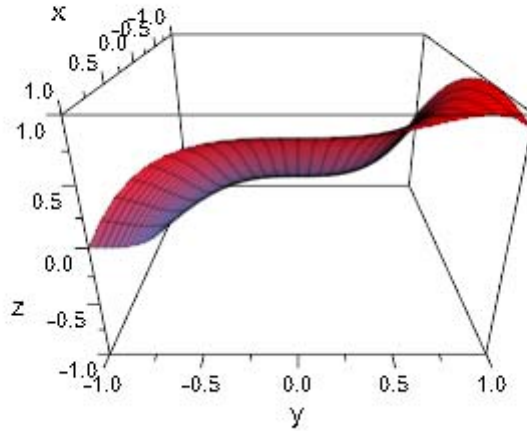
```
plot(plot::Function3d(sin(x + y^3), x = -1..1, y = -1..1),  
      camera):
```



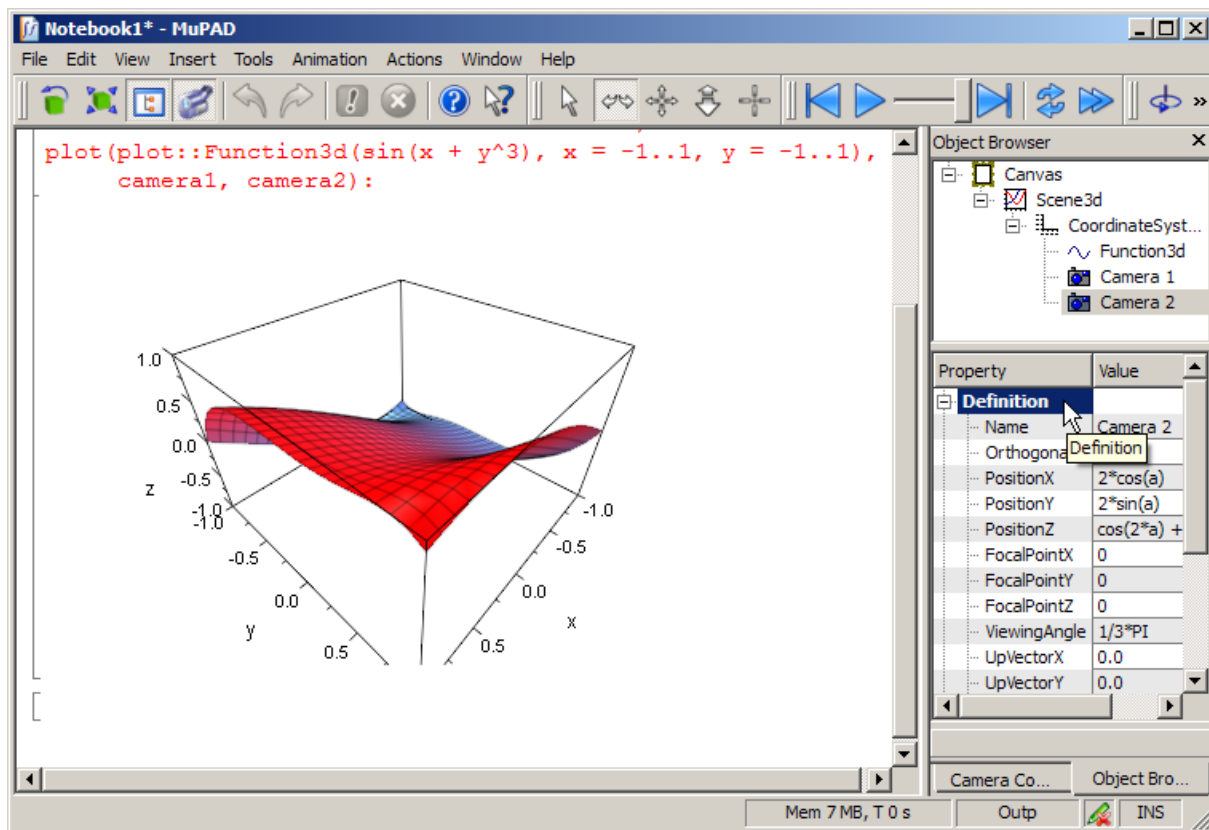


In fact, several cameras can be installed simultaneously in a scene:

```
camera1 := plot::Camera([3*cos(a), 3*sin(a), 1 + cos(2*a)],
                        [0, 0, 0], PI/3, a = 0..2*PI,
                        Name = "Camera 1"):
camera2 := plot::Camera([2*cos(a), 2*sin(a), 2 + cos(2*a)],
                        [0, 0, 0], PI/3, a = 0..2*PI,
                        Name = "Camera 2"):
plot(plot::Function3d(sin(x + y^3), x = -1..1, y = -1..1),
     camera1, camera2):
```



Per default, the first camera produces the view rendered. After clicking on another camera in the object browser of the viewer (see section Viewer, Browser, and Inspector: Interactive Manipulation), the selected camera takes over and the new view is shown.



Next, we have a look at a more appealing example: the so-called “Lorenz attractor.” The Lorenz ODE is the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p(y-x) \\ -xz + rx - y \\ xy - bz \end{pmatrix}$$

with fixed parameters  $p$ ,  $r$ ,  $b$ . As a dynamic system for  $Y = [x, y, z]$ , we have to solve the ODE  $\frac{dY}{dt} = f(t, Y)$  with the following vector field:

```
f := proc(t, Y)
```

```
local x, y, z;  
begin  
  [x, y, z] := Y:  
  [p*(y - x), -x*z + r*x - y, x*y - b*z]  
end_proc:
```

Consider the following parameters and the following initial condition  $Y_0$ :

```
p := 10: r := 28: b := 1: Y0 := [1, 1, 1]:
```

The routine `plot::Ode3d` serves for generating a graphical 3D solution of a dynamic system. It solves the ODE numerically and generates graphical data from the numerical mesh. The plot data are specified by the user via “generators” (procedures) that map a solution point  $(t, Y)$  to a point  $(x, y, z)$  in 3D.

The following generator `Gxyz` produces a 3D phase plot of the solution. The generator `Gyz` projects the solution curve to the  $(y, z)$  plane with  $x = -20$ ; the generator `Gxz` projects the solution curve to the  $(x, z)$  plane with  $y = -20$ ; the generator `Gxy` projects the solution curve to the  $(x, y)$  plane with  $z = 0$ :

```
Gxyz := (t, Y) -> Y:  
Gyz := (t, Y) -> [-20, Y[2], Y[3]]:  
Gxz := (t, Y) -> [Y[1], -20, Y[3]]:  
Gxy := (t, Y) -> [Y[1], Y[2], 0]:
```

With these generators, we create a 3D plot object consisting of the phase curve and its projections. The following command calls the numerical solver `numeric::odesolve` to produce the graphical data. It takes about half a minute on a 1 GHz computer:

```
object := plot::Ode3d(f, [i/10 $ i=1..500], Y0,  
  [Gxyz, Style = Splines, Color = RGB::Red],  
  [Gyz, Style = Splines, Color = RGB::LightGrey],  
  [Gxz, Style = Splines, Color = RGB::LightGrey],  
  [Gxy, Style = Splines, Color = RGB::LightGrey]):
```

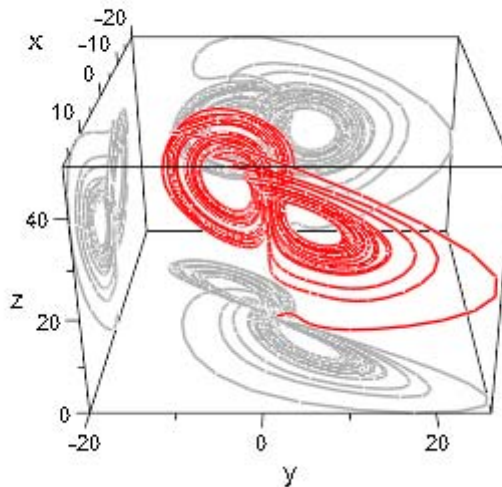
We define an animated camera moving around the scene:

```
camera := plot::Camera([-1 + 100*cos(a), 6 + 100*sin(a), 120],
```

```
[-1, 6, 25], PI/6, a = 0..2*PI,
Frames = 120):
```

The following plot call also takes about half a minute on a 1 GHz computer:

```
plot(object, camera, Axes = Boxed, TicksNumber = Low):
```



Next, we wish to fly *along the Lorenz attractor*. We cannot use `plot::Ode3d`, because we need access to the numerical data of the attractor to build a suitable animated camera object. We use the numerical ODE solver `numeric::odesolve2` and compute a list of numerical sample points on the Lorenz attractor. This takes about half a minute on a 1 GHz computer:

```
Y := numeric::odesolve2(f, 0, Y0, RememberLast):
timemesh := [i/50 $ i = 0..2000]:
Y := [Y(t) $ t in timemesh]:
```

Similar to the picture above, we define a box around the attractor with the projections of the solution curve:

```

box := [-15, 20, -20, 26, 1, 50]:
Yyz := map(Y, pt -> [box[1], pt[2], pt[3]]):
Yxy := map(Y, pt -> [pt[1], pt[2], box[5]]):
Yxz := map(Y, pt -> [pt[1], box[3], pt[3]]):

```

We create an animated camera using an animation parameter  $a$  that corresponds to the index of the list of numerical sample points. The following procedure returns the  $i$ -th coordinate ( $i = 1, 2, 3$ ) of the  $a$ -th point in the list of sample points:

```

Point := proc(a, i)
begin
  if domtype(float(a)) <> DOM_FLOAT then
    procname(args());
  else Y[round(a)][i];
  end_if;
end_proc:

```

In the  $a$ -th frame of the animation, the camera is positioned at the  $a$ -th sample point of the Lorenz attractor, pointing toward the next sample point. Setting `TimeRange = 0..n/10`, the camera visits about 10 points per second:

```

n := nops(timemesh) - 1:
plot(plot::Scene3d(
  plot::Camera([Point(a, i) $ i = 1..3],
    [Point(a + 1, i) $ i = 1..3],
    PI/4, a = 1..n, Frames = n,
    TimeRange = 0..n/10),
  plot::Polygon3d(Y, LineColor = RGB::Red,
    PointsVisible = TRUE),
  plot::Polygon3d(Yxy, LineColor = RGB::DimGrey),
  plot::Polygon3d(Yxz, LineColor = RGB::DimGrey),
  plot::Polygon3d(Yyz, LineColor = RGB::DimGrey),
  plot::Box(box[1]..box[2], box[3]..box[4], box[5]..box[6],
    LineColor = RGB::Black, Filled = TRUE,
    FillColor = RGB::Grey.[0.1]),
  BackgroundStyle = Flat)
):

```



## Possible Strange Effects in 3D

Starting with the MuPAD Release 3.0, the rendering engine for 3D plots uses the OpenGL library. The OpenGL is a widely used standard graphics library and (almost) any computer has appropriate drivers installed in its system.

By default, the MuPAD Graphics Tool uses a software OpenGL driver provided by the operating system. Depending on the graphics card of your machine, you may also have further OpenGL drivers on your system, maybe using hardware support to accelerate OpenGL. (On most Apple Macintosh machines, no software OpenGL is available. MuPAD uses “accelerated” OpenGL on these machines automatically.)

The MuPAD 3D graphics was written and tested using certain standard OpenGL drivers. The numerous drivers available on the market have different rendering quality and differ slightly which may lead to some unexpected graphical effects on your machine.

After clicking on a 3D plot, a “Help” menu is visible in a MuPAD notebook. The item “OpenGL Info ...” provides the information which OpenGL driver you are currently using. By default, MuPAD uses the software driver provided by the operating system, indicated as “Renderer: GDI Generic” on Microsoft Windows, “Apple Software Renderer” on Macintosh systems with software OpenGL, and “Renderer: Mesa GLX Indirect” on a typical Linux system, followed by a line “Direct: No” to indicate that no hardware acceleration is used. You also get the information how many light sources and how many clipping planes this driver supports.

If you wish to switch to another driver, use the item “Configure ...” of the “View” menu. (“Preferences ...” of the “MuPAD” menu on the Macintosh.) Picking “User Interface” on the left, you get a dialog that allows to enable “accelerated” OpenGL using hardware drivers for your graphics card (if installed).

If you encounter strange graphical effects in 3D, we recommend to use the menu item “Help”/“OpenGL Info ...” to check which OpenGL driver you are currently using. Switch to the alternative driver via “View”/“Configure ...”



# Quick Reference

---

## Glossary

This glossary explains some of the terms that are used throughout the MuPAD documentation.

*arithmetical expression*

Syntactically, this is an object of `Type::Arithmetical`. In particular, this type includes numbers, identifiers and expressions of domain type `DOM_EXPR`.

*domain*

The phrase “domain” is synonymous with “data type.” Every MuPAD object has a data type referred to as its “domain type.” It may be queried via the function `domtype`.

There are “basic domains” provided by the system kernel. These include various types of numbers, sets, lists, arrays, hfarrays, tables, expressions, polynomials etc. The documentation refers to these data types as “kernel domains.” The name of the kernel domains are of the form `DOM_XXX` (e.g., `DOM_INT`, `DOM_SET`, `DOM_LIST`, `DOM_ARRAY`, `DOM_HFARRAY`, `DOM_TABLE`, etc.). In addition, the MuPAD programming language allows to introduce new data types via the keyword `domain` or the function `newDomain`. The MuPAD library provides many such domains. For example, `series` expansions, `matrices`, `piecewise` defined objects etc. are domains implemented in the MuPAD language. The documentation refers to such data types as “library domains.” In particular, the library `Dom` provides

---

	<p>a variety of predefined data types such as matrices, residue classes, intervals etc.</p> <p>See <code>DOM_DOMAIN</code> for general explanations on data types. Here you also find some simple examples demonstrating how the user can implement her own domains.</p>
<i>domain element</i>	<p>The phrase “<math>x</math> is an element of the domain <math>d</math>” is synonymous with “<math>x</math> is of domain type <math>d</math>,” i.e., “<code>domtype(x) = d</code>”. In many cases, the help pages refer to “domain elements” as objects of library domains, i.e., the corresponding domain is implemented in the MuPAD language.</p>
<i>domain type</i>	<p>The domain type of an object is the data type of the object. It may be queried via <code>domtype</code>.</p>
<i>flattening</i>	<p>Sequences such as <code>a := (x, y)</code> or <code>b := (u, v)</code> consist of objects separated by commas. Several sequences may be combined to a longer sequence: <code>(a, b)</code> is “flattened” to the sequence <code>(x, y, u, v)</code> consisting of 4 elements. Most functions flatten their arguments, i.e., the call <code>f(a, b)</code> is interpreted as the call <code>f(x, y, u, v)</code> with 4 arguments. Note, however, that some functions (e.g., the operand function <code>op</code>) do not flatten their arguments: <code>op(a, 1)</code> is a call with 2 arguments.</p> <p>The concept of flattening also applies to some functions such as <code>max</code>, where it refers to simplification rules such</p>

*function*

as  $\max(a, \max(b, c)) = \max(a, b, c)$ .

Typically, functions are represented by a procedure or a function environment. Also functional expressions such as `sin@exp + id^2`:  $x \rightarrow \sin(e^x) + x^2$  represent functions. Also numbers can be used as (constant) functions. For example, the call `3(x)` yields the number 3 for any argument `x`.

*number*

A number may be an integer (of type `DOM_INT`), or a rational number (of type `DOM_RAT`), or a real floating-point number (of type `DOM_FLOAT`), or a complex number (of type `DOM_COMPLEX`).

The type `DOM_COMPLEX` encompasses the Gaussian integers such as  $3 + 7*I$ , the Gaussian rationals such as  $3/4 + 7/4*I$ , and complex floating point numbers such as  $1.2 + 3.4*I$ .

*numerical expression*

This is an expression that does not contain any symbolic variable apart from the special constants `PI`, `E`, `EULER`, and `CATALAN`. A numerical expression such as  $I^{(1/3)} + \text{sqrt}(PI) * \text{exp}(17)$  is an exact representation of a real or a complex number; it may be composed of numbers, radicals and calls to special functions. It may be converted to a real or complex floating-point number via `float`.

*overloading*

The help page of a system function only documents the admissible arguments that are of some basic type provided by the MuPAD kernel. If the system function  $f$ , say, is declared as “overloadable,” the user may extend its functionality. He can implement his own domain or function environment with a corresponding slot “ $f$ ”. An element of this domain is then accepted by the system function  $f$  which calls the user-defined slot function.

*polynomial*

Syntactically, a polynomial such as `poly(x^2 + 2, [x])` is an object of type `DOM_POLY`. It must be created by a call to the function `poly`. Most functions that operate on such polynomials are overloaded by other polynomial domains of the MuPAD library.

*polynomial expression*

This is an arithmetical expression in which symbolic variables and combinations of such variables only enter via positive integer powers. Examples are  $x^2 + 2$  or  $x*y + (z + 1)^2$ .

*rational expression*

This is an arithmetical expression in which symbolic variables and combinations of such variables only enter via integer powers. Examples are  $x^{(-2)} + x + 2$  or  $x*y + 1/(z + 1)^2$ . Every polynomial expression is also a rational expression, but the two previous expressions are not polynomial expressions.



# More Information About Some of the MuPAD Libraries

---

- “Abstract Data Types Library” on page 7-2
- “Axioms” on page 7-4
- “Categories” on page 7-5
- “Combinatorics” on page 7-7
- “Functional Programming” on page 7-8
- “Gröbner bases” on page 7-10
- “The import Library” on page 7-11
- “Integration Utilities” on page 7-12
- “Linear Algebra Library” on page 7-16
- “Linear Optimization” on page 7-24
- “The misc Library” on page 7-26
- “Numeric Algorithms Library” on page 7-27
- “Orthogonal Polynomials” on page 7-28
- “Properties” on page 7-29
- “Typeset Symbols” on page 7-32
- “Type Checking and Mathematical Properties” on page 7-41

## Abstract Data Types Library

This library contains MuPAD implementations of abstract data types.

Every instance of these data types is realized as a MuPAD domain.

The usage of this library is completely different from the rest of the MuPAD library: An object of an `adt` data type is a domain, so that by using the methods described here, you change the object itself as a side-effect. No assignment is necessary to keep your changes. Compare this to usual MuPAD functions, where you have to always use, for example,

```
f := fourier(f, x, y)
```

The data types are implemented completely within the MuPAD programming language. Keeping this in mind, the performance is excellent.

### Example

We create an object of the abstract data type “stack” and perform the standard operations.

The stack will be initialized with the characters "a", "b" and "c":

```
S := adt::Stack("a", "b", "c")
```

```
Stack1
```

To handle the stack, it must be assigned to an identifier.

The depth (number of elements) and the top of the stack:

```
S::depth(), S::top()
```

```
3, "c"
```

Push an element, control the depth and then revert the stack. You can see that `S` is changed, e.g., when the method "push" is called:



```
S::push("d"):  
S::depth(), S::top()
```

```
4, "d"
```

The stack is now reverted (although this is not a standard operation for abstract stacks, it comes in handy in many uses). After that, we pop all elements until the stack is empty:

```
S::reverse():  
  
while not S::empty() do  
    print(S::pop())  
end_while;  
S::depth(), S::top()
```

```
"a"
```

```
"b"
```

```
"c"
```

```
"d"
```

```
0, FAIL
```

## Axioms

In MuPAD, an algebraic structure may be represented by a *domain*. Parameterized domains may be defined by *domain constructors*. Many domain constructors are defined in the library package Dom.

Domains which have a similar mathematical structure may be members of a *category*. A category adds a level of abstraction because it postulates conditions which must hold for a domain in order to become a valid member of the category. Operations may be defined for all members of a category based on the assumptions and basic operations of that category, as long as they make no assumptions about the representation of the elements of the domain that belong to the category. Categories may also depend on parameters and are created by *category constructors*. The category constructors of the MuPAD library are contained in the library package Cat.

Attributes of domains and categories are defined in terms of so-called *axioms*. Axioms state properties of domains or categories. They may also depend on parameters and are defined by *axiom constructors*.

Please note that most axioms of the domains and categories defined in the MuPAD library are not stated explicitly. Only axioms which are not implied by the definition of a category are stated explicitly. The category of groups for example has no axiom stating that the multiplication is invertible because that is implied by the definition of a group. Most axioms defined in this package are of technical (i.e. algorithmic nature).

## Bibliography

K. Drescher. Axioms, Categories and Domains. *Automath Technical Report* No. 1, Univ. GH Paderborn 1995.

## Categories

In this section...
“Introduction” on page 7-5
“Category Constructors” on page 7-6
“Bibliography” on page 7-6

### Introduction

In MuPAD, an algebraic structure may be represented by a *domain*. Parametrized domains may be defined by *domain constructors*. Many domain constructors are defined in the library package `Dom`.

Domains which have a similar mathematical structure may be members of a *category*. A category adds a level of abstraction because it postulates conditions which must hold for a domain in order to become a valid member of the category. Operations may be defined for all members of a category based on the assumptions and basic operations of that category, as long as they make no assumptions about the representation of the elements of the domains that belong to the category. Categories may also depend on parameters and are created by *category constructors*.

Attributes of domains and categories are defined in terms of so-called *axioms*. Axioms state properties of domains or categories.

This paper describes the category constructors which are part of the `Cat` library package.

The categories defined so far in general follow the conventions of algebra. There are some properties of the categories which differ from the ‘classical’ nonconstructive theory of algebra because these properties are not constructive or can not be constructed efficiently.

The category hierarchy of the `Cat` package is quite similar to (part of) the category hierarchy of AXIOM JeSu (see DaTr for a description of the basic categories of Scratchpad, the predecessor of AXIOM).

### Category Constructors

For each category constructor only the entries defined directly by the constructor are described. Entries which are inherited from super-categories are not described.

Please note that most axioms of the categories are not stated explicitly. Only axioms which are not implied by the definition of a category are stated explicitly. The category of groups for example has no axiom stating that the multiplication is invertible because that is implied by the definition of a group.

### Bibliography

J.H. Davenport and B.M. Trager. Scratchpad's View of Algebra I: Basic Commutative Algebra. *DISCO '90* (Springer LNCS 429, ed. A. Miola):40–54, 1990.

K. Drescher. Axioms, Categories and Domains. *Automath Technical Report* No. 1, Univ. GH Paderborn 1995.

R.D. Jenks and R.S. Sutor. *AXIOM, The Scientific Computation System*. Springer, 1992.

## Combinatorics

The `combinat` library provides combinatorial functions.

Many more combinatorial functions can be found in the algebraic combinatorics contribution package which can be downloaded at <http://sourceforge.net/>. Note: Computed results may differ after an external package is installed compared to those computed with the original software installation.

The library functions are called using the library name `combinat` and the name of the function. E.g., use

```
combinat::bell(5)
```

to compute the 5-th Bell number. This mechanism avoids naming conflicts with other library functions.

If this is found to be inconvenient, then the routines of the `combinat` library may be exported via `use`. E.g., after calling

```
use(combinat, bell)
```

the function `combinat::bell` may be called directly:

```
bell(5)
```

All routines of the `combinat` library are exported simultaneously by

```
use(combinat)
```

The functions available in the `combinat` library can be listed using

```
info(combinat)
```

## Functional Programming

The functions of the `fp` package are higher order functions and other utilities useful for functional programming. Some other functions useful for functional programming are already contained in the MuPAD standard library, like `map`, `select` and `zip`.

For a more detailed description of concepts like “higher order function”, “currying” and “fixed points” see for example the textbook “Computability, Complexity and Languages” by M. Davis, R. Sigal, and E. J. Weyuker, Academic Press (1994).

Most of the functions of the `fp` package take functions as arguments and return other functions. In this context a function may be a functional environment, a procedure, a kernel function or any other object which may be regarded as a function (i.e. applied to arguments). Note that almost all MuPAD objects are functions in this sense.

The rational integer  $2/3$  for example may be regarded as a constant function returning the value  $2/3$ :

$2/3(x)$

$$\frac{2}{3}$$

The list `[sin, cos, 2/3]` may be regarded as a unary function mapping  $x$  to `[sin(x), cos(x), 2/3]`:

$[sin, cos, 2/3](x)$

$$\left[ \sin(x), \cos(x), \frac{2}{3} \right]$$

The library functions are called in the form `fp::fixedpt(f)`. By this mechanism, naming conflicts with other library functions are avoided. If this is found to be too awkward the methods of the `fp` package may be exported. After calling `use(fp, fixedpt)` the function `fixedpt` is also directly available, i.e. `fixedpt(f)` may also be called. If a variable with the name

`fixedpt` already exists then `use` raises an error. The value of the identifier `fixedpt` must then be deleted in order to be exported. With `use(fp)` all methods of the `fp` package are exported.

## Gröbner bases

The groebner package contains some functions dealing with ideals of multivariate polynomial rings over a field. In particular, Gröbner bases of such ideals can be computed.

An ideal is given by a list of generators. They must all be polynomials of the same type, i.e., for all of them, the coefficient ring (third operand) and list of unknowns (second operand) must be the same. The generators may also be expressions (all of them must be, if any of them is).

Gröbner bases and related notions depend on the monomial ordering (also called term ordering) under consideration. MuPAD knows the following orderings:

- the lexicographical ordering, denoted by the identifier `LexOrder`.
- the ordering by total degree, with the lexicographical ordering used as a tie-break; it is denoted by the identifier `DegreeOrder`.
- the ordering by total degree, with the opposite of the lexicographical ordering for the reverse order of unknowns used as a tie-break (i.e., the monomial that is lexicographically smaller if the order of variables is reversed, is considered the bigger one); this one is denoted by `DegInvLexOrder`.
- user-defined orderings. They constitute a domain `Dom::MonomOrdering` of their own.

Orderings always refer to the order of the unknowns of the polynomial; e.g.,  $x$  is lexicographically bigger than  $y$  in  $F[x, y]$ , but smaller than  $y$  when regarded as an element of  $F[y, x]$ .



## The import Library

The import library provides functions for importing external data into MuPAD.

The package functions are called using the package name `import` and the name of the function. E.g., use

```
import::readdata("datafile")
```

for reading external data from the file 'datafile.' This mechanism avoids naming conflicts with other library functions. If this is found to be inconvenient, then the routines of the `import` package may be exported via `use`. E.g., after calling

```
use(import, readdata)
```

the function `import::readdata` may be called directly:

```
readdata("datafile")
```

All routines of the `import` package are exported simultaneously by

```
use(import)
```

The functions available in the `import` library can be listed with:

```
info(import)
```

## Integration Utilities

### In this section...

“First steps” on page 7-12

“Integration by parts and by change of variables” on page 7-14

This library contains functions for manipulating and solving integrals. Currently there are only described interfaces for the well-known integration methods change of variables and integration by parts. In addition, a function for integrating over arbitrary subsets of the real numbers exists. In future versions more interfaces will be added.

### First steps

Integration is the process inverse to differentiation. Any function  $F$  in the

variable  $x$  with  $\frac{\partial}{\partial x} F = f$  is an integral of  $f$ :

```
f := cos(x)*exp(sin(x))
```

$$e^{\sin(x)} \cos(x)$$

```
F := int(f,x)
```

$$e^{\sin(x)}$$

```
diff(F,x)
```

$$e^{\sin(x)} \cos(x)$$

No constant is added to the integral or, in other words, a special integration constant is chosen automatically. With MuPAD it is possible to determine integrals of elementary functions, of many special functions and, with some restrictions, of algebraic functions:

```
int(sin(x)^4*cos(x),x)
```

$$\frac{\sin(x)^5}{5}$$

`int(1/(2+cos(x)),x)`

$$\frac{2\sqrt{3}\left(\frac{x}{2} - \arctan\left(\tan\left(\frac{x}{2}\right)\right)\right)}{3} + \frac{2\sqrt{3}\arctan\left(\frac{\sqrt{3}\tan\left(\frac{x}{2}\right)}{3}\right)}{3}$$

`int(exp(-a*x^2),x)`

$$\frac{\sqrt{\pi}\operatorname{erf}(\sqrt{a}x)}{2\sqrt{a}}$$

`int(x^2/sqrt(1-5*x^3),x)`

$$-\frac{2\sqrt{1-5x^3}}{15}$$

`normal(simplify(diff(% , x)))`

$$\frac{x^2}{\sqrt{1-5x^3}}$$

It is also possible to compute definite and multiple integrals:

`int(exp(-x^2)*ln(x)^2, x=0..infinity)`

$$\frac{\sqrt{\pi}\left((\text{EULER} + \ln(4))^2 + \frac{\pi^2}{2}\right)}{8}$$

`int(sin(x)*dirac(x+2)-heaviside(x+3)/x, x=1..4)`

$$-\ln(4)$$

```
int(int(int(1, z=0..c*(1-x/a-y/b)), y=0..b*(1-x/a)), x=0..a)
```

$$\frac{abc}{6}$$

## Integration by parts and by change of variables

Typical applications for the rule of integration by parts

$$\int u'(x) v(x) dx = u(x) v(x) - \int u(x) v'(x) dx$$

are integrals of the form  $\int p(x) \cos(x) dx$  where  $p(x)$  is polynomial. Thereby one has to use the rule in the way that the polynomial is differentiated. Thus one has to choose  $u'(x) = \cos(x)$ .

```
intlilb::byparts(hold(int)((x-1)*cos(x),x),cos(x))
```

$$\sin(x) (x-1) - \int \sin(x) dx$$

In particular with the guess  $u'(x) = 1$  it is possible to compute a lot of the well-known standard integrals, like e.g.  $\int \arcsin(x) dx$ .

```
intlilb::byparts(hold(int)(arcsin(x),x),1)
```

$$x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx$$

In order to determine the remaining integral one may use the method change of variable

$$\int f(g(x)) g'(x) dx = F(g(x)) + c$$

with  $g(x) = 1 - x^2$ .

```
F:=intlib::changevar(hold(int)(x/sqrt(1-x^2),x), t=1-x^2)
```

$$\int \left( -\frac{1}{2\sqrt{t}} \right) dt$$

Via backsubstitution into the solved integral F one gets the requested result.

```
hold(int)(arcsin(x),x) = x*arcsin(x) - subs(eval(F),t=1-x^2)
```

$$\int \arcsin(x) dx = x \arcsin(x) + \sqrt{1-x^2}$$

Applying change of variable with the integrator is problematic, since it may occur that the integrator will never terminate. For that reason this rule is used within the integrator only on certain secure places. On the other hand, this may also lead to the fact that some integrals cannot be solved directly.

```
f:= sqrt(x)*sqrt(1+sqrt(x)):
int(f,x)
```

$$\int \sqrt{x} \sqrt{\sqrt{x}+1} dx$$

```
eval(intlib::changevar(hold(int)(f,x),t=sqrt(x))) | t=sqrt(x)
```

$$-\frac{4(\sqrt{x}+1)^{3/2}(42\sqrt{x}-15(\sqrt{x}+1)^2+7)}{105}$$

## Linear Algebra Library

### In this section...

“Introduction” on page 7-16

“Data Types for Matrices and Vectors” on page 7-17

### Introduction

An overview of all the available functions can be obtained by using the MuPAD function `info`. Here we see an extract of the functions available in the linear algebra library (we do not state the whole output generated by the call `info(linalg)`, since the library contains more than 40 different functions):

```
info(linalg)
```

```
Library 'linalg': the linear algebra package
```

```
-- Interface:
linalg::addCol,          linalg::addRow,
linalg::adjoint,        linalg::angle,
linalg::basis,          linalg::charmat,
linalg::charpoly,      linalg::col,
...
```

After being exported, library functions can also be used by their short notation. The function call `use(linalg)` exports all functions of `linalg`. After that one can use the function name `gaussElim` instead of `linalg::gaussElim`, for example.

Please note that user-defined procedures that use functions of the library `linalg` should always use the long notation `linalg::functionname`, in order to make sure that the unambiguity of the function name is guaranteed.

The easiest way to define a matrix  $A$  is using the command `matrix`. The following defines a  $2 \times 2$  matrix:

```
A := matrix([[1, 2], [3, 2]])
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Now, you can add or multiply matrices using the standard arithmetical operators of MuPAD:

`A * A, 2 * A, 1/A`

$$\begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}$$

Further, you can use functions of the `linalg` library:

`linalg::det(A)`

`-4`

The domain type returned by `matrix` is `Dom::Matrix()`:

`domtype(A)`

`Dom::Matrix()`

which is introduced in the following section.

## Data Types for Matrices and Vectors

The library `linalg` is based on the domains `Dom::Matrix` and `Dom::SquareMatrix`. These constructors enable the user to define matrices and they offer matrix arithmetic and several functions for matrix manipulation.

A domain created by `Dom::Matrix` represents matrices of arbitrary rows and columns over a specified ring. The domain constructor `Dom::Matrix` expects a coefficient ring of category `Cat::Rng` (a ring without unit) as argument. If no argument is given, the domain of matrices is created that

represents matrices over the field of arithmetical expressions, i.e., the domain `Dom::ExpressionField()`.

Be careful with calculations with matrices over this coefficient domain, because their entries usually do not have a unique representation (e.g., there is more than one representation of zero). You can normalize the components of such a matrix `A` with the command `map(A, normal)`.

The library `Dom` offers standard coefficient domains, such as the field of rational numbers (`Dom::Rational`), the ring of integers (`Dom::Integer`), the residue classes of integers (`Dom::IntegerMod(n)`) for an integer `n`, or even the rings of polynomials (such as `Dom::DistributedPolynomial(ind,R)` or `Dom::Polynomial(R)`, where `ind` is the list of variables and `R` is the coefficient ring).

A domain created by the domain constructor `Dom::SquareMatrix` represents the ring of square matrices over a specified coefficient domain. `Dom::SquareMatrix` expects the number of rows of the square matrices and optionally a coefficient ring of category `Cat::Rng`.

There are several possibilities to define matrices of a domain created by `Dom::Matrix` or `Dom::SquareMatrix`. A matrix can be created by giving a two-dimensional array, a list of the matrix components, or a function that generates the matrix components:

```
delete a, b, c, d:  
A := matrix([[a, b], [c, d]])
```

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The command `matrix` actually is an abbreviation for the domain `Dom::Matrix()`.

To create diagonal matrices one should use the option `Diagonal` (the third argument of `matrix` is either a function or a list):

```
B := matrix(2, 2, [2, -2], Diagonal)
```



$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

The following two examples show the meaning of the third argument:

```
delete x: matrix(2, 2, () -> x), matrix(2, 2, x)
```

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}, \begin{pmatrix} x(1, 1) & x(1, 2) \\ x(2, 1) & x(2, 2) \end{pmatrix}$$

The MuPAD arithmetical operators are used to perform matrix arithmetic:

```
A * B - 2 * B
```

$$\begin{pmatrix} 2a-4 & -2b \\ 2c & 4-2d \end{pmatrix}$$

```
1/A
```

$$\begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Next we create the 2×2 generalized Hilbert matrix (see also `linalg::hilbert`) as a matrix of the ring of two-dimensional square matrices:

```
MatQ2 := Dom::SquareMatrix(2, Dom::Rational)
```

```
Dom::SquareMatrix(2, Dom::Rational)
```

```
H2 := MatQ2((i, j) -> 1/(i + j - 1))
```

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

A *vector* with  $n$  components is a  $1 \times n$  matrix (a row vector) or a  $n \times 1$  matrix (a column vector).

The components of a matrix or a vector are accessed using the index operator, i.e.,  $A[i, j]$  returns the component of the row with index  $i$  and column with index  $j$ .

The input  $A[i, j] := x$  sets the  $(i, j)$ -th component of the matrix  $A$  to the value of  $x$ .

The index operator can also be used to extract sub-matrices by giving ranges of integers as its arguments:

```
A := Dom::Matrix(Dom::Integer)(
  [[1, 2, 3], [4, 5, 6], [7, 8, 9]]
)
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

```
A[1..3, 1..2], A[3..3, 1..3]
```

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}, (7 \ 8 \ 9)$$

See also the function `linalg::submatrix`.

### Remarks on Improving Runtime

The runtime of user-defined procedures that use functions of the `linalg` library and methods of the constructors `Dom::Matrix` and `Dom::SquareMatrix` can be considerably improved in certain cases.

- 1 Some of the functions of the `linalg` library correspond to certain methods of the domain constructor `Dom::Matrix` in their name and functionality. These functions are implemented by calling relevant methods of the domain to that they belong, apart from additional argument checking. These functions enable an user-friendly usage on the interactive level after exporting.

However, in user-defined procedures the methods of the corresponding domain should be used directly to avoid additionally calls of procedures.

For example standard matrix manipulation functions such as deleting, extracting or swapping of rows and columns are defined as methods of the domain constructors `Dom::Matrix` and `Dom::SquareMatrix`.

The method "`gaussElim`" offers a Gaussian elimination process for each domain created by these constructors.

- 2 When creating a new matrix the method "`new`" is called. It converts each component of the matrix explicitly into a component the component ring, which may be time consuming.

However, matrices and vectors are often the results of computations, whose components already are elements of the component ring. Thus, the conversion of the entries is not necessary. To take this into account, the domain constructors `Dom::Matrix` and `Dom::SquareMatrix` offer a method "`create`" to define matrices in the usual way but without the conversion of the components.

Please note that this method does not test its arguments. Thus it should be used with caution.

- 3 A further possibility of achieving better runtimes using functions of `linalg` or methods of the constructor `Dom::Matrix` is to store functions and methods that are called more than once in local variables. This enables a faster access of these functions and methods.

The following example shows how a user-defined procedure using functions of `linalg` and methods of the domain constructor `Dom::Matrix` may look like. It computes the adjoint of a square matrix defined over a commutative ring (see `Cat::CommutativeRing`):

```

adjoint := proc(A)
  local n, R, i, j, a, Ai, Mat,
    // local variables to store often used methods
    det, delRow, delCol, Rnegate;
begin
  if args(0) <> 1 then
    error("wrong number of arguments")
  end_if;

  Mat := A::dom;          // the domain of A
  R := Mat::coeffRing;   // the component ring of A
  n := Mat::matdim(A);   // the dimension of A; faster than calling
                        // 'linalg::matdim!'

  if testargs() then
    if Mat::hasProp(Cat::Matrix) <> TRUE then
      error("expecting a matrix")
    elif not R::hasProp( Cat::CommutativeRing ) then
      error("expecting matrix over a 'Cat::CommutativeRing'")
    elif n[1] <> n[2] then
      error("expecting a square matrix")
    end_if
  end_if;

  // store often used methods in local variables:
  det := linalg::det;
  delRow := Mat::delRow; // faster than calling 'linalg::delRow!'
  delCol := Mat::delCol; // faster than calling 'linalg::delCol!'
  Rnegate := R::_negate; // faster than using the '-' operator!

  n := Mat::matdim(A)[1]; // faster than calling 'linalg::nrows!'
  a := array(1..n, 1..n);

  for i from 1 to n do
    Ai := delCol(A, i);
    for j from 1 to n do
      a[i, j] := det(delRow(Ai, j));
      if i + j mod 2 = 1 then
        a[i, j] := Rnegate(a[i, j])
      end_if
    end_for
  end_for
end

```

```
end_for;  
  
// create a new matrix: use Mat::create instead of Mat::new  
// because the entries of the array are already elements of R  
return(Mat::create(a))  
end_proc;
```

We give an example:

```
MatZ6 := Dom::Matrix(Dom::IntegerMod(6)):  
adjoint(MatZ6([[1, 5], [2, 4]]))
```

$$\begin{pmatrix} 4 \bmod 6 & 1 \bmod 6 \\ 4 \bmod 6 & 1 \bmod 6 \end{pmatrix}$$

## Linear Optimization

The `linopt` library provides algorithms for linear and integer programming. The routines in this library can be used in order to minimize and maximize a linear function subject to the set of linear constraints. It is possible to get only integer solutions. The routines for linear optimization are based on the two phase simplex algorithm. The algorithm of Land-Doig is used to find integer solutions.

The library functions are called using the library name `linopt` and the name of the function. E.g., use

```
c := [{3*x + 4*y - 3*z <= 23, 5*x - 4*y - 3*z <= 10,
      7*x + 4*y + 11*z <= 30}, - x + y + 2*z, {x, y, z}]:
linopt::maximize(c)
```

```
[OPTIMAL, {x = 0, y = 49/8, z = 1/2}, 57/8]
```

to solve the linear optimization problem defined in the variable `c`. This mechanism avoids naming conflicts with other library functions. If this is found to be inconvenient, then the routines of the `linopt` package may be exported via `use`. E.g., after calling

```
use(linopt, maximize):
```

the function `linopt::maximize` may be called directly:

```
c := [{3*x + 4*y - 3*z <= 23, 5*x - 4*y - 3*z <= 10,
      7*x + 4*y + 11*z <= 30}, - x + y + 2*z, {x, y, z}]:
maximize(c)
```

```
[OPTIMAL, {x = 0, y = 49/8, z = 1/2}, 57/8]
```

All routines of the `linopt` package are exported simultaneously by

```
use(linopt):
```

The functions available in the `linopt` library can be listed with:

```
info(linopt)
```

```
Library 'linopt': a package for linear optimization
```

```
-- Interface:
```

```
linopt::Transparent, linopt::corners, linopt::maximize, linopt::minimize
```

## The misc Library

The `misc` library contains some miscellaneous utility functions.

The package functions are called using the package name `misc` and the name of the function. E.g., use

```
myplus := misc::genassop(_plus, 0)
```

to create your own  $n$ -ary plus operator. (This is not really useful, since `_plus` is an  $n$ -ary operator, anyway.)

This mechanism avoids naming conflicts with other library functions. If this is found to be inconvenient, then the routines of the `misc` package may be exported via `use`. E.g., after calling

```
use(misc, genassop)
```

the function `misc::genassop` may be called directly:

```
myplus := genassop(_plus, 0)
```

All routines of the `misc` package are exported simultaneously by

```
use(misc)
```

The functions available in the `misc` library can be listed with:

```
info(misc)
```

We would especially like to thank Raymond Manzoni for contributing the function `misc::pslq`.



## Numeric Algorithms Library

The `numeric` package provides algorithms from various areas of numerical mathematics.

The package functions are called using the package name `numeric` and the name of the function. E.g., use

```
numeric::solve(equations, unknowns)
```

to call the numerical solver. This mechanism avoids naming conflicts with other library functions. If this is found to be inconvenient, the routines of the `numeric` package may be exported via `use`. E.g., after calling

```
use(numeric, fsolve)
```

the function `numeric::fsolve` may be called directly:

```
fsolve(equations, unknowns)
```

All routines of the `numeric` package are exported simultaneously by

```
use(numeric)
```

Note, however, that presently naming conflicts with the functions `indets`, `int`, `linsolve`, `rationalize`, `solve` and `sort` of the standard library exist. The corresponding functions of the `numeric` package are not exported. Further, if the identifier `fsolve`, say, already has a value, then `use` returns a warning and does not export `numeric::fsolve`. The value of the identifier `fsolve` must be deleted before it can be exported successfully from the `numeric` package.

## Orthogonal Polynomials

The `orthpoly` package provides some standard orthogonal polynomials.

The package functions are called using the package name `orthpoly` and the name of the function. E.g., use

```
orthpoly::legendre(5, x)
```

to generate the fifth degree Legendre polynomial in the indeterminate `x`. This mechanism avoids naming conflicts with other library functions. If this is found to be inconvenient, then the routines of the `orthpoly` package may be exported via `use`. E.g., after calling

```
use(orthpoly, legendre)
```

the function `orthpoly::legendre` may be called directly:

```
legendre(5, x)
```

All routines of the `orthpoly` package are exported simultaneously by

```
use(orthpoly)
```

If the identifier `legendre` already has a value, then `use` returns a warning and does not export `orthpoly::legendre`. The value of the identifier `legendre` must be deleted before it can be exported successfully from the `orthpoly` package.

## Properties

Three types of mathematical properties are available in MuPAD:

- Basic number domains, such as the domain of integers, the domain of rational numbers, the domain of real numbers, the domain of positive real numbers, or the domain of complex numbers
- Intervals in basic number domains
- Residue classes of integers

Often, there are several equivalent ways to specify a property. For example, `>= 0`, `Type::NonNegative`, and `Type::Interval([0], infinity)` are equivalent properties. Similarly, `Type::Odd` is equivalent to `Type::Residue(1, 2)`.

Some members of the `Type` library do not correspond to mathematical properties, for example, `Type::ListOf`.

This table shows some examples of how to set properties on the identifier or expression `expr`.

---

**Note** If `expr` is a list, vector, or matrix, then only the syntax `(expr, set)` is valid. Do not use the syntaxes `(expr in set)` or relations, such as `expr > 0` or `expr < 0`, for nonscalar `expr`.

---

Property	Recommended Way to Specify a Property of <code>expr</code>	Alternative Way to Specify a Property of <code>expr</code>
Real value, $\mathbb{R}$	<code>(expr, R_)</code>	<code>(expr, Type::Real)</code>
Imaginary value, $\mathbb{R}i$	<code>(expr, R_*I)</code> or <code>(expr in R_*I)</code>	<code>(expr, Type::Imaginary)</code>
Complex value, $\mathbb{C}$ . This is a default value for most MuPAD objects.	<code>(expr, C_)</code> or <code>(expr in C_)</code>	<code>(expr, Type::Complex)</code>

<b>Property</b>	<b>Recommended Way to Specify a Property of expr</b>	<b>Alternative Way to Specify a Property of expr</b>
Integer value, $\mathbb{Z}$	(expr, Z_) or (expr in Z_)	(expr, Type::Integer)
Rational value, $\mathbb{Q}$	(expr, Q_)	(expr, Type::Rational)
Positive value, $\mathbb{R}_{>0}$	(expr > 0)	(expr, Type::Positive)
Negative value, $\mathbb{R}_{<0}$	(expr < 0)	(expr, Type::Negative)
Nonnegative value, $\mathbb{R}_{\geq 0}$	(expr >= 0)	(expr, Type::NonNegative)
Zero, {0}	(expr = 0)	(expr, Type::Zero)
Nonzero value, $\mathbb{C} \setminus \{0\}$	(expr <> 0)	(expr, Type::NonZero)
Even value, $2\mathbb{Z}$	(expr, 2*Z_) or (expr in 2*Z_)	(expr, Type::Even)
Odd value, $2\mathbb{Z} + 1$	(expr, 2*Z_ + 1)	(expr, Type::Odd)
Positive integer value, $\mathbb{Z}_{>0}$	(expr in Z_ and expr > 0)	(expr, Type::PosInt)
Negative integer value, $\mathbb{Z}_{<0}$	(expr in Z_ and expr < 0)	(expr, Type::NegInt)
Nonnegative integer value, $\mathbb{Z}_{\geq 0}$	(expr in Z_ and expr >= 0)	(expr, Type::NonNegInt)
Positive rational value, $\mathbb{Q}_{>0}$	(expr in Q_ and expr > 0)	(expr, Type::PosRat)
Negative rational value, $\mathbb{Q}_{<0}$	(expr in Q_ and expr < 0)	(expr, Type::NegRat)

<b>Property</b>	<b>Recommended Way to Specify a Property of expr</b>	<b>Alternative Way to Specify a Property of expr</b>
Nonnegative rational value, $\mathbb{Q}_+$	(expr in $\mathbb{Q}_+$ and expr >= 0)	(expr, Type::NonNegRat)
$\{expr \in T \mid a < expr < b\}$ Here, a,b are expressions, and T is a basic number domain.	(expr in T and a < expr < b)	(expr, Type::Interval(a, b, T))
$\{expr \in T \mid a \leq expr < b\}$ Here, a,b are expressions, and T is a basic number domain.	(expr in T and a <= expr < b)	(expr, Type::Interval([a], b, T))
$\{expr \in T \mid a < expr \leq b\}$ Here, a,b are expressions, and T is a basic number domain.	(expr in T and a < expr <= b)	(expr, Type::Interval(a, [b], T))
$\{expr \in T \mid a \leq expr \leq b\}$ Here, a,b are expressions, and T is a basic number domain.	(expr in T and a <= expr <= b)	(expr, Type::Interval([a], [b], T))
$b + a$ Here, a and b are integers.	(expr, b* $\mathbb{Z}$ + a) or (expr in b* $\mathbb{Z}$ + a)	(expr, Type::Residue(a, b)) or (expr, b*Type::Integer + a)

## Typeset Symbols

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`Symbol` provides access to typesetting symbols. All available symbols are shown below, sorted in appropriate groups. The symbols can be accessed via slots of `Symbol` (e.g. `Symbol::alpha` for the symbol  $a$ ) or via function calls (e.g. `Symbol("alpha")`). Some symbols can be accessed only via function calls. For details, see `Symbol::new`.

### Greek Letters

$a$	alpha	$\beta$	beta
$\gamma$	gamma	$\delta$	delta
	epsi, epsilon	$\varepsilon$	varepsilon, epsiv
$\zeta$	zeta	$\eta$	eta
$\theta$	theta		thetasym, thetav, vartheta

$\iota$	iota	$\kappa$	kappa
$\lambda$	lambda	$\mu$	mu, micro
$\nu$	nu	$\xi$	xi
$\omicron$	omicron	$\pi$	pi
$\varpi$	varpi, piv	$\rho$	rho
$\sigma$	sigma	$\varsigma$	sigmaf, sigmav, varsigma
$\tau$	tau	$\upsilon$	upsilon, upsi
$\varphi$	straightphi, phi	$\phi$	phiv, varphi
$\chi$	chi	$\psi$	psi
$\omega$	omega	$A$	Alpha
$B$	Beta	$\Gamma$	Gamma
$\Delta$	Delta	$E$	Epsi, Epsilon
$Z$	Zeta	$H$	Eta
$\Theta$	Theta	$I$	Iota
$K$	Kappa	$\Lambda$	Lambda
$M$	Mu	$N$	Nu
$\Xi$	Xi	$O$	Omicron
$\Pi$	Pi	$P$	Rho
$\Sigma$	Sigma	$T$	Tau
$Y$	Upsi, upsih, Upsilon	$\Phi$	Phi
$X$	Chi	$\Psi$	Psi
$\Omega$	Omega, ohm		

## Open Face Letters

Aopf	Bopf
Copf	Dopf
Eopf	Fopf
Gopf	Hopf
Iopf	Jopf
Kopf	Lopf
Mopf	Nopf
Oopf	Popf
Qopf	Ropf
Sopf	Topf
Uopf	Vopf
Wopf	Xopf
Yopf	Zopf

## Arrows

↑	uarr, UpArrow	Uparrow, DoubleUpArrow, uArr
↓	darr, DownArrow	Downarrow, dArr, DoubleDownArrow
←	LeftArrow, larr, ⇐ leftarrow, lArr	DoubleLeftArrow, Leftarrow, lArr
↔	leftrightarrow, ⇔ harr, LeftRightArrow	Leftrightarrow, DoubleLeftRightArrow, iff, hArr



$\rightarrow$	RightArrow, rarr, rightarrow	$\Rightarrow$	Rightarrow, Implies, rArr, DoubleRightArrow
---------------	---------------------------------	---------------	---

$\lrcorner$	ldsh
-------------	------

## Operators

$+$	plus	$-$	minus
-----	------	-----	-------

$\pm$	PlusMinus, plusmn, pm		times
-------	--------------------------	--	-------

$\wedge$	Hat, circ	$\cdot$	dot
----------	-----------	---------	-----

	prod		coprod, Coproduct, amalg
--	------	--	--------------------------------

$\prod$		$\amalg$	
$*$	ast	$*$	lowast

$\bullet$	bull, bullet	$\cdot$	cdot, middot, CenterDot, centerdot
-----------	--------------	---------	--

	circ, compfn, SmallCircle	$\circ$	deg
--	------------------------------	---------	-----

$\oplus$	oplus	$\otimes$	otimes
	CirclePlus, bigoplus, xoplus		CircleTimes, bigotimes, xotime

$\oplus$		$\otimes$	
	bigodot, xodot, CircleDot	$\dagger$	dagger

$\ddagger$	Dagger	$\nabla$	Del
------------	--------	----------	-----

$/$	div, divide		Backslash, setminus, setmn, bsol
-----	-------------	--	--

	Element, isin, isin, in	✓	or, vee
∧	wedge, and		Vee, bigvee, xvee
	Wedge, xwedge, bigwedge, And	∨	cap
∆	cup		Intersection, bigcap, xcap
	xcup, bigcup, Union	∩	biguplus, xuplus, UnionPlus
∪	bigsqcup, xsqcup,	⊕	
⊔	SquareUnion		

### Comparison Operators

≈	ap, approx, asyp, TildeTilde	~	tilde, Tilde, sim
≅	TildeFullEqual, cong		bot, bottom, UpTee, perp
≡	Congruent, equiv	=	Equal, equals
≥	ge, geq, GreaterEqual	>	gt
≤	le, leq	≤	NotEqual, ne
<	lt	⊂	Subset, sub, subset
⊆	subsetq, sube, SubsetEqual	⊃	Superset, sup, supset

$\supseteq$	SupersetEqual, supe, supseteq	$\notin$	NotElement, notin
	NotSubset, nsub, nsubset, vnsup		

## Other Symbols

	brvbar	—	UnderBar, UnderLine, lowbar
—	OverBar	#	num, sharp
&	amp	<i>true</i>	true
<i>false</i>	false	<i>unknown</i>	UNKNOWN
<i>Nil</i>	NIL	<i>Fail</i>	FAIL
<i>Null</i>	NULL	<i>NaN</i>	NotANumber
<i>D</i>	CapitalDifferentialD, D, DD		PartialD, part
<i>d</i>	DifferentialD, dd	@	commat
©	copy	™	trade
®	reg	—	die, uml
♦	diam, diams, diamond, diamondsuit	♥	hearts, heartsuit
♠	spades, spadesuit	♣	clubs
<i>E</i>	E	<i>I</i>	I
$\frac{1}{2}$	half, frac12	$\frac{1}{4}$	frac14
$\frac{3}{4}$	frac34	<sup>1</sup>	sup1
<sup>2</sup>	sup2	<sup>3</sup>	sup3

¶	para	‰	permil
%	percent, percent	§	sect

## Whitespaces

“	ApplyFunction	”	InvisibleComma
“	InvisibleTimes	”	Tab
“	blank	”	MediumSpace
“	Space	”	ThickSpace
‘ ’	NonBreakingSpace, nbsp		

## Braces

	lang, langle, LeftAngleBracket	{	lbrace, lcub
/	lbrack, lsqb		lceil, LeftCeiling
	LeftFloor, lfloor	(	lpar
	verbar, vert		Verbar, Vert
)	rpar		rfloor, RightFloor
	rceil, RightCeiling	/	rsqb, rbrack
}	rbrace, rcub		rang, rangle, RightAngleBracket

## Punctuation Marks

::	Colon	:	colon
,	comma	;	semi

.	period	·	sdot
..	Dot, DoubleDot	...	hellip, tdot, dots
...	cdots	—	horbar, mdash
—	hyphen, ndash	’	apos, quot, rsquo
′	prime	˘	backprime, bprime
”	lsquor, sbquo	‘	lsquo, OpenCurlyQuote
„	ldquor, bdquo	“	ldquo, OpenCurlyDoubleQuote
’	rsquor, CloseCurlyQuote	”	Prime, rdquo
”	rdquor, CloseCurlyDoubleQuote	“	laquo
»	raquo	!	excl
¡	iexcl	?	quest
	iquest		

## Umlauts

ä	auml	Ä	Auml
ö	ouml	Ö	Ouml
ü	uuml	Ü	Uuml
ß	szlig		

## Currency

¢	cent	\$	dollar
€	euro	£	pound
¥	yen		

## Math Symbols

∇	nabla	∞∞	vprop, varpropto, vprop
℘	weierp, wp	¬	not
	aleph, alefsym	◇	loz, lozenge
√	Sqrt, radic		Sum, sum
	int, Integral	∑	conint, oint, ContourIntegral
∫	cauchypv, pvint, PrincipalValueIntegral	∮	there4, Therefore, therefore
∫			
∃	exist, Exists	∀	ForAll, forall
∞	infin	∅	emptyv, varnothing
∠	ang, angle	Re	Re
ℜ	real	ℑ	Im, image
e	ee, ExponentialE	i	ImaginaryI, ii

## Type Checking and Mathematical Properties

This library contains several objects to perform syntactical tests with `testtype` (see “Example 1” on page 7-43).

Some of the objects in this library depend on arguments that must be given by the user (see “Example 2” on page 7-43).

Some of the objects can be used as mathematical properties within `assume` and `is` (see “Example 3” on page 7-44).

---

**Note** All other objects that are not properties cannot be used within `assume` and `is` (see “Example 4” on page 7-44).

---

The next tables gives an overview of all objects in this library:

<b>Name</b>	<b>syntactical test</b>	<b>is a property</b>	<b>has arguments</b>
Type::AlgebraicConstant	no	no	no
Type::AnyType	yes	no	no
Type::Arithmetical	yes	no	no
Type::Boolean	yes	no	no
Type::Complex	yes	yes	no
Type::Constant	yes	no	no
Type::ConstantIdentifiers	no	no	no
Type::Equation	yes	no	yes
Type::Even	yes	yes	no
Type::Function	yes	no	no
Type::Imaginary	yes	yes	no
Type::IndepOf	yes	no	yes
Type::Integer	yes	yes	no
Type::Interval	no	yes	yes

<b>Name</b>	<b>syntactical test</b>	<b>is a property</b>	<b>has arguments</b>
Type::ListOf	yes	no	yes
Type::ListProduct	yes	no	yes
Type::NegInt	yes	yes	no
Type::NegRat	yes	yes	no
Type::Negative	yes	yes	no
Type::NonNegInt	yes	yes	no
Type::NonNegRat	yes	yes	no
Type::NonNegative	yes	yes	no
Type::NonZero	yes	yes	no
Type::Numeric	yes	no	no
Type::Odd	yes	yes	no
Type::PolyOf	yes	no	yes
Type::PosInt	yes	yes	no
Type::PosRat	yes	yes	no
Type::Positive	yes	yes	no
Type::Prime	yes	no	no
Type::Product	yes	no	yes
Type::Property	yes	no	no
Type::Rational	yes	yes	no
Type::Real	yes	yes	no
Type::Relation	yes	no	no
Type::Residue	yes	yes	yes
Type::SequenceOf	yes	no	yes
Type::Series	yes	no	yes
Type::Set	yes	no	no
Type::SetOf	yes	no	yes



Name	syntactical test	is a property	has arguments
Type::Singleton	yes	no	no
Type::TableOfEntries	yes	no	yes
Type::TableOfIndices	yes	no	yes
Type::Union	yes	no	yes
Type::Unknown	yes	no	no
Type::Zero	yes	yes	no

### Example 1

testtype performs syntactical tests:

```
testtype([1, 2, 3], Type::ListOf(Type::PosInt))
```

TRUE

```
testtype(3 + 4*I, Type::Constant)
```

TRUE

### Example 2

Some types depends on parameters and cannot be used without parameters:

```
testtype([1, 2, 3], Type::ListOf)
```

FALSE

```
testtype(x = 0, Type::Equation(Type::Unknown, Type::Zero))
```

TRUE

An interval must be given with borders, otherwise it is not a property:

```
assume(x, Type::Interval)
```

```
Error: The second argument must be a property. [assume]
```

```
assume(x, Type::Interval(0, infinity))
```

### **Example 3**

is derives mathematical properties:

```
assume(x > 0):  
is(sqrt(x^2), Type::NonNegative)
```

```
TRUE
```

```
is(-(2*x + 1) < 0)
```

```
TRUE
```

### **Example 4**

Type::Property and Type::Constant are not properties:

```
assume(x, Type::Property)
```

```
Error: The second argument must be a property. [assume]
```

```
is(x, Type::AnyType)
```

```
Error: The type of the second argument is invalid. It must be a property.
```